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Cutting Mutually Congruent Pieces from Convex Regions

R Nandakumar
R.K.M.Vivekananda University, Belur Math, Howrah, India (nandacumar@gmail.com)

ABSTRACT

What is the shape of the 2D convex region P from which, when 2 mutually congruent convex pieces with maximum possible area are cut out, the highest fraction of the area of P is left over? When P is restricted to the set of all possible triangular shapes, our computational search yields an approximate upper bound of 5.6% on the minimum area wasted - when any triangle is given *its* best (most area utilizing) partition into 2 convex pieces. We then produce evidence for the general convex region which wastes the most area for its best convex 2-partition *not* being a triangle.

1 Introducing the Problem

Two planar regions are *congruent* if one can be made to perfectly coincide with the other by translation, rotation or reflection (flipping over).

The Problem: Which is the convex shape P for which the largest fraction of its area gets left over ('wasted') under a partition of it into 2 congruent pieces with the largest possible area?

Define the fraction of the area of a given P covered by 2 mutually congruent pieces cut from it as the 2-coverage of that partition. Any P will have some such partition that maximizes this 2-coverage. We try to find P which has the *least maximum 2-coverage*. We consider only convex pieces; our approach is strongly experimental. To our knowledge, this problem was first stated in [1] and [2].

Aside: We note, in passing, a claim from [1] which remains unexplored, to our knowledge: for any given N, if any given convex region P allows a partition into N non-convex congruent pieces with zero wastage, then P also allows a partition into N convex congruent pieces with zero wastage.

2 A Special Case - Triangles

We first try to find the triangular shape of least maximum 2-coverage. It is easy to construct triangles for which, the maximum 2-coverage cannot be a perfect 1. It can be seen, if the pieces have to be convex, that the maximum 2-coverage cannot be made arbitrarily close to 1 for all possible triangles.

Among the infinitely many ways in which any given triangle T can be partitioned into 2 convex congruent pieces, we (mainly) consider only *3 separate sets of partitions*:

1. *Method 1:* using an angular bisector to cut 2 congruent triangular pieces from T - a total of 3 such candidate partitions. In each partition, a small triangular bit goes waste.

2. *Method 2:* with a suitable line parallel to an edge of T as reference, cut 2 congruent quadrilaterals - this method yields a further 3 candidates. 2 triangular pieces are left over in each candidate partition.

