Groups of formal diffeomorphisms in several complex variables and closed one-forms

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Abstract

We study groups of formal diffeomorphisms in several complex variables. For abelian, metabelian or nilpotent groups we investigate the existence of suitable formal vector fields and closed differential forms which exhibit an invariance property under the group action. Our results are applicable in the construction of suitable integrating factors for holomorphic foliations with singularities. We believe they are a starting point in the study of the connection between Liouvillian integration and transverse structures of holomorphic foliations with singularities in the case of arbitrary codimension.

1 Introduction and main results

The study of groups and germs of complex diffeomorphisms fixing the origin is an important tool in the Theory of Holomorphic Foliations, through the study of holonomy groups of its leaves. Indeed, the holonomy groups of (the leaves) of a codimension $n \geq 1$ holomorphic foliation are (identified with) groups of germs of complex diffeomorphisms fixing the origin of $\mathbb{C}^n$. In the codimension $n = 1$ case these are subgroups of germs of one variable holomorphic maps and there is a well-established dictionary relating topological and dynamical properties of (the leaves of) the foliation to algebraic properties of the group. This is clear in works as [2], [7], [9], [10] and [15].

All these facts are compiled in some works relating the existence of suitable "transverse structures" for the foliation with the transverse dynamics of the foliation ([3], [12], [13]). Further, relations with the existence of suitable formal or analytic objects invariant by the holonomy group are obtained (cf. [12]).

Let us be more precise. Denote by $\operatorname{Diff}(\mathbb{C}^n, 0)$ the group of germs of complex diffeomorphisms fixing the origin $0 \in \mathbb{C}^n$. To each germ $f \in \operatorname{Diff}(\mathbb{C}^n, 0)$ we associate its (convergent) power series $f(z) = \sum\limits_{I\mathbb{N}^n} a_I z^I$, $a_I \in \mathbb{C}$. This gives an embedding of $\operatorname{Diff}(\mathbb{C}^n, 0)$ into the group $\widehat{\operatorname{Diff}}(\mathbb{C}^n, 0)$ of formal complex diffeomorphisms in n variables, consisting of all formal series $\hat{f} = \sum\limits_{I\in\mathbb{N}^n} a_I z^I$ with complex coefficients $a_I \in \mathbb{C}$ with $\hat{f}'(0) := a_0 \neq 0$. We denote by $\mathcal{O}_n$ the ring of germs at the origin of holomorphic functions of n variables and by $\hat{\mathcal{O}}_n$ its formal counterpart. Also denote by $\mathfrak{X}(\mathbb{C}^n, 0)$ the $\mathcal{O}_n$-module of germs of complex vector fields vanishing at the origin $0 \in \mathbb{C}^n$ and by $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ its formal counterpart. Compiling results of various authors we have for the one-dimensional case ($n = 1$):

Theorem 1 ([8])**.** *Let $G \subset \widehat{\operatorname{Diff}}(\mathbb{C}, 0)$ be a subgroup.*

1. *G is abelian if, and only if, G admits a formal invariant vector field: $\exists \hat{\xi} \in \hat{\mathfrak{X}}(\mathbb{C}, 0)$ such that $g_* \hat{\xi} = \hat{\xi}$, $\forall g \in G$.*

2. *G is solvable if, and only if, G admits a formal vector field which is projectively invariant by G: $\exists\, \hat{\xi} \in \hat{\mathfrak{X}}(\mathbb{C},0)$ such that for each $g \in G$ we have $g_*\hat{\xi} = c_{\hat{g}}.\hat{\xi}$ for some $c_{\hat{g}} \in \mathbb{C}^*$.*

We also quote that *a subgroup $G < \hat{\text{Diff}}(\mathbb{C},0)$ is solvable if and only if its subgroup G_1 of flat elements is abelian, that is, if and only if, G is metabelian.* This important fact is essentially a consequence of two other facts:

1. For a subgroup $G < \hat{\text{Diff}}(\mathbb{C},0)$ the *derivative group* $DG = \{g'(0) :\ \hat{g} \in G\}$ is abelian and therefore the group of commutators $G^{(1)} := [G,G] =< \hat{g}\hat{h}\hat{g}^{-1}\hat{h}^{-1} :\ \hat{g},\hat{h} \in G >$ is a flat subgroup, i.e., a subgroup of G_1.

2. Two flat elements $\hat{f},\hat{g} \in G_1$ commute only if they have the same order of tangency to the identity.

As we shall see, none of the above facts holds for subgroups of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ when $n \geq 2$. Therefore it is quite natural to expect that the above mentioned dictionary is much different or much harder to find, in the $n \geq 2$ case. To begin this study is one of the main goals of this work. We also aim on possible applications of our results to the framework of holomorphic foliations.

For some of the reasons mentioned above we divide this work in two parts. The first is concerned with the study of flat subgroups, i.e., groups with all elements tangent to the identity. The second is about not necessarily flat groups, but we require the existence of suitable *dicritic* ("radial type") elements in the group.

1.1 Part I - Flat groups

As mentioned above, in the first part we focus on the study of subgroups $G < \hat{\text{Diff}}(\mathbb{C}^n,0)$ under the hypothesis that G is abelian and flat or metabelian with abelian derivative group. Before stating our main results we observe that in some main applications of the results in Theorem 1 (case $n = 1$), the winning strategy is to construct from the information on the holonomy groups of the foliation, some suitable differential forms which allow to "integrate" the foliation (as for instance a foliation admitting a Liouvillian first integral). More precisely, in dimension $n = 1$ a formal vector field $\hat{\xi} \in \hat{\mathfrak{X}}(\mathbb{C},0)$ can be written as

$$\hat{\xi}(z) = \frac{z^{k+1}}{1+\lambda z^k}\frac{d}{dz}$$

for some $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$.

The duality equation $\hat{\omega}\cdot\hat{\xi} = 1$ has, in this dimension one case, a single solution

$$\hat{\omega} = \lambda\frac{dz}{z} + \frac{dz}{z^{k+1}}$$

This expression, is the expression of general closed meromorphic one-form with an isolated pole of order $k+1$ at the origin $0 \in \mathbb{C}$, residue λ, in a suitable coordinate system. It is a particular case of the so called Integration Lemma (see for instance [12] Example 1.6 page 174, or Proposition 5 in Section 5).

Given a formal diffeomorphism $\hat{g} \in \hat{\text{Diff}}(\mathbb{C}^n,0)$ and $\hat{\xi}$ and $\hat{\omega}$ satisfying the duality equation as above, we have:

1. $\hat{g}_*\hat{\xi} = \hat{\xi} \iff \hat{g}^*\hat{\omega} = \hat{\omega}$

2. $\hat{g}_*\hat{\xi} = c_{\hat{g}}\hat{\xi}$ for some $c_{\hat{g}} \in \mathbb{C}^* \iff g^*\hat{\omega} = \frac{1}{c_{\hat{g}}}\hat{\omega}$

Finally, notice that, in dimension-one each formal or meromorphic one-form is closed. This suggests, in view of Theorem 1 and all the above, that one may expect to obtain results relating algebraic properties of subgroups of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$ with the existence of suitable *closed* one-forms.

In Section 3 we prove (cf. Proposition 4) that *an abelian subgroup of formal diffeomorphisms admits an invariant formal vector field.* Nevertheless, unlike the one-dimensional case, in general the existence of such an invariant vector field is not enough to assure that the group is abelian (see Remark 4).

We will adopt the following convention. Denote by $\hat{\text{Diff}}_1(\mathbb{C}^n, 0) < \hat{\text{Diff}}(\mathbb{C}^n, 0)$ the subgroup of *flat* elements, i.e., the subgroup of formal complex diffeomorphisms $\hat{f}$ with $\hat{f}'(0) = \text{Id}$. Given a flat subgroup $G < \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$, by the *dimension* of the associate Lie algebra we mean the dimension of Lie algebra $\exp(G)$, viewed as vector space over $\hat{K}(\mathbb{C}^2)$, the fraction field of $\mathcal{O}(\mathbb{C}^2)$. In [1] it is proved (Proposition 4.1) that every nilpotent subalgebra $\mathcal{L}$ of $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ is metabelian. This proposition implies a characterization of abelian subgroups of $\text{Diff}_1(\mathbb{C}^2, 0)$ (cf. [1] Corollary 4.4, and Proposition 3 in this paper). Applying this to our framework we obtain:

Theorem A. *Let $G < \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ be an abelian flat subgroup. We have the following possibilities:*

(i) *G leaves invariant an exact rational one-form, say $\hat{\omega} = dT$ for some rational function T.*

(ii) *G embeds into the flow of a formal vector field $\hat{\xi}$.*

(iii) *The Lie algebra of G has dimension two. There are two invariant independent commuting formal vector fields.*

We shall see (cf. Theorem **B** below) that a subgroup $G \subset \hat{\text{Diff}}(\mathbb{C}, 0)$ admitting two commuting formal invariant vector fields, exhibits two invariant closed formal meromorphic one-forms.

Unlike the one-dimensional case, the fact that the group is flat and abelian does not imply that it embeds into the flow of a formal vector field. Indeed, the point is that there are flat commuting vector fields of different orders of tangency to the identity (Remark 4 (2)).

The derivative map $D\colon \hat{\text{Diff}}(\mathbb{C}^n, 0) \to \text{GL}(\mathbb{C}, k)$, $\hat{f} \mapsto D\hat{f} := \hat{f}'(0)$, induces by restriction to any subgroup $G < \hat{\text{Diff}}(\mathbb{C}^n, 0)$ a linear map $D\colon G \to \text{GL}(\mathbb{C}, k)$. The kernel of this linear map is the subgroup $G_1 := G \cap \hat{\text{Diff}}_1(\mathbb{C}^n, 0)$ and the image is the *derivative subgroup* $DG < \text{GL}(\mathbb{C}, n)$. If G_1 is trivial then we have an embedding $G \hookrightarrow \text{GL}(\mathbb{C}, n)$.

Let now $G < \hat{\text{Diff}}(\mathbb{C}^2, 0)$ be an abelian subgroup. The Lie algebra of $G_1 = G \cap \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ has dimension ≤ 2. If the dimension is zero then $G_1 = \{\text{Id}\}$ and the map $G \to \text{GL}(2, \mathbb{C})$ embeds G into an abelian linear group. Then, in this case, either G is finite or its image in $\text{GL}(2, \mathbb{C})$ contains a flow in its closure (Remark 1).

As a converse of (iii) in Theorem **A** we have:

Theorem B. *Let $G \leq \hat{\text{Diff}}(\mathbb{C}^2, 0)$ be a subgroup admitting two invariant commuting formal vector fields. Then G admits two closed, independent, formal meromorphic, invariant one-forms. If G is flat or exhibits two formal transverse separatrices then G is abelian.*

The notions of formal closed meromorphic one-form and other formal objects are clearly stated in Section 2 (Remark 2). As a spolium of the proof of the second part of Theorem **B** we obtain normal forms for abelian subgroups admitting two transverse separatrices and having Lie algebra of dimension two (cf. Remark 7).

All the above is concerned with the abelian case. As for the metabelian case we have:

Theorem 2. *Let $G < \hat{\text{Diff}}(\mathbb{C}^2, 0)$ be a metabelian not-abelian subgroup. Assume that $G^{(1)}$ is flat, for instance if the derivative group $DG < \text{GL}(2, \mathbb{C})$ is abelian. Denote by $\mathcal{L}(G^{(1)})$ the Lie algebra of the group of commutators $G^{(1)} = [G, G]$. We have the following possibilities:*

(i) *$\mathcal{L}(G^{(1)})$ is one-dimensional. There is a formal vector field $\hat{\xi}$ such that, for each $\hat{g} \in G$ there is a rational function $T_{\hat{g}}$ that satisfies: $\hat{\xi}(T_{\hat{g}}) = 0$ and $\hat{g}^*(\hat{\xi}) = T_{\hat{g}}\hat{\xi}$.*

(ii) *$\mathcal{L}(G^{(1)})$ is two-dimensional. There are two formal vector fields $\hat{\xi}, \hat{\zeta}$, linearly independent such that*

  (ii.1) $[\hat{\xi}, \hat{\zeta}] = 0$

  (ii.2) *For each $\hat{g} \in G$ there are $(s_1, t_1), (s_2, t_2) \in \mathbb{C}^2$ linearly independent that satisfy $\hat{g}^*(\hat{\xi}) = s_1\hat{\xi} + t_1\hat{\zeta}$ and $\hat{g}^*(\hat{\zeta}) = s_2\hat{\xi} + t_2\hat{\zeta}$.*

In this last case there are two linearly independent closed formal meromorphic one-forms $\hat{\omega}_j$, $(j = 1, 2)$ and $a_j, b_j \in \mathbb{C}^*$ such that

$$\hat{g}^*(\hat{\omega}_j) = a_j\hat{\omega}_1 + b_j\hat{\omega}_2, \ \forall \hat{g} \in G.$$

Groups as in (ii.2) above are studied in Section 6 (cf. Remark 7).

1.2 Part II - Dicritic diffeomorphisms, vector fields and groups

The second part of this work is dedicated to the study of subgroups of formal diffeomorphisms under the hypothesis of existence of a suitable dicritic (radial type) element. Let us introduce the main notion we use. According to [1] a flat diffeomorphism $\hat{f} \in \hat{\text{Diff}}(\mathbb{C}, 0)$ is *dicritic* it writes as $\hat{f}(z) = z + f_{k+1}(z) + \text{h.o.t.}$, where $f_{k+1}(z) = p_f(z)z$ and p_f is a homogeneous polynomial of degree k. This is equivalent (by Proposition 1) to say that $\hat{f}$ is of the form $\hat{f} = \exp(\hat{\xi})$ where $\hat{\xi} \in (\mathbb{C}^n, 0)$ is a formal vector field having as first jet the product of a homogeneous polynomial by the radial vector field. In this paper we introduce a useful subclass of dicritic diffeomorphisms. A formal vector field $\hat{\xi} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ is *dicritic* (respectively, *regular dicritic*) if the formal diffeomorphisms $\hat{f} = \exp(\hat{\xi})$ is dicritic (respectively, regular dicritic). We shall say that $\hat{f}$ is *regular dicritic* if $\hat{\xi}$ has an isolated singularity at the origin. The above concepts are studied in our next results. A subgroup $G < \hat{\text{Diff}}(\mathbb{C}^n, 0)$ is *quasi-abelian* if its subgroup G_1 of flat elements is abelian. A formal vector field $\hat{\xi} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ is *projectively invariant* by $G < \hat{\text{Diff}}(\mathbb{C}^n, 0)$ if for each $\hat{g} \in G$ we have $\hat{g}_*\hat{\xi} = c_{\hat{g}}.\hat{\xi}$ for some $c_{\hat{g}} \in \mathbb{C}^*$.

Our next results are analogous to those in Theorem 1, for groups containing a regular dicritic element.

Theorem C. *A subgroup of formal diffeomorphisms containing a regular dicritic diffeomorphism is quasi-abelian if and only if it admits a projectively invariant regular dicritic formal vector field.*

In particular we obtain:

Corollary 1. *A flat subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$ containing a regular dicritic diffeomorphism is abelian if and only if it admits an invariant formal vector field.*

The proof of Theorem **C** also shows that *a subgroup containing its derivative group and with a regular dicritic diffeomorphism is abelian if and only if the group leaves invariant a formal vector field and its derivative group is abelian* (cf. Proposition 11). Regarding the case of metabelian groups we have:

Theorem 3. *A subgroup of formal diffeomorphisms containing a regular dicritic diffeomorphism and with abelian derivative group is metabelian provided that it admits a projectively invariant formal vector field.*

As a converse of Theorem 3, it is proved in Proposition 13 that a metabelian group $G < \hat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ of formal diffeomorphisms, containing a regular dicritic diffeomorphism $\hat{f}$ of order k and a linear diffeomorphism $\hat{h}(z)\lambda z$ with $\lambda^k \neq 1, \lambda^{k+1} \neq 1$, admits a projectively invariant formal vector field.

As an application we study the case where a group with two generators, one of which is linear, is metabelian (cf. Corollary 4).

Recall that a group G is called *nilpotent* if there is $n \in \mathbb{N}$, such that $\gamma_n(G) = \{\mathrm{Id}\}$, where $\gamma_k(G) = [\gamma_{k-1}(G), G]$ and $\gamma_0(G) = G$. Finally, G is *solvable*, if there is $n \in \mathbb{N}$, such that the n-th commutator is trivial, *i.e.*, $G^{(n)} = \{\mathrm{Id}\}$, where $G^{(k)} = [G^{(k-1)}, G^{(k-1)}]$ and $G^{(0)} = G$.

Next we state an equivalence similar to the dimension one case, but for groups that contain some dicritic diffeomorphism. Theorem **D** below is related to Theorem 4.1 and Corollary 4.2 in [1] and to our Example 1 of a solvable flat group of formal diffeomorphisms which is not metabelian. This example shows the need of our assumption of existence of a dicritic element in the group of commutators in any extension of Theorem 1 to higher dimension.

Theorem D. *For a flat subgroup of formal diffeomorphisms containing a dicritic diffeomorphism with order of tangency $k+1$, the following statements are equivalent:*

(1) *The group is abelian.*

(2) *The group is nilpotent.*

(3) *Every nontrivial element in the group is tangent to the identity with order $k+1$.*

Theorem **D** has the following consequence:

Corollary 2. *A subgroup of formal diffeomorphisms containing a dicritic element in its commutators group and such that the group of commutators is a flat nilpotent group, is metabelian, i.e., its group of commutators is abelian.*

Our results apply to the study of foliations on complex projective spaces and other ambient manifolds as well. The class of singularities which correspond, via the holonomy of its separatrices, to the class of regular dicritic diffeomorphisms is to be formally introduced and studied in a forthcoming work. Using an adaptation of a classical result due to Hironaka and Matsumara ([6], [5]) we may be able to move from the formal world (considered is this paper) to the analytic/convergent world, which is the natural ambient to the study of holomorphic foliations with singularities.

A final word should be said about the possible applications of our results. We are interested in the study of Liouvillian integration for holomorphic foliations of codimension $n \geq 1$. As suggested by the codimension one cases (see for instance [14]), this passes through the comprehension of algebraic, geometric and formal structures of subgroups of $\hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ in terms we propose in this work.

2 Preliminaries

In this paper we write $z = (z_1, ..., z_n)$ as several complex variables. We denote by $\mathbb{C}^2[[z]]_i$ is the set of vectors of $\mathbb{C}^2$, whose coordinates are homogeneous polynomials of degree i. The subgroup of formal diffeomorphisms of two variables, tangent to the identity with order k, is defined as $\hat{\mathrm{Diff}}_k(\mathbb{C}^2, 0) = \{\hat{h}(z) = z + P_k(z) + \cdots \mid \hat{h} \in \hat{\mathrm{Diff}}(\mathbb{C}^2, 0)\}$, $P_k \neq 0$. Similarly the group of germs of holomorphic diffeomorphisms at the origin $0 \in \mathbb{C}^2$, tangent to the identity with order k is defined as $\mathrm{Diff}_k(\mathbb{C}^2, 0) = \hat{\mathrm{Diff}}_k(\mathbb{C}^2, 0) \cap \mathrm{Diff}(\mathbb{C}^2, 0)$. The Lie algebra of formal vectors field of $\mathbb{C}^2$ of order k is defined by $\hat{\mathfrak{X}}_k(\mathbb{C}^2, 0) = \{\hat{f}_1(z)\frac{\partial}{\partial z_1} + \hat{f}_2(z)\frac{\partial}{\partial z_2} \mid \hat{f}_k \in \bigoplus_{i=k}^{\infty} \mathbb{C}^2[[z]]_i\}$ where $\hat{f}_1$ or $\hat{f}_2$ has order k.

Proposition 1 ([8])**.** *The exponential map* $\exp : \hat{\mathfrak{X}}_k(\mathbb{C}^2, 0) \to \hat{\mathrm{Diff}}_k(\mathbb{C}^2, 0)$ *is a bijection.*

According to [1] Theorem 3.3 *a subgroup* $G \subset \mathrm{Diff}(\mathbb{C}^n, 0)$ *is analytically linearizable if and only if it admits an analytic invariant vector field of the form* $z = R + \mathrm{h.o.t.}$ *where* R *is the radial vector field.* In this same work it is proved (Proposition 4.1) that every nilpotent subalgebra $\mathcal{L}$ of $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ is metabelian. This proposition implies the following characterization of abelian subgroups of $\mathrm{Diff}_1(\mathbb{C}^2, 0)$ (cf. [1] Corollary 4.4).

Proposition 2. *If* $G < \mathrm{Diff}_1(\mathbb{C}^2, 0)$ *is a flat abelian* (*convergent*) *group, then one of the following items is true:*

(1) *There is a formal vector field* $\hat{\xi}$ *with* $\exp(\hat{\xi}) \in G$ *such that for each* $g \in G$ *we have* $g = \exp(T_g\hat{\xi})$ *where* T_g *is a rational holomorphic function such that* $\hat{\xi}(T_g) = 0$*;*

(2) $G < \langle f^{[t]} \circ g^{[s]} | t, s \in \mathbb{C}\rangle$, *where* $f, g \in G$ *and* $[f, g] = \mathrm{Id}$.

Notice that Proposition 2 above is for convergent (analytic) objects. The formal version is easily obtained by a mimic of the proof, and reads as:

Proposition 3 ([1])**.** *Let* $G \leq \hat{\mathrm{Diff}}_1(\mathbb{C}^2, 0)$ *be an abelian flat subgroup, we have the two following possibilities:*

1. *There is a formal vector field* $\hat{\xi}$*, invariant by* G*, such that for each element* $\hat{f} \in G$ *there is a rational function* T *depending on* $\hat{f}$*, such that* $\hat{\xi}(N) = 0$ *and* $\hat{f} = \exp(N\hat{\xi})$*.*

2. *There are formal commuting vector fields* $\hat{\xi}$ *and* $\hat{\zeta}$ *such that* $\exp(\hat{\xi}), \exp(\hat{\zeta}) \in G$ *and* $G < \langle \exp(t\hat{\xi}) \circ \exp(s\hat{\zeta}) \mid t, s \in \mathbb{C}\rangle$*.*

Remark 1 (infinite linear groups)**.** Let $G < \mathrm{GL}(k, \mathbb{C})$ be an infinite linear algebraic group. Then its Lie Algebra $\mathcal{L}(G)$ is not trivial and we may choose a (linear) vector field $z \in \mathcal{L}(G)$. The Zariski closure $\overline{\{X_t\}}$ of the flow z_t of z in $\mathbb{C}^n$ is a closed abelian subgroup of the closure $\bar{G}$. Since $\overline{\{X_t\}}$ is abelian, there is a closed one-parameter subgroup H, which is a one-dimensional linear algebraic subgroup of G.

Remark 2 (formal meromorphic objects)**.** By a formal meromorphic function of n complex variables we shall mean a formal quotient $\hat{R} = \frac{\hat{P}}{\hat{Q}}$ of two formal power series with positive exponents $\hat{P}, \hat{Q} \in \hat{\mathcal{O}}_n = \mathbb{C}[[z]]$. In other words, the field of formal meromorphic functions of n variables $\hat{\mathcal{M}}_n$ will be the fraction field of the domain of integrity $\hat{\mathcal{O}}_n$. By a meromorphic one-form we mean a formal expression $\hat{\omega} = \sum_{j=1}^{n} \hat{R}_j \, dz_j$ where each $\hat{R}_j$ is a formal meromorphic function as defined above. The exterior derivative, wedge product and other concepts are defined for meromorphic formal one-forms in the same way as for analytic one-forms.

Remark 3 (formal separatrices)**.** A *formal curve* of n complex variables is defined as follows: denote by $\hat{\mathcal{O}}_n$ the ring of formal functions of n complex variables. In $\hat{\mathcal{O}}_n$ we introduce the equivalence relation $\hat{f} \sim \hat{g} \iff \hat{\varphi} = \hat{u}.\hat{\psi}$ for some unit $\hat{u} \in \hat{\mathcal{O}}_n$, i.e., for some power series $\hat{u}$ with first coefficient $u_0 \neq 0$. By a formal curve we mean an equivalence class of a function $\hat{\varphi} \in \hat{\mathcal{O}}_n$ that satisfies $\hat{\varphi}(0) = 0$, that is, a non-invertible formal power series. Such a formal curve is called invariant by a formal complex diffeomorphism $\hat{f} \in \hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ if $\hat{f}^*\varphi = \varphi \circ \hat{f}$ is equivalent to φ in the above sense. Such a formal curve will be called a separatrix of a subgroup $G < \hat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ if it is invariant by each element of this group. The tangent space of a formal curve with representative $\hat{\varphi}$ is defined as the linear subspace of $\mathbb{C}^2$ given by the kernel of $D\hat{\varphi}(0)\colon \mathbb{C}^2 \to \mathbb{C}$. Two formal curves with representatives $\hat{\varphi}$ and $\hat{\psi}$ are called transverse if their tangent spaces span $\mathbb{C}^2$.

3 Construction of a formal invariant vector field

In this section we prove the existence of an invariant formal vector field for an abelian group:

Proposition 4. *An abelian subgroup of formal diffeomorphisms admits an invariant formal vector field.*

Some steps in the proof of the following well-known lemma will be used later on this paper:

Lemma 1. *If $\hat{f} \in \hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ commutes with the time one flow map of a formal vector field $\hat{\xi} \in \hat{\mathfrak{X}}_k(\mathbb{C}^n, 0)$, $k \geq 2$ then $\hat{f}$ commutes with the flow of $\hat{\xi}$ for all time $t \in \mathbb{C}$.*

Proof. Let $\hat{\Phi}_t$ be the (formal) flow of $\hat{\xi}$, which is defined by $\hat{\Phi}_t := \exp(t\hat{\xi}) \in \hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. Then $\hat{\Phi}_1 \circ \hat{f} = \hat{f} \circ \hat{\Phi}_1$. We claim that $\hat{\Phi}_t \circ \hat{f} = \hat{f} \circ \hat{\Phi}_t$ for all $t \in \mathbb{Z}$. First we prove this by induction, $\forall t \in \mathbb{N}$. In fact, this is true for $t = 1$, Suppose that equality holds for $n \in \mathbb{N}$. Then

$$\hat{\Phi}_{n+1} \circ \hat{f} = \hat{\Phi} \circ \hat{\Phi}_n \circ \hat{f} = \hat{\Phi} \circ \hat{f} \circ \hat{\Phi}_n = \hat{f} \circ \hat{\Phi} \circ \hat{\Phi}_n = \hat{f} \circ \hat{\Phi}_{n+1}$$

thus $\hat{\Phi}_t \circ \hat{f} = \hat{f} \circ \hat{\Phi}_t$, $\forall t \in \mathbb{N}$. Now to show that $\hat{\Phi}_t \circ \hat{f} = \hat{f} \circ \hat{\Phi}_t$, $\forall t \in \mathbb{C}$, is sufficient to prove this equality in the spaces of jets, i.e. in $\mathcal{J}^k(\mathbb{C}^n, 0) = \mathbb{C}[[z]]/\mathfrak{m}^{k+1}$ (this has a natural identification with the space of polynomials of degree less than or equal to k), where $\mathfrak{m} = \{\hat{f} \in \mathbb{C}[[z]]/\hat{f}(0) = 0\}$ is the maximal ideal of $\mathbb{C}[[z]]$. Indeed, given $k \in \mathbb{N}$ we have that $j^k \circ \hat{\Phi}_t \circ \hat{f} = (f_1, \ldots, f_n)$, where the truncation of formal series

$$j^k : \mathbb{C}[[z]] \to \mathcal{J}^k(\mathbb{C}^n, 0),$$

is defined by $j^k(\hat{f}) = \hat{f} \mod \mathfrak{m}^{k+1}$, we have that

$$f_l(z) = \sum_{|N| \leq k} P^l_N(t) z^N$$

e $P^l_N(t)$ is a polynomial of degree less than or equal to $|N|$. Similarly, we have
$j^k \circ \hat{f} \circ \hat{\Phi}_t = (\widetilde{f}_1, \ldots, \widetilde{f}_n)$ where

$$\widetilde{f}_l(z) = \sum \widetilde{P}^l_N(t) z^N$$

e $\widetilde{P}^l_N(t)$ is a polynomial of degree less than or equal to $|N|$. now, as $\hat{\Phi}_t \circ \hat{f} = \hat{f} \circ \hat{\Phi}_t$, $\forall t \in \mathbb{Z}$, for each $N \in \mathbb{N}^n$ with $|N| \leq k$, we have that $P^l_N(t) \mid_{\mathbb{Z}} = \widetilde{P}^l_N(t) \mid_{\mathbb{Z}}$, now as these are polynomial and coincide in $\mathbb{Z}$, we have that

$$P^l_N(t) = \widetilde{P}^l_N(t), \forall t \in \mathbb{C}.$$

in consequence $f_l(z) = \widetilde{f}_l(z)$ $\forall X \in \mathbb{C}^n$, $l \in \{1, \ldots, n\}$. therefore $j^k \circ \hat{\Phi}_t \circ \hat{f} = j^k \circ \hat{f} \circ \hat{\Phi}_t$, $\forall t \in \mathbb{C}$ e $k \in \mathbb{N}$. So

$$\hat{\Phi}_t \circ \hat{f} = \hat{f} \circ \hat{\Phi}_t, \forall t \in \mathbb{C}.$$

On the other hand, if $\hat{\Phi}_t \circ \hat{f} = \hat{f} \circ \hat{\Phi}_t$ then $\hat{\Phi}_{-t} \circ \hat{f} = \hat{f} \circ \hat{\Phi}_{-t}$. Consequently $\hat{\Phi}_t \circ \hat{f} = \hat{f} \circ \hat{\Phi}_t$, $\forall t \in \mathbb{Z}$. Now note that $\hat{\Phi}_t \circ \hat{f} = (f_1, \ldots, f_n)$ where $f_k \in \mathbb{C}[t][[z]]$. Thus $f_k = \sum P_N^k(t) z^N$ where $P_N^k(t)$ is a polynomial of degree less or equal to $|N|$. Similarly $\hat{f} \circ \hat{\Phi}_t = (\widetilde{f}_1, \ldots, \widetilde{f}_n)$ where $\widetilde{f}_k \in \mathbb{C}[t][[z]]$, thus $\widetilde{f}_k = \sum \widetilde{P}_N^k(t) z^N$. Since $\hat{\Phi}_t \circ \hat{f} = \hat{f} \circ \hat{\Phi}_t$, $\forall t \in \mathbb{Z}$, then $P_N^k(t) \mid_{\mathbb{Z}} = \widetilde{P}_N^k(t) \mid_{\mathbb{Z}}$ for z Fixed. Since they are polynomials and coincide in $\mathbb{Z}$, we have $P_N^k(t) = \widetilde{P}_N^k(t)$, $\forall t \in \mathbb{C}$. consequently $f_k(z) = \widetilde{f}_k(z)$ $\forall X \in \mathbb{C}^n$ and $k \in \{1, \ldots, n\}$. Therefore $\hat{\Phi}_t \circ \hat{f} = \hat{f} \circ \hat{\Phi}_t$, $\forall t \in \mathbb{C}$. □

Proof of Proposition 4. First we assume that G_1 is nontrivial. Thus there is $\hat{f} \in G_1$ which is of the form $\hat{f} = \exp(\hat{\xi})$ for some formal vector field $\hat{\xi} \in \hat{\mathfrak{X}}_j(\mathbb{C}^n, 0)$, $j \geq 2$. Since G is abelian, for any $\hat{g} \in G$, $\hat{g} \circ \hat{f}(z) = \hat{f} \circ \hat{g}(z)$, *i.e*, $\hat{g} \circ \exp(\hat{\xi})(z) = \exp(\hat{\xi}) \circ \hat{g}(z)$. Thus, from the previous lemma, we have $\hat{g} \circ \exp(t\hat{\xi})(z) = \exp(t\hat{\xi}) \circ \hat{g}(z)$, $\forall t \in \mathbb{C}$ or equivalently $\hat{g} \circ \exp(t\hat{\xi}) \circ \hat{g}^{-1} = \exp(t\hat{\xi})$, $\forall t \in \mathbb{C}$. Therefore $\hat{g}^* \hat{\xi} = \hat{\xi}$, $\forall \hat{g} \in G$. In case G is abelian and the identity is the only flat element, the map $\hat{g} \mapsto D\hat{g}(0)$ gives a natural group isomorphism $G \cong DG$, *i.e.*, G is *algebraically linearizable.* According then to a classical theorem on linear groups ([16]) either G is finite (and therefore analytically conjugated to a finite group of diagonal periodic linear maps) or the Zariski closure $\overline{G}$ contains a linear flow. In this last case, as in Remark 1, there is a (linear) vector field $\hat{\xi}$ which is invariant under the action of G.

□

Remark 4. Now we give some examples showing that the conditions in our main results, cannot be dropped.

1. The converse of Proposition 4 is not always true for dimension ($n \geq 2$). In fact if $\hat{f}(x, y) = (2x, 4y)$ and $\hat{g}(x, y) = (x, x + y)$ then $G = \langle \hat{f}, \hat{g} \rangle$ is not abelian, however, G is invariant by $\hat{\xi}$, where $\exp(\hat{\xi}) = (x, y + x^2)$. As for the flat case, let $\hat{f}(x, y) = \exp(x^2 y \frac{\partial}{\partial x})$ and $\hat{g}(x, y) = \exp(x^3 y^2 \frac{\partial}{\partial x})$, then $G =< \hat{f}, \hat{g} >$ is not abelian, however, G is invariant by $\hat{X} = -xy\frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$.

2. In dimension $k = 1$, we have that a group $G < \widehat{\mathrm{Diff}}_1(\mathbb{C}, 0)$ of diffeomorphisms tangent to the identity is abelian if and only if there is a formal vector field $\hat{\xi} \in \hat{\mathfrak{X}}_k(\mathbb{C}, 0)$ ($k \geq 1$), such that $G < \langle \exp(t\hat{\xi}) \mid t \in \mathbb{C} \rangle$. For ($n \geq 2$) if there is a formal vector field $\hat{\xi} \in \hat{\mathfrak{X}}_k(\mathbb{C}^n, 0)$ ($n \geq 2$), such that $G < \langle \exp(t\hat{\xi}) \mid t \in \mathbb{C} \rangle$ then G is abelian, however again the converse is not always true. This is due to the fact that for $n = 1$, if the lie bracket of two vector field $\hat{\xi} \in \hat{\mathfrak{X}}_k(\mathbb{C}, 0)$ and $\hat{\zeta} \in \hat{\mathfrak{X}}_r(\mathbb{C}, 0)$ is zero ($[\hat{\xi}, \hat{\zeta}] = 0$) then $r = k$ and there is $c \in \mathbb{C}^*$ such that $\hat{\xi} = c\,\hat{\zeta}$. However this last fact is not always true in dimension $n \geq 2$ as can be seen in the following examples:

   (a) Let $a \in \mathbb{C}^*$ be constant and take $\hat{\xi}(x, y) = xy\frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}$, $\hat{\zeta}(x, y) = ax^2 y^2 \frac{\partial}{\partial x} - axy^3 \frac{\partial}{\partial y}$.

   (b) $\hat{\xi}(x, y) = (x^2 + 3xy)\frac{\partial}{\partial x} + (3xy + y^2)\frac{\partial}{\partial y}$, $\hat{\zeta}(x, y) = (3x^3 - 5x^2 y + xy^2 + y^3)\frac{\partial}{\partial x} + (x^3 + x^2 y - 2xy^2 + 3y^3)\frac{\partial}{\partial y}$.

   (c) For $k \geq 1$, we have: $\hat{\xi} = (x^{k+1})\frac{\partial}{\partial x} + (x^k.y)\frac{\partial}{\partial y}$, $\hat{\zeta} = (y^k.x)\frac{\partial}{\partial x} + (y^{k+1})\frac{\partial}{\partial y}$

   (d) $\hat{\xi}(x, y) = x^2 \frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$, $\hat{\zeta}(x, y) = xy\frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$.

4 Flat abelian groups

Now we study the characterization and classification of flat abelian groups, which is the subject of Theorem **A**.

Proof of Theorem **A**. Assume that the Lie algebra $\mathcal{L}(G_1)$ has dimension one. By Proposition 3 there is a formal vector field $\hat{\xi}$, invariant by G, such that for each $\hat{f} \in G$ there is a rational function $T = T_{\hat{f}}$ with $\hat{\xi}(T) = 0$ and $\hat{f} = \exp(T\hat{\xi})$. Suppose that for some $\hat{f} \in G$ the function $T = T_{\hat{f}}$ is not constant. We consider the one-form $\hat{\omega} := dT$. This is a closed rational one-form and we claim that this is G-invariant. In fact, take $\hat{g} \in G$ and write $G = \exp(S\hat{\xi})$ for some rational function S such that $\hat{\xi}(S) = 0$. Then $(S\hat{\xi})(T) = dT(S\hat{\xi}) = SdT(\hat{\xi}) = 0$. Therefore $T \circ \exp(S\hat{\xi}) = T$. This gives $\hat{g}^*(\omega) = \hat{g}^*(dT) = d(T \circ \hat{g}) = d(T \circ \exp(S\hat{\xi})) = dT = \omega$, proving the claim. This corresponds to (i) in Theorem **A**.

Now we consider the where $T_{\hat{f}}$ is constant for each $\hat{f} \in G$. In this case each element $\hat{f} \in G$ writes as $\hat{f} = \exp(c_{\hat{f}}\hat{\xi})$ for some constant $c_{\hat{f}} \in \mathbb{C}$. In other words, G embeds into the flow of $\hat{\xi}$ as in (ii) in the statement.

Suppose now that G is as in (2) in Proposition 3. There are two linearly independent, formal commuting vector fields $\hat{\xi}_j$, invariant by G, such that $\exp(\hat{\xi}_j) \in G$ and $G < \langle \exp(t\hat{\xi}_1) \circ \exp(s\hat{\xi}_2) \mid t, s \in \mathbb{C} \rangle$. We can write $\hat{\xi}_j = A_j \frac{\partial}{\partial x} + B_j \frac{\partial}{\partial y}$. Since $\hat{\xi}_j$ is G invariant, we have

$$\begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix} \begin{bmatrix} A_1(z) & A_2(z) \\ B_1(z) & B_2(z) \end{bmatrix} = \begin{bmatrix} A_1(\hat{g}) & A_2(\hat{g}) \\ B_1(\hat{g}) & B_2(\hat{g}) \end{bmatrix}$$

Taking transposes, we obtain:

$$\begin{bmatrix} A_1(z) & B_1(z) \\ A_2(z) & B_2(z) \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} \end{bmatrix} = \begin{bmatrix} A_1(\hat{g}) & B_1(\hat{g}) \\ A_2(\hat{g}) & B_2(\hat{g}) \end{bmatrix}$$

thus

$$\begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} \end{bmatrix} \begin{bmatrix} A_1(\hat{g}) & B_1(\hat{g}) \\ A_2(\hat{g}) & B_2(\hat{g}) \end{bmatrix}^{-1} = \begin{bmatrix} A_1(z) & B_1(z) \\ A_2(z) & B_2(z) \end{bmatrix}^{-1}$$

so that, take:

$$\begin{bmatrix} C_1(z) & C_2(z) \\ D_1(z) & D_2(z) \end{bmatrix} = \begin{bmatrix} A_1(z) & B_1(z) \\ A_2(z) & B_2(z) \end{bmatrix}^{-1} = \frac{1}{Q(z)} \begin{bmatrix} B_2(z) & -B_1(z) \\ -A_2(z) & A_1(z) \end{bmatrix}$$

Where $Q(z) = A_1B_2 - A_2B_1$. Thus, take $\hat{\omega}_j = C_j dx + D_j dy$ the above relationship clearly $\hat{\omega}_j$ are invariant for G and are linearly independent. In order to finish we show that $\hat{\omega}_j$ are closed forms, i,e. $\frac{\partial D_j}{\partial x} - \frac{\partial C_j}{\partial y} = 0$. As $[\hat{X}_1, \hat{\xi}_2] = 0$ then

$$\begin{aligned} \frac{\partial A_2}{\partial x} A_1 + \frac{\partial A_2}{\partial y} B_1 &= \frac{\partial A_1}{\partial x} A_2 + \frac{\partial A_1}{\partial y} B_2 \\ \frac{\partial B_2}{\partial x} A_1 + \frac{\partial B_2}{\partial y} B_1 &= \frac{\partial B_1}{\partial x} A_2 + \frac{\partial B_1}{\partial y} B_2 \end{aligned}$$

Thus

$$\begin{aligned}
Q^2(\frac{\partial D_1}{\partial x}-\frac{\partial C_1}{\partial y}) &= Q^2(-\frac{1}{Q}\frac{\partial A_2}{\partial x}+\frac{A_2}{Q^2}\frac{\partial Q}{\partial x}-\frac{1}{Q}\frac{\partial B_2}{\partial y}+\frac{B_2}{Q^2}\frac{\partial Q}{\partial y}) \\
&= A_2\frac{\partial Q}{\partial x}-Q\frac{\partial A_2}{\partial x}+B_2\frac{\partial Q}{\partial y}-Q\frac{\partial B_2}{\partial y} \\
&= A_2B_2\frac{\partial A_1}{\partial x}+A_2A_1\frac{\partial B_2}{\partial x}-A_2B_1\frac{\partial A_2}{\partial x}-A_2^2\frac{\partial B_1}{\partial x}-A_1B_2\frac{\partial A_2}{\partial x}+A_2B_1\frac{\partial A_2}{\partial x}+ \\
&\quad B_2^2\frac{\partial A_1}{\partial y}+B_2A_1\frac{\partial B_2}{\partial y}-B_2B_1\frac{\partial A_2}{\partial y}-B_2A_2\frac{\partial B_1}{\partial y}-A_1B_2\frac{\partial B_2}{\partial y}+A_2B_1\frac{\partial B_2}{\partial y} \\
&= B_2(\frac{\partial A_1}{\partial x}A_2+\frac{\partial A_1}{\partial y}B_2-\frac{\partial A_2}{\partial x}A_1-\frac{\partial A_2}{\partial y}B_1)+ \\
&\quad A_2(\frac{\partial B_2}{\partial x}A_1+\frac{\partial B_2}{\partial y}B_1-\frac{\partial B_1}{\partial x}A_2-\frac{\partial B_1}{\partial y}B_2) \\
&= B_2.0+A_2.0=0.
\end{aligned}$$

Therefore $\hat{\omega}_1$ is closed. Analogously $\hat{\omega}_2$ is closed and we are in case (iii) in Theorem **A**. □

Applying Remark 1 we have the following immediate consequence of Theorem **A**.

Corollary 3. *Let $G < \hat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ be a (not necessarily flat) commutative subgroup. There are four possibilities:*

(i) *G is finite and linearizable.*

(ii) *G embeds into a linear flow.*

(iii) *The Lie algebra of G_1 has dimension one: G_1 leaves invariant an exact rational one-form, say $\hat{\omega} = dT$ for some rational function T.*

(iv) *The Lie algebra of G_1 has dimension two: G_1 admits two closed, independent, formal meromorphic, invariant one-forms.*

As already mentioned in the Introduction, (iii) in Theorem **A** admits a converse, proved as follows:

First part of the proof of Theorem **B**. Suppose that G is flat. Let $\hat{\omega}_j$ two linearly independent invariant closed formal meromorphic one-forms invariant by G, i,e, we have $\hat{g} \in G$, $\hat{g}^*(\hat{\omega}_j) = \hat{\omega}_j$ $(j = 1, 2)$. Write $\hat{\omega}_j = A_j dx + B_j dy$ then,

$$\hat{\omega}_j = \hat{g}^*(\hat{\omega}_j) = A_j(\hat{g})dg_1 + B_j(\hat{g})dg_2 = (A_j(\hat{g})\frac{\partial g_1}{\partial x} + B_j(\hat{g})\frac{\partial g_2}{\partial x})dx + (A_j(\hat{g})\frac{\partial g_1}{\partial y} + B_j(\hat{g})\frac{\partial g_2}{\partial y})dy$$

Thus

$$\begin{bmatrix} A_1(\hat{g}) & B_1(\hat{g}) \\ A_2(\hat{g}) & B_2(\hat{g}) \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} \end{bmatrix} = \begin{bmatrix} A_1(z) & B_1(z) \\ A_2(z) & B_2(z) \end{bmatrix}$$

consequently

$$\begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} \end{bmatrix} \begin{bmatrix} A_1(z) & B_1(z) \\ A_2(z) & B_2(z) \end{bmatrix}^{-1} = \begin{bmatrix} A_1(\hat{g}) & B_1(\hat{g}) \\ A_2(\hat{g}) & B_2(\hat{g}) \end{bmatrix}^{-1}$$

Let us introduce $\hat{\xi}_1, \hat{\xi}_2$ as follows:

$$\hat{\xi}_1 = \frac{1}{Q(z)}(B_2\frac{\partial}{\partial x} - A_2\frac{\partial}{\partial y}), \qquad \hat{\xi}_2 = \frac{1}{Q(z)}(-B_1\frac{\partial}{\partial x} + A_1\frac{\partial}{\partial y})$$

where $Q(z) = A_1(z)B_2(z) - A_2(z)B_1(z)$. Since the $\hat{\omega}_j$ are closed one-forms we have $\hat{g}^*(\hat{\xi}_j) = \hat{\xi}_j$, for all $\hat{g} \in G$ and $[\hat{\xi}_1, \hat{\xi}_2] = 0$. Also $\hat{\xi}_1, \hat{\xi}_2$ are linearly independent in $\hat{K}(\mathbb{C}^2)$, the fraction field of $\mathcal{O}(\mathbb{C}^2)$. Note that $\{\hat{\xi}_1, \hat{\xi}_2\}$ is a basis for the vector space $\hat{\mathfrak{X}}_M(\mathbb{C}^2, 0) \otimes \hat{K}(\mathbb{C}^n)$ and, since for $\hat{g} \in G$ we can write $\hat{g} = \exp(\hat{\zeta}_{\hat{g}})$ with $\hat{\zeta}_{\hat{g}} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0) \subset \hat{\mathfrak{X}}_M(\mathbb{C}^2, 0)$, then $\hat{\zeta}_{\hat{g}} = u_1\hat{\xi}_1 + u_2\hat{\xi}_2$, where $u_j \in \hat{K}(\mathbb{C}^2)$. On the other side $\hat{g}^*(\hat{\xi}_j) = \hat{\xi}_j$ we have that $[\hat{\zeta}_{\hat{g}}, \hat{\xi}_1] = 0$ and $[\hat{\zeta}_{\hat{g}}, \hat{\xi}_2] = 0$, then $\hat{\xi}_j(u_k) = 0$ $(j, k = 1, 2)$, consequently u_j are constant in C^*. Now if $G < \langle \exp(t\hat{\zeta}_{\hat{g}})) \mid t \in \mathbb{C}\rangle$ there is nothing left to prove, thus suppose that there is $\hat{h} \in G$, $\hat{h} = \exp(\hat{\zeta}_{\hat{h}})$ with $\hat{\zeta}_{\hat{g}}$ and $\hat{\zeta}_{\hat{h}}$ linearly independent in $\hat{K}(\mathbb{C}^2)$ then there are $v_j \in \mathbb{C}$ such that $\hat{\zeta}_{\hat{h}} = v_1\hat{\xi}_1 + v_2\hat{\xi}_2$ and (u_1, u_2),(v_1, v_2) are linearly independent in $\mathbb{C}^2$, therefore $\hat{\xi}_j \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ and $G = \exp(a\hat{\xi}_1) \circ \exp(b\hat{\xi}_2)$, with $a, b \in C^*$, therefore there are formal vector fields $\hat{\xi}$ and $\hat{\zeta}$ such that $[\hat{\xi}, \hat{\zeta}] = 0$ and $G < \langle \exp(t\hat{\xi}) \circ \exp(s\hat{\zeta}) \mid t, s \in \mathbb{C}\rangle$. Therefore G is abelian, this proves the first part of Theorem **B**. $\square$

5 Groups preserving closed one-forms

In this section we proceed studying the classification of groups of formal diffeomorphisms preserving closed meromorphic one-forms in $(\mathbb{C}^2, 0)$. Special attention is given to the "generic" case where the group exhibits two transverse formal separatrices. Before going further into the main subject we recall some classical facts about integration of closed meromorphic one-forms in several complex variables.

Proposition 5 (Integration Lemma)**.** *Let ω be a closed meromorphic one-form on M where M is a polydisc in $\mathbb{C}^n$. Then there are irreducible holomorphic functions $f_1, .., f_r \in \mathcal{O}(M)$, $n_1, ..., n_r \in \mathbb{N}$, complex numbers $\lambda_1, ..., \lambda_r$ and a holomorphic function $g \in \mathcal{O}(M)$ such that*

$$\omega = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + d(\frac{g}{f_1^{n_1} \cdots f_r^{n_r}})$$

The polar set of ω is given in irreducible components by $\bigcup_{j=1}^{r} \{f_j = 0\}$, n_j is the order of $\{f_j = 0\}$ as a component of the polar set of ω, λ_j is the residue of ω at the component $\{f_j = 0\}$ and the function g has no common factors with f_j in $\mathcal{O}(M)$.

If $M = \mathbb{C}^n$ and ω is rational then we have the same result, where the f_j are irreducible polynomials and g is a polynomial without common factors with the f_j. The proof of Theorem 5 relies on integration and the fact that the first homology group of the complement of a pure codimension one analytic subset $\Lambda = \bigcup_{j=1}^{r} \Lambda_j$, where each Λ_j is an irreducible component, of a polydisc M as above, is generated by small loops around the components Λ_j, contained in transverse discs circulating the component. Then a standard argument involving Laurent series implies the result. This cannot be repeated in the formal case, because we cannot rely on integration processes, at first glance. Nevertheless, we still have a formal version of Theorem 5 as follows:

Proposition 6 (Formal integration lemma)**.** *Let $\hat\omega$ be a closed formal meromorphic one-form in n complex variables. Denote by $\hat f_j \in \hat{\mathcal O}_k$, $j = 1, ..., r$ the formal equations of the set of poles of $\hat\omega$, in independent terms. Then, there are $\lambda_j \in \mathbb C$ and $n_j \in \mathbb N$ and a formal function $\hat g \in \hat{\mathcal O}_k$ such that*

$$\hat\omega = \sum_{j=1}^{r} \lambda_j \frac{d\hat f_j}{\hat f_j} + d\Big(\frac{\hat g}{\hat f_1^{n_1} \cdots \hat f_j^{n_r}}\Big)$$

The proof is somehow similar to the proof of the local analytic version and it is based on the following:

Lemma 2. *A closed formal meromorphic one-form $\hat\omega$ in n complex variables, without residues is exact: $\hat\omega = d\hat f$ for some meromorphic formal function $\hat f \in \hat{\mathcal M}_n$.*

This lemma is proved similarly to the following particular case:

Lemma 3. *Let $\hat\omega$ be a closed formal meromorphic one-form in two complex variables and assume that the polar set of $\hat\omega$ consists of two transverse formal curves, and that the residues of $\hat\omega$ are all zero. