A remark on the definitions of viscosity solutions for the integro-differential equations with Lévy operators.

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1 Introduction.

In this note, we shall consider the following problem

$$F(x,u,\nabla u,\nabla^2 u)-\int_{\mathbf{R^N}}[u(x+z)-u(x)-\mathbf{1}_{|z|\leq 1}\langle\nabla u(x),z\rangle]q(z)dz=0 \qquad x\in\Omega, \tag{1}$$

where $\Omega\subset\mathbf{R^N}$, $F\in C(\Omega\times\mathbf{R}\times\mathbf{R^N}\times\mathbf{S^N})$ is a second-order fully nonlinear elliptic operator, and the Lévy measure $q(z)dz$ is a positive Radon measure such that

$$\int_{|z|<1}|z|^2q(z)dz+\int_{|z|\geq 1}1q(z)dz<\infty. \tag{2}$$

The above type of problems is interested from the view point of the application in the mathematical finances (see Cont and Tankov [7], Sulem and Oksendel [11]). The comparison and the existence results have been studied in some frameworks of the viscosity solutions. However, the equivalence between these notions of viscosity solutions for (1) are not trivial. Here, we would like to give some remarks on the relationships between viscosity solutions defined in different manners.

For an upper (resp. lower) semicontinuous function $u \in USC(\mathbf{R^N})$ (resp. $LSC(\mathbf{R^N})$), we say that $(p, X) \in \mathbf{R^N} \times \mathbf{S^N}$ a subdifferential (resp. superdifferential) of u at x, if for any ¿0 there exists $\varepsilon > 0$ such that

$$u(x+z) - u(x) \leq \quad \langle p, z \rangle + \frac{1}{2}\langle Xz, z \rangle + \bar{.} z|^2 \quad \forall |z| \leq \varepsilon, \tag{3}$$

(resp.

$$u(x+z) - u(x) \geq \quad \langle p, z \rangle + \frac{1}{2}\langle Xz, z \rangle - \bar{.} z|^2 \quad \forall |z| \leq \varepsilon. \tag{4}$$

) We denote the set of all subdifferentials (resp. superdifferentials) of u at x $J^{2,+}_{\mathbf{R^N}} u(x)$ (resp. $J^{2,-}_{\mathbf{R^N}} u(x)$). As is well-known (see Crandall, Ishii and Lions [8]), if (p, X) is a subdifferential (resp. superdifferential) of u at x, then there exists $\phi \in C^2(\mathbf{R^N})$ such that $u(x) = \phi(x)$, $u - \phi$ takes a global maximum (resp. minimum) at x, and for any ¿0 there exists $\varepsilon > 0$ such that

$$u(x+z) - u(x) \leq \phi(x+z) - \phi(x) \leq \langle \nabla\phi(x), z \rangle + \frac{1}{2}\langle \nabla^2\phi(x)z, z \rangle + \bar{.} z|^2 \quad \forall |z| \leq \varepsilon. \tag{5}$$

(resp.

$$u(x+z) - u(x) \geq \phi(x+z) - \phi(x) \geq \langle \nabla\phi(x), z \rangle + \frac{1}{2}\langle \nabla^2\phi(x)z, z \rangle - \bar{.} z|^2 \quad \forall |z| \leq \varepsilon. \tag{6}$$

) In Arisawa [1], [2], [3], the following definition of the viscosity solutions for (1) was introduced.

Definition A. *Let $u \in USC(\mathbf{R^N})$ (resp. $v \in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x} \in \Omega$, any $(p, X) \in J^{2,+}_{\mathbf{R^N}} u(\hat{x})$ (resp. $\in J^{2,-}_{\mathbf{R^N}} v(\hat{x})$), and any pair of numbers (ε, δ) satisfying (3) (resp. (4)), the following holds*

$$F(\hat{x}, u(\hat{x}), p, X) - \int_{|z|<\varepsilon} \frac{1}{2}\langle (X + 2I)z, z \rangle q(z) dz$$
$$- \int_{|z| \geq \varepsilon} [u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z| \leq 1}\langle z, p \rangle] q(z) dz \leq 0. \tag{7}$$

(resp.

$$F(\hat{x}, v(\hat{x}), p, X) - \int_{|z|<\varepsilon} \frac{1}{2}\langle (X - 2I)z, z \rangle q(z) dz$$

$$-\int_{|z|\geq\varepsilon}[v(\hat{x}+z)-v(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,p\rangle]q(z)dz\geq 0. \tag{8}$$

) If u is both a viscosity subsolution and a viscosity supersolution , it is called a viscosity solution.

We can rephrase Definition A by using the test functions in (5) (resp. (6)) as follows.

Definition A'. *Let $u\in USC(\mathbf{R^N})$ (resp. $v\in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x}\in\Omega$ and for any $\phi\in C^2(\mathbf{R^N})$ such that $u(\hat{x})=\phi(\hat{x})$ and $u-\phi$ takes a global maximum (resp. minimum) at $\hat{x}$, and for any pair of numbers $(\varepsilon,\dot{)}$ satisfying (5) (resp. (6)), the following holds*

$$F(\hat{x},u(\hat{x}),\nabla\phi(\hat{x}),\nabla^2\phi(\hat{x}))-\int_{|z|<\varepsilon}\frac{1}{2}\langle(\nabla^2\phi(\hat{x})+2\dot{I})z,z\rangle q(z)dz$$

$$-\int_{|z|\geq\varepsilon}[u(\hat{x}+z)-u(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\phi(\hat{x})\rangle]q(z)dz\underline{\leq}0. \tag{9}$$

(resp.

$$F(\hat{x},v(\hat{x}),\nabla\phi(\hat{x}),\nabla^2\phi(\hat{x}))-\int_{|z|<\varepsilon}\frac{1}{2}\langle(\nabla^2\phi(\hat{x})-2\dot{I})z,z\rangle q(z)dz$$

$$-\int_{|z|\geq\varepsilon}[v(\hat{x}+z)-v(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\phi(\hat{x})\rangle]q(z)dz\geq 0. \tag{10}$$

) If u is both a viscosity subsolution and a viscosity supersolution , it is called a viscosity solution.

We remark that the "global" maximality (resp. minimality) of $u-\phi$ at $\hat{x}$ in Definition A' can be replaced by the "local" maximality (resp. minimality), without changing any meaning of the definition. It is also clear that Definitions A and A' are equivalent. Next, we state the following definition of the viscosity solution in Barles, Buckdahn and Pardoux [4], Jacobsen and Karlsen [10], Barles and Imbert [5].

Definition B. *Let $u\in USC(\mathbf{R^N})$ (resp. $v\in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any*

$\hat{x} \in \Omega$ *and for any* $\phi \in C^2(\mathbf{R^N})$ *such that* $u(\hat{x}) = \phi(\hat{x})$ *and* $u - \phi$ *takes a global maximum (resp. minimum) at* $\hat{x}$*,*

$$F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{z\in\mathbf{R^N}} [\phi(\hat{x}+z) - \phi(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \leq 0. \tag{11}$$

(resp.

$$F(\hat{x}, v(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{z\in\mathbf{R^N}} [\phi(\hat{x}+z) - \phi(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \geq 0. \tag{12}$$

) If u *is both a viscosity subsolution and a viscosity supersolution , it is called a viscosity solution.*

Remark 1. In the above cited works, Definition B was claimed to be equivalent to the following definition.

Definition B'. *Let* $u \in USC(\mathbf{R^N})$ *(resp.* $v \in LSC(\mathbf{R^N})$*). We say that* u *(resp.* v*) is a viscosity subsolution (resp. supersolution) of (1), if for any* $\hat{x} \in \Omega$ *and for any* $\phi \in C^2(\mathbf{R^N})$ *such that* $u(\hat{x}) = \phi(\hat{x})$ *and* $u - \phi$ *takes a global maximum (resp. minimum) at* $\hat{x}$*, and for any* $\varepsilon > 0$*,*

$$\begin{aligned} F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) &- \int_{|z|<\varepsilon} [\phi(\hat{x}+z) - \phi(\hat{x}) - \langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \\ &- \int_{|z|\geq\varepsilon} [u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \leq 0. \end{aligned} \tag{13}$$

(resp.

$$\begin{aligned} F(\hat{x}, v(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) &- \int_{|z|<\varepsilon} [\phi(\hat{x}+z) - \phi(\hat{x}) - \langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \\ &- \int_{|z|\geq\varepsilon} [v(\hat{x}+z) - v(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \geq 0. \end{aligned} \tag{14}$$

) If u *is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.*

The existence of the approximating sequence of test functions $\phi_n(x)$ ($\phi_n(x) \to \phi(x)$ as $n \to \infty$, *a.e.x*; $u(x) \leq \phi_n(x) \leq \phi(x)$ $\forall x \in \mathbf{R^N}$, $\forall n \in \mathbf{N}$, in the case of the subsolution) was used in the argument. Here, we shall consider Definition B, but not B'.

In this paper, thirdly we are interested in the following definition of the viscosity solution, which seems to be stronger than others at a first glance.

Definition C. *Let $u \in USC(\mathbf{R^N})$ (resp. $v \in LSC(\mathbf{R^N})$). We say that u (resp. v) is a viscosity subsolution (resp. supersolution) of (1), if for any $\hat{x} \in \Omega$ and for any $\phi \in C^2(\mathbf{R^N})$ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ takes a global maximum (resp. minimum) at $\hat{x}$, the function $h(z) = u(\hat{x}+z) - u(\hat{x}) - \langle z, \nabla\phi((\hat{x})\rangle \in L^1(\mathbf{R^N}, q(z)dz)$ and*

$$F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{z \in \mathbf{R^N}} [u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z| \leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \leq 0. \tag{15}$$

(resp.

$$F(\hat{x}, v(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{z \in \mathbf{R^N}} [v(\hat{x}+z) - v(\hat{x}) - \mathbf{1}_{|z| \leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \geq 0. \tag{16}$$

) If u is both a viscosity subsolution and a viscosity supersolution , it is called a viscosity solution.

We state the following results on the relationships between Definitions A, B and C.

Theorem 1.

(i) If u is the viscosity subsolution (resp. supersolution) of (1) in the sense of Definition B, then u is the viscosity subsolution (resp. supersolution) of (1) in the sense of Definition C.
(ii) If u is the viscosity subsolution (resp. supersolution) of (1) in the sense of Definition C, then u is the viscosity subsolution (resp. supersolution) of (1) in the sense of Definition B.

Theorem 2.

(i) If u is the viscosity subsolution (resp. supersolution) of (1) in the sense of Definition A, then u is the viscosity subsolution (resp. supersolution) of (1) in the sense of Definition B.

(ii) If u is the viscosity subsolution (resp. supersolution) of (1) in the sense of Definition C, then u is the viscosity subsolution (resp. supersolution) of (1) in the sense of Definition A.

Theorem 3.

The definitions A, B, and C are equivalent.

In the following section 2, we first solve a technical problem, i.e. the construction of the sequence of test fuctions approximating the subsolution u from above. Then, in section 3 the above theorems will be proved by using the result of section 2.

We denote $B_s(x) = \{y| \quad |y-x| < s\} \subset \mathbf{R^N}$ the ball centered at x with the radius s, and $C_s(x) = \{y| \quad |y_i - x_i| < s \quad 1 \underline{<} i \underline{<} N\} \subset \mathbf{R^N}$ (where $x = (x_1, ..., x_N)$, $y = (y_1, ..., y_N)$) the cube centered at x with the length of the edge $2s$. Moreover, we denote

$$B_{s,s'}(x) = \{y| \quad s < |y-x| < s'\} \subset B_{s'}(x) \subset \mathbf{R^N},$$

and denote

$$C_{s,s'}(x) = \{y| \quad s < |y_i - x_i| < s' \quad 1 \underline{<} i \underline{<} N\} \subset C_{s'}(x) \subset \mathbf{R^N}.$$

In the above notations, when $x = 0$ we abbreviate as follows: $B_s = B_s(0)$, $C_s = C_s(0)$, $B_{s,s'} = B_{s,s'}(0)$, and $C_{s,s'} = C_{s,s'}(0)$. Let P be a parallelotope which is the image of a linear transformation T $(rankT = N)$ of a cube C_s, i.e. $P = TC_s$. For $0 < t < t'$ we denote

$$P_t = TC_{ts}, \quad P_{t,t'} = TC_{ts,t's}.$$

We denote $P(x) = x + P$, $P_t(x) = x + P_t$, and $P_{t,t'}(x) = x + P_{t,t'}$.

2 Approximating sequence of test functions.

Let $u(x)$ be an upper semi-continuous function. Assume that there exists $\phi(x) \in C^2(\mathbf{R^N})$, such that $u - \phi$ takes a global maximum at a point $\hat{x} \in \mathbf{R^N}$ and $u(\hat{x}) = \phi(\hat{x})$. In this situation, we would like to construct a sequence of test functions $\psi_n \in C^2(\mathbf{R}^\mathbf{N})$ $(n \in \mathbf{N})$, which roughly speaking, converges to u as $n \to \infty$, by preserving the following properties of ϕ at $\hat{x}$: for any $n \in \mathbf{N}$

$$u - \psi_n \quad \text{takes a global maximum at} \quad \hat{x},$$

$$\nabla \psi_n(\hat{x}) = \nabla \phi(\hat{x}), \quad \nabla^2 \psi_n(\hat{x}) \geq \nabla^2 \phi(\hat{x}), \quad \nabla^2 \psi_n(\hat{x}) \downarrow \nabla^2 \phi(\hat{x}) \quad \text{as} \quad n \to \infty.$$

The construction of such a sequence is not trivial, and we obtained the following very near result.

Proposition 1.

Let $u(x) \in USC(\mathbf{R}^\mathbf{N})$. Assume that there exists $\phi(x) \in C^2(\mathbf{R}^\mathbf{N})$, such that $u - \phi$ takes a global maximum at a point $\hat{x} \in \mathbf{R}^\mathbf{N}$ and $u(\hat{x}) = \phi(\hat{x})$. Then, the following hold.

(i) For any $r \in (0,1)$ there exists $\psi^r(x) \in C^2(\mathbf{R}^\mathbf{N})$ and $P^r(\hat{x})$ a parallelotrope centered at $\hat{x}$, such that $u - \psi^r$ takes a global maximum at $\hat{x}$, $u(\hat{x}) = \psi^r(\hat{x})$, $\nabla \psi^r(\hat{x}) = \nabla \phi(\hat{x})$, $\nabla^2 \psi^r(\hat{x}) = \nabla^2 \phi(\hat{x}) + rI$,

$$\psi^r(x) = \psi^r(\hat{x}) + \langle \nabla \phi(\hat{x}), x - \hat{x} \rangle + \frac{1}{2} \langle (\nabla^2 \phi(\hat{x}) + rI)(x - \hat{x}), x - \hat{x} \rangle \quad in \quad P^r_{\frac{1}{3}}(\hat{x}), \tag{17}$$

$$\nabla^2 \psi^r(x) \geq O \quad in \quad P^r_{\frac{2}{3},1}(\hat{x}), \tag{18}$$

$$P^r(\hat{x}) \subset P^{r'}(\hat{x}) \quad if \quad r < r'; \quad \lim_{r \to 0} \max_{x,y \in P_r(\hat{x})} |x - y| = 0. \tag{19}$$

Moreover, there exists a constant $C > 0$ independent on $r > 0$ such that

$$\psi^r(x) - \psi^r(\hat{x}) - \langle \nabla \psi^r(\hat{x}), x - \hat{x} \rangle \leq C|x - \hat{x}|^2. \quad in \quad P^r(\hat{x}). \tag{20}$$

(ii) For each $r > 0$ there exists a sequence of functions $\psi^r_n(x) \in C^2(\mathbf{R}^\mathbf{N})$ $(n \in \mathbf{N})$ such that

$$\psi^r_n(x) = \psi^r(x) \quad in \quad P^r(\hat{x}), \quad \lim_{n \to \infty} \psi^r_n(x) = u(x) \quad in \quad \mathbf{R}^\mathbf{N} \backslash P^r(\hat{x}), \tag{21}$$

$$\psi^r_{n+1}(x) < \psi^r_n(x) \quad in \quad \mathbf{R}^\mathbf{N} \backslash P^r(\hat{x}). \tag{22}$$

Proof of Proposition 1. Without any loss of generality we may assume that $\hat{x} = 0$, $u(0) = \phi(0) = 0$, $\nabla\phi(0) = 0$.
(i) We shall first construct $\psi^r(x)$ for $r > 0$. Put

$$\psi_0^r(x) = \langle\nabla\phi(0), x\rangle + \frac{1}{2}\langle\nabla^2\phi(0)x, x\rangle + \frac{r}{2}|x|^2. \tag{23}$$

Since $\phi(0) = 0$, there exists a number $s(r) > 0$ such that

$$\phi(x) \leq \psi_0^r(x) \quad x \in B_{2s(r)},$$

and that $u - \psi_0^r$ takes the global strict maximum at 0 in $B_{2s(r)}$. We shall extend ψ_0^r on $\mathbf{R^N}$, so that for the extended new function (by keeping the same notation) $\psi_0^r \in C^2(\mathbf{R^N})$, $u - \psi_0^r$ takes its global strict maximum at 0 in $\mathbf{R^N}$. Remark that the equation: $x_{N+1} = \psi_0^r(x)$ defined in $B_{2s(r)} \subset \mathbf{R^N}$ gives a quadratic surface in $B_{2s(r)} \times \mathbf{R} \subset \mathbf{R^{N+1}}$. Therefore, by the elementary result on the classification of the quadratic surface in the linear algebra, by changing the coordinate system $x = (x_1, ..., x_N)$ if necessary, the quadratic surface given by (23) can be written in the following way

$$\psi_0^r(x) = \sum_{i=1}^{N} l_i x_i^2 \quad x \in B_{2s(r)}, \tag{24}$$

where l_i $(1 \leq i \leq N)$ are the eigenvalues of the matrix $\nabla^2\phi(0) + rI$, we still use the notation $(x_1, ..., x_N)$ for the new coordinate system, and the equation may be considered to hold in $B_{2s(r)}$ of the new coordinate system. We need the following lemma.

Lemma 1.

Let $l < 0$, $s > 0$, and consider $f(x) = lx^2$ in the interval $-s \leq x \leq s$. Let $g(x) = a\exp(-\frac{c}{|x-\alpha|^2}) + b$, where $\alpha = \frac{2s}{3}$, $a = -\frac{els^2}{9}$, $b = \frac{2ls^2}{9}$, and $c = \frac{s^2}{9}$. Define

$$\psi(x) = f(x) \quad 0 \leq x \leq \frac{s}{3}; \quad = g(x) \quad \frac{s}{3} \leq x < \frac{2s}{3}; \quad = \frac{2ls^2}{9} \quad \frac{2s}{3} \leq x \leq s,$$

and $\psi(x) = \psi(-x)$ $(-s \leq x \leq 0)$. Then, $\psi(x)$ is C^2 in $-s \leq x \leq s$, $\psi(x) = f(x)$ in $|x| \leq \frac{s}{3}$ and $\psi(x)$ is convex in $\frac{2s}{3} \leq |x| \leq s$.

Proof of Lemma 1. By the elementary calculation, we see that $f(\frac{s}{3}) = g(\frac{s}{3}) = \frac{ls^2}{9}$, $f'(\frac{s}{3}) = g'(\frac{s}{3}) = \frac{2ls}{3}$, $f''(\frac{s}{3}) = g''(\frac{s}{3}) = 2l$, $\lim_{x\uparrow\frac{2s}{3}} g(x) = \frac{2ls^2}{9}$,

$\lim_{x\uparrow\frac{2s}{3}} g'(x) = \lim_{x\uparrow\frac{2s}{3}} g''(x) = 0$. Thus, we get the function ψ as in the claim.

Assume that $l_i < 0$ $(1\underline{<}i\underline{<}n)$, and $l_i \geq 0$ $(n+1\underline{<}i\underline{<}N)$ in (24). Remark that $C_{s(r)} \subset B_{2s(r)}$, and by using the above lemma for $l = l_i$ $(1\underline{<}i\underline{<}n)$, put

$$\psi_i(x) = \psi(x_i) \quad \text{for} \quad x \in C_{s(r)} \subset \mathbf{R^N}.$$

Define

$$\psi^r(x) = \sum_{i=1}^{n} \psi_i(x) + \sum_{i=n+1}^{N} l_i x_i^2 \quad \text{for} \quad x \in C_{s(r)}. \tag{25}$$

Then from Lemma 1, $\psi^r(x) = \psi_0^r(x)$ in $C_{\frac{s(r)}{3}}$, $\psi^r(x)$ is convex in $C_{\frac{2s(r)}{3},s(r)}$. Consider now the original coordinate system, by putting $P^r = TC_{s(r)}$, where T represents the linear transformation to the original coordinate system. The above argument leads (17) and (18) in the corresponding parallerotope $P^r_{\frac{1}{3}}$, and the doughnut type region $P^r_{\frac{2}{3},1}$. As for (19), if $r < r'$ holds then we can take $s(r) < s(r')$, and the claim is clear from the above argument. From Lemma 1,

$$0 \geq \psi_i(x) \geq \min_{|x_i|<s(r)} l_i x_i^2 \quad x \in C_{s(r)}, \quad 1\underline{<}i\underline{<}n.$$

Therefore, by taking account the way that $\psi^r(x)$ is constructed from $\psi_0^r(x)$ (quadratic in $C_{s(r)}$) in (24) and (25), it is clear that the following holds.

$$\psi^r(x) - \psi^r(0) - \langle \nabla\psi^r(0), x\rangle \underline{<} max_{1\underline{<}i\underline{<}N} l_i |x|^2 \underline{<} C|x|^2 \quad x \in C_{s(r)},$$

where $C = |\nabla^2\phi(0)| + 1$. We consider the above inequality in the original coordinate system, and see that (20) holds in $P^r = TC_{s(r)}$. Therefore, we get the function $\psi^r(x)$ in (i).

(ii) Let $\rho_n > 0$ $(n \in \mathbf{N})$ be a sequence of numbers such that $\lim_{n\to\infty} \rho_n = 0$. From (i) $\psi^r(x)$ is convex in $P^r_{\frac{2}{3},1}$, and $\psi^r(x) > u(x)$ in $P^r\backslash\{0\}$. Thus, for each $n \in \mathbf{N}$ we can extend $\psi^r(x)$ on $P^r_{1+\rho_n}(\supset P^r)$ so that the extended function $\psi_n^r(x)$ is C^2, satisfying

$$\psi_n^r(x) = \psi^r(x) \quad \text{in} \quad P^r, \quad \psi_n^r \quad \text{is convex in} \quad P^r_{\frac{2}{3},1+\rho_n},$$

and $u - \psi_n^r$ takes a global strict maximum at 0 in $P^r_{1+\rho_n}$. Furthermore, since $P^r_{1+\rho_{n+1}} \subset P^r_{1+\rho_n}$, we can extend ψ_n^r on $\mathbf{R^N}$ $(n \in \mathbf{N})$ so that the extended functions (keeping the same notations) $\psi_n^r(x) \in C^2(\mathbf{R^N})$, and

$$\psi_{n+1}^r(x) < \psi_n^r(x) \quad \text{in} \quad \mathbf{R^N} \backslash P^r, \quad \forall n \in \mathbf{N}, \tag{26}$$

$$\lim_{n\to\infty} \psi_n^r(x) = u(x) \quad \text{in} \quad \mathbf{R^N} \backslash P^r. \tag{27}$$

Remark that (26), (27) are possible, for ψ_n^r ($n \in \mathbf{N}$) are convex on ∂P^r. Therefore, we have constructed the sequence $\psi_n^r(x)$ ($n \in \mathbf{N}$) in (ii).

If we do not need the convergence of the second-order derivatives of the test functions: $\nabla^2\psi_n(\hat{x}) \downarrow \nabla^2\phi(\hat{x}) \quad \text{as} \quad n \to \infty$, the construction of the approximating sequence is much simpler. The idea of the following comes from a result in Evans [9].

Proposition 2.

Let $u(x) \in USC(\mathbf{R^N})$. Assume that there exists $\phi(x) \in C^2(\mathbf{R^N})$, such that $u - \phi$ takes a global maximum at a point $\hat{x} \in \mathbf{R^N}$ and $u(\hat{x}) = \phi(\hat{x})$. Then, there exists a sequence of functions $\psi_n(x) \in C^2(\mathbf{R^N})$ such that $u - \psi_n$ takes the global maximum at $\hat{x}$, $u(\hat{x}) = \psi_n(\hat{x})$, $\nabla\phi(\hat{x}) = \nabla\psi_n(\hat{x})$, and

$$\lim_{n\to\infty} \psi_n(x) = u(x), \quad \psi_n(x) \geq u(x) \quad \forall x \in \mathbf{R^N}.$$

Proof of Proposition 2. We may assume that $\hat{x} = 0$, $u(\hat{x}) = \phi(\hat{x}) = 0$, $\nabla\phi(\hat{x}) = 0$, without any loss of the generality. Now, since $\phi \in C^2$, we can take $M_n = \sup_{|x|\leq n^{-1}} |\nabla^2\phi(0)|$ for any $n \in \mathbf{N}$. Put $\psi_n^0(x) = 2M_n|x|^2$ in $\{|x| \leq n^{-1}\}$, and extend it to $\mathbf{R^N}$ so that $\psi_n^0(x) \geq \phi(x)$, $\psi_n^0(x) \in C^2$ on $\mathbf{R^N}$. Remark that $\psi_n^0 - u$ takes its global maximum at 0 for any $n \in \mathbf{N}$. Since ψ_n^0 is convex and radially symmetric in $\{|x| \leq n^{-1}\}$, we can take ψ_n such that

$$\psi_n(x) = \psi_n^0(x) \quad \text{for} \quad |x| \leq (2n)^{-1}; \quad u(x) \leq \psi_n(x) \leq u(x) + n^{-1} \quad \text{for} \quad |x| \geq 2n^{-1},$$

$$\psi_{n+1}(x) \leq \psi_n(x) \quad \text{on} \quad \mathbf{R^N}, \quad \text{for} \quad \forall n \in \mathbf{N}.$$

The sequence of functions $\{\psi_n\}$ ($n \in \mathbf{R^N}$) satisfies the claim, clearly.

Remark 2. From the above construction of $\psi_n(x)$, we only have

$$\nabla^2\phi(\hat{x}) \leq \nabla^2\psi_n(\hat{x}) \quad \text{for} \quad \forall \quad n.$$

Proposition 2 can be used to prove the equivalence of the definitions of viscosity solutions for (1), when F is the first-order Hamiltonian.

Remark 3. The construction of the approximating sequence of test functions for the supersolution can be done similarly.

3 Proofs of the main results.

We use the following well-known elementary theorem of the monotone convergence of Beppo-Levi.

Lemma 2. (Beppo-Levi, see H. Brezis [6].)

Let $f_n(x)$ $(n \in \mathbf{N})$ *be a sequence of increasing functions in* $L^1(\mathcal{O}, d\mu(x))$ $(\mathcal{O} \in \mathbf{R^N})$, *such that* $\sup_n \int_O f_n d\mu(x) < \infty$. *Then,* $f_n(x)$ *converges almost everywhere in* $\mathcal{O}$ *to a function* $f(x)$. *Moreover* $f(x) \in L^1$ *and* $||f_n - f||_{L^1} \to 0$ *as¡* $n \to \infty$.

We begin with the proof of Theorem 1.

Proof of Theorem 1. (i) Let u be a viscosity subsolution (resp. supersolution) of (1) in the sense of Definition B. Assume that there exists $\phi \in C^2(\mathbf{R^N})$ such that $u - \phi$ takes a global maximum at $\hat{x} \in \Omega$, and $u(\hat{x}) = \phi(\hat{x})$. Let $r > 0$ be an arbitrary small number. Then from Lemma 1, there exists a parallelotrope $P^r(\hat{x})$, a function $\psi^r \in C^2$, and a sequence of functions $\psi^r_n \in C^2$ $(n \in \mathbf{N})$ having the properties in (i), (ii) of Lemma 1. Since $u - \psi^r_n$ $(n \in \mathbf{N})$ takes a global maximum at $\hat{x}$, from Definition B

$$F(\hat{x}, u(\hat{x}), \nabla\psi^r_n(\hat{x}), \nabla^2\psi^r_n(\hat{x}))$$

$$-\int_{z\in\mathbf{R^N}}[\psi^r_n(\hat{x}+z) - \psi^r_n(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\psi^r_n(\hat{x})\rangle]q(z)dz \leq 0 \quad \forall n.$$

From Lemma 1 (ii) (21), the above can be written as follows.

$$F(\hat{x}, u(\hat{x}), \nabla\psi^r_n(\hat{x}), \nabla^2\psi^r_n(\hat{x}))$$

$$-\int_{\hat{x}+z\in P^r(\hat{x})}[\psi^r(\hat{x}+z) - \psi^r(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\psi^r(\hat{x})\rangle]q(z)dz$$

$$-\int_{\hat{x}+z\in(P^r(\hat{x}))^c}[\psi_n^r(\hat{x}+z)-\psi_n^r(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\psi_n^r(\hat{x})\rangle]q(z)dz\leq 0 \quad \forall n.$$

Put

$$h_n(z)=\psi_n^r(\hat{x}+z)-\psi_n^r(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\psi_n^r(\hat{x})\rangle \quad \forall n.$$

Then, from the continuity of F and (20), we have

$$\sup_n[-\int_{\hat{x}+z\in(P^r(\hat{x}))^c}h_n(z)q(z)dz]$$

$$\leq\sup_n[-F(\hat{x},u(\hat{x}),\nabla\psi_n^r(\hat{x}),\nabla^2\psi_n^r(\hat{x}))+C\int_{\hat{x}+z\in P^r(\hat{x})}|z|^2q(z)dz]<\infty.$$

From (21), (22), $h_n(z)$ is monotone decreasing as $n\to\infty$ and

$$\lim_{n\to\infty}h_n(z)=u(\hat{x}+z)-u(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\phi(\hat{x})\rangle \quad z\in\{z|\hat{x}+z\in(P^r(\hat{x}))^c\}.$$

Thus, from Lemma 2 (Beppo-Levi) we see $u(\hat{x}+z)-u(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\phi(\hat{x})\rangle\in L^1(\mathbf{R^N},q(z)dz)$, and

$$-\int_{\hat{x}+z\in(P^r(\hat{x}))^c}[u(\hat{x}+z)-u(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\phi(\hat{x})\rangle]q(z)dz$$

$$\leq -F(\hat{x},u(\hat{x}),\nabla\phi(\hat{x}),\nabla^2\phi(\hat{x})+rI)+C_r<\infty,$$

where $C_r>0$ is a constant such that $\lim_{r\to 0}C_r=0$ from (19). Now, by letting $r\to 0$ in the above inequality, from the continuity of F and (19)

$$F(\hat{x},u(\hat{x}),\nabla\phi(\hat{x}),\nabla^2\phi(\hat{x}))-\int_{z\in\mathbf{R^N}}[u(\hat{x}+z)-u(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\phi(\hat{x})\rangle]q(z)dz\leq 0$$

holds. Therefore, u is the viscosity subsolution in the sense of Definition C.

(ii) Let u be a viscosity subsolution (resp. supersolution) of (1) in the sense of Definition C. Assume that there exists $\phi\in C^2(\mathbf{R^N})$ such that $u-\phi$ takes a global maximum at $\hat{x}\in\Omega$, and $u(\hat{x})=\phi(\hat{x})$. From Definition C,

$$F(\hat{x},u(\hat{x}),\nabla\phi(\hat{x}),\nabla^2\phi(\hat{x}))-\int_{z\in\mathbf{R^N}}[u(\hat{x}+z)-u(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\phi((\hat{x})\rangle]q(z)dz\leq 0.$$

Since $u(\hat{x}+z)\leq\phi(\hat{x}+z)$ for any $z\in\mathbf{R^N}$, it is clear hat the above leads

$$F(\hat{x},u(\hat{x}),\nabla\phi(\hat{x}),\nabla^2\phi(\hat{x}))-\int_{z\in\mathbf{R^N}}[\phi(\hat{x}+z)-\phi(\hat{x})-\mathbf{1}_{|z|\leq 1}\langle z,\nabla\phi(\hat{x})\rangle]q(z)dz\leq 0.$$

Therefore, u is the viscosity subsolution in the sense of Definition B.

Remark 4. If F is the first-order Hamiltonian, the approximating sequence ψ_n ($n \in \mathbf{N}$) in Proposition 2 serves to prove the claim in Theorem 1.

Next, we shall prove Theorems 2 and 3.

Proof of Theorem 2. (i) Let u be a viscosity subsolution (resp. supersolution) of (1) in the sense of Definition A. Remark that Definition A is equivalent to Definition A'. Assume that there exists $\phi \in C^2(\mathbf{R}^\mathbf{N})$ such that $u - \phi$ takes a global maximum at $\hat{x} \in \Omega$, and $u(\hat{x}) = \phi(\hat{x})$. Then, for any pair of numbers $(\varepsilon,)$ such that (5) holds,

$$F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{|z|<\varepsilon} \frac{1}{2}\langle(\nabla^2\phi(\hat{x}) + 2I)z, z\rangle q(z)dz$$

$$- \int_{|z|\geq\varepsilon} [u(\hat{x} + z) - u(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \leq 0.$$

Then, since $u(\hat{x} + z) \leq \phi(\hat{x} + z)$ for any $z \in \mathbf{R}^\mathbf{N}$,

$$F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{|z|<\varepsilon} \frac{1}{2}\langle(\nabla^2\phi(\hat{x}) + 2I)z, z\rangle q(z)dz$$

$$- \int_{|z|\geq\varepsilon} [\phi(\hat{x} + z) - \phi(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \leq 0.$$

By tending $\varepsilon \to 0$, this shows that u is the viscosity solution in the sense of Definition B.

(ii) Let u be a viscosity subsolution (resp. supersolution) of (1) in the sense of Definition C. Assume that there exists $\phi \in C^2(\mathbf{R}^\mathbf{N})$ such that $u - \phi$ takes a global maximum at $\hat{x} \in \Omega$, and $u(\hat{x}) = \phi(\hat{x})$. We have

$$F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{z\in\mathbf{R}^\mathbf{N}} [u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \leq 0.$$

Since

$$u(\hat{x} + z) - u(\hat{x}) - \langle z, \nabla\phi(\hat{x})\rangle \leq \phi(\hat{x} + z) - \phi(\hat{x}) - \langle z, \nabla\phi(\hat{x})\rangle$$

$$\leq \frac{1}{2}\langle \nabla^2\phi(\hat{x})z, z\rangle + \ z|^2 \quad |z| \leq \varepsilon,$$

we have

$$F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{|z|<\varepsilon} \frac{1}{2}\langle(\nabla^2\phi(\hat{x}) + 2I)z, z\rangle q(z)dz$$

$$- \int_{|z|\geq\varepsilon} [u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leq 1}\langle z, \nabla\phi(\hat{x})\rangle] q(z)dz \leq 0.$$

That is, Definition C implies Definition A.

Remark 5. For the viscosity supersolutions, the similar claims to those in Theorems 1 and 2 hold, too.

Proof of Theorem 3. The claim comes directly from Theorems 1 and 2.

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