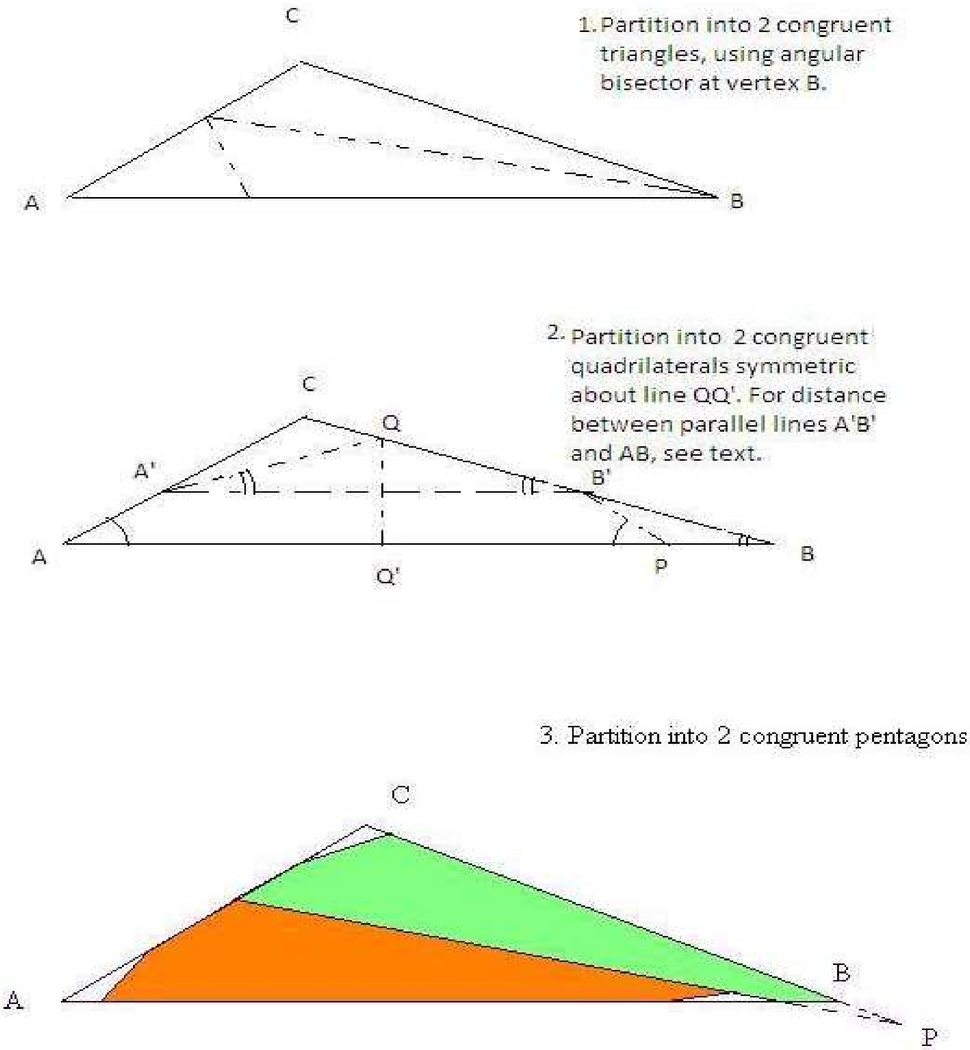

Figure 1:

3. *Method 3:* partition into 2 congruent pentagons, 3 more candidates. Each candidate wastes 3 small triangles.

Figure 1 shows one candidate of each type. For methods 1 and 2, the 'cut lines' are shown dashed. Exception: line $A'B'$ in method 2 (partition into quads) is not a partitioning line but a reference line for the partition. The two quadrilateral pieces in method 2 are $AQ'QA'$ and $PB'QQ'$. In method 3, the 2 congruent pentagons cut from triangle ABC are shown in orange and green colors.

Description of Method 2: if α and β are the angles at base vertices A and B of the full triangle and h, the perpendicular distance of vertex C from base AB, line $A'B'$ is parallel to the base AB at a perpendicular distance d above AB, given by:

$$d = h/(c+1) \text{ where } c = 2\sin\alpha\cos\beta/\sin(\alpha+\beta).$$

The above expression gives the separation between the base and the reference line for which the coverage of the triangle by the 2 congruent quadrilaterals is the most. The derivation: Consider line $A'B'$ passing through ABC at any given distance from AB and parallel to it. If $A'B'$ is displaced by a small distance perpendicular to itself from this reference position, the area of one of the 2 triangular bits left out in the partition generated by $A'B'$ increases and the area of the other left out triangle decreases. The above value of h is such that these two changes cancel - for this h, if $A'B'$ is parallel-displaced, the first order change in the 2-coverage of full triangle ABC vanishes.

Short Description of Method 3: The 2 congruent pentagons are calculated by an exhaustive search within the triangle. The 2 pentagons are not mirror images of each other. The search first finds two congruent triangles, one of which lies entirely inside the triangle ABC and the other has a small portion projecting out. The point P in figure 3 is this external vertex of one of these triangles. These 2 congruent triangles are then trimmed (where needed) and expanded (where possible) resulting in 2 congruent pentagons. There are 2 other candidate partitions of this type - in those cases, the outer point P lies close to the other two vertices of the full triangle.

The Setup:

All triangular *shapes* can be generated by fixing the longest side (equivalently 2 of the vertices) of the triangle and only varying the position of third vertex within a finite region. We fix the longest side of the triangle to run from $(0,0)$ (vertex A in the earlier discussion) to $(10,0)$ - the vertex B. The third vertex $(x_3, y_3) = C$ varies inside a portion of a circle with radius 10 and centered at B (10, 0) as shown in figure 2. It is easy to see that placing the third vertex within the yellow region exhausts all possible *triangle shapes* up to reflections.

For any position of C (x_3, y_3) in above yellow region (ie. for every shape of the triangle), we partition the resulting triangle ABC using all candidates from all methods 1, 2, 3 and select the partition which gives the maximum 2-coverage of that triangle. Then, from all these maximum 2-coverages, we select the minimum; the corresponding triangular shape is output.

Findings:

For each candidate partition, we find that the 2-coverage of triangle ABC for that partition as a function of the position of C has a regular behavior with no multiple local maxima and minima as C varies within its domain. So there is no threat of making qualitative errors if we restrict C to a closely spaced grid of points.

1. For each triangle, if we consider only the 3 candidates from method 1 (using angular bisectors), the least maximum 2-coverage (equivalently, the maximum of the least wastage) is given a sliver (degenerate) triangle which has the highest possible value of *scalenity*; the maximum 2-coverage is $1 - 1/\phi^3 = 0.763..$ (ϕ is the golden ratio). In our setup, the corresponding position of vertex $C = (x_3, y_3)$ is $(3.82.., \delta)$ with δ tending to 0. However, this sliver has a partition by method 2 into congruent quadrilaterals resulting in max 2-coverage of almost 0.9 and indeed, it is not the triangle we are looking for.

2. If we only consider candidate partitions selected from methods 1 and 2, we find another sliver with

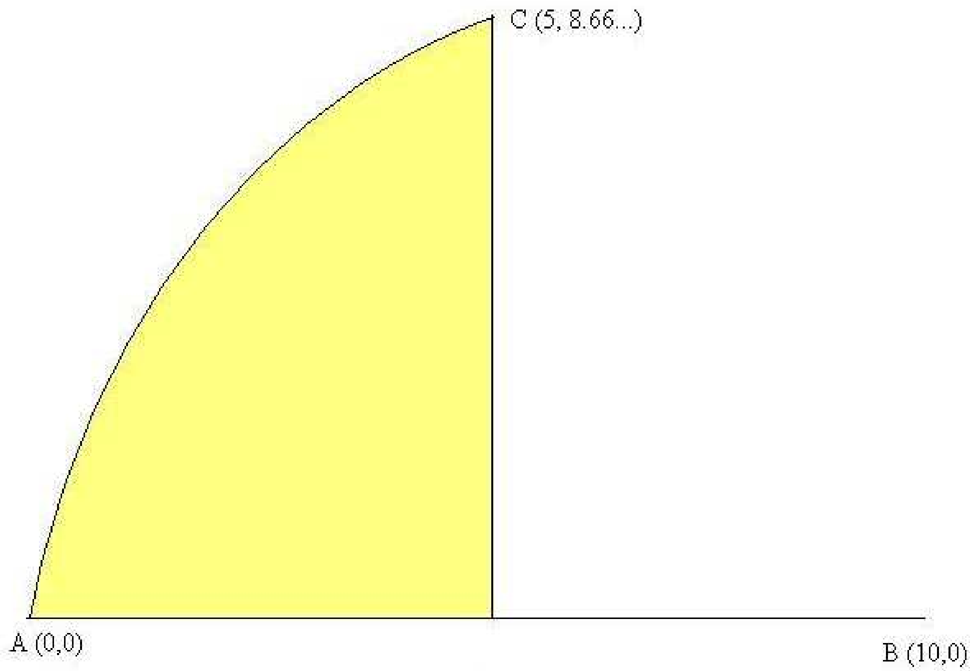

Figure 2:

sides tending to the ratio: $1 : 1/\sqrt{2} : (1 - 1/\sqrt{2})$ to have the highest least wastage. The best 2-partition of this shape has gives a 2-coverage of 0.8284...(a considerable increase from 0.763..) The corresponding Vertex C (x_3, y_3) is at $(2.92..., \delta)$ where δ tends to 0.

3. Finally, when we try all candidates from all the 3 methods for every triangular shape, the triangle with least maximum 2-coverage turns out to be 'fat'. For C at $(4.2, 6.7)$, the best partition of the resulting fat triangle gives a maximum 2-coverage of approximately 0.942 - only just under 6% of the area of this triangle goes waste - and for every other triangle (got by varying C in a lattice), the best 2-coverage is even higher than 0.942.. (and wastage, correspondingly less).

For this most wasteful triangle (which does not waste too much!), the best 2-coverage is given by 2 different candidate partitions - the pentagonal (method 3) partition with B being the closest vertex to the external point P and the partition into 2 congruent triangles given by the bisectors of the angle at C. Both best congruent partitions are shown in Figure 3 - with the left out bits colored (not to scale).

The vertices A and C of the full triangle are not part of either congruent piece in the best pentagonal partition of this triangle; vertex $B = (10, 0)$ is part of one of the pentagonal pieces. On the other hand, vertices A and C are part of the best partition into 2 congruent triangles but vertex B is left out by that partition.

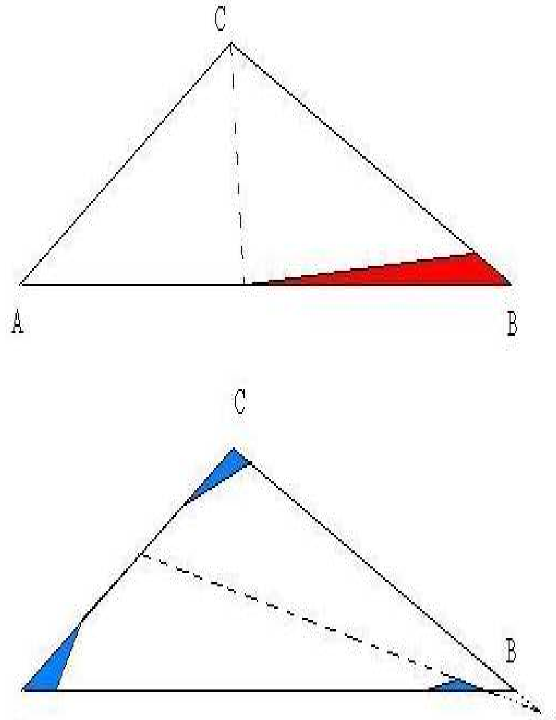

Figure 3:

Note 1: For some positions of vertex $C = (x_3, y_3)$, the congruent partition with maximum 2-coverage (and least wastage) is given by the method 1 above (the one using angular bisectors, yielding 2 triangle pieces); for some positions of C, the maximum 2-coverage is given by method 2 (that gives quadrilaterals); for the other positions of C, method 3 (partition into pentagons) gives the best 2-coverage. Indeed, we have a division of the yellow zone above into 3 different regions and these regions are separated by curves which converge on a 'triple point' which is approximately $(4.5, 5.3)$.

Note 2: If we were to try still more partition schemes, for any given triangular shape, the maximum 2-coverage obviously cannot decrease. So the minimum among the max 2-coverages of all triangles can only increase from 0.942... As noted earlier, with only convex pieces, the least max 2-coverage cannot arbitrarily approach 1 for every triangle so 0.942 ... is very close to the final answer.

Note 3: In the 'neighborhood' of the partition of a triangle into 2 pentagons (method 3), there are partitions into convex polygons with more sides (for example, the pentagonal pieces could be deformed into hexagons leaving out 4 tiny bits from the full triangle). Searching for these partitions and finding the best among them for each triangle could be computationally very expensive and unlikely to improve the bound substantially. However, for the most wasteful triangle, all such close variations on the best pentagonal partition necessarily leave out a small neighborhood of vertex A - the point $(0, 0)$.

3 Generalization

We now consider the wider question: the general convex 2D shape that minimizes the maximum 2-coverage on congruent partitioned into 2 pieces of maximum area.

Claim: In the immediate neighborhood of the most wasteful triangle found by our search, we can find a convex shape with lower maximum 2-coverage than the triangle itself.

Proof: All convex shapes in the immediate neighborhood of a triangle can be got by suitably deforming it at one or more of its vertices. Consider again, the most wasteful triangle with C at (4.2, 6.7) and its 2 best partitions - one into congruent triangles using the angular bisector at vertex C and another into congruent pentagons with the external corner point P lying outside vertex B (figure 3).

Let the most wasteful triangle have area 1 in suitable units. Let a total area of α be covered by the 2 pieces in both the best partitions as in figure 3(α is nearly .942). Trim this triangle slightly at both vertices A and B causing a loss of area of say, ϵ near each A and B. Since the resulting convex polygon with area $1-2\epsilon$ is in the immediate neighborhood of the most wasteful triangle, the partitions of this new polygon which maximize 2-coverage will be suitable slight deformations of the pieces of the best partitions of the triangle. It is easily seen that for the polygon after trimming, its area covered by the two congruent pieces under suitable deformations of either best partition is $\alpha - 2\epsilon$. Then, the 2-coverage for the trimmed triangle under *its* best partitions is $(\alpha - 2\epsilon)/(1-2\epsilon)$. This is slightly less than α, the maximum 2-coverage of the most wasteful triangle. Thus we have a convex shape in the immediate neighborhood of the most wasteful triangle with lower maximum coverage under *its* best possible congruent 2-partition.$\diamond$

We said above that some slight modifications of the pentagonal partition may increase the 2-coverage for triangles. But these new partitions also leave out vertex A of the full triangle ABC and include B; so, we could still make a convex polygon with lower maximum 2-coverage than the resulting wasteful triangle.

4 Conclusions

The main question of the convex 2D shape that has the least maximum 2-coverage remains open. From above arguments, such a convex shape is very likely to be non-triangular. We do not have much of an idea about number of pieces being larger than 2, even for partitioning triangles (except for some special cases) or about higher dimensions. Allowing the pieces to be non-convex could further increase the least maximum coverage. Guess: it may even be possible, given any convex polygon (not only triangles) and any number of pieces, to approach a perfect congruent partition (zero wastage) arbitrarily closely with non-convex pieces; this issue was briefly mentioned in [2].

References

[1] http://maven.smith.edu/ orourke/TOPP/P73.html

[2] R.Nandakumar 'Congruent Partitions of Polygons - a Short Introduction' (http://arxiv.org/abs/1002.0122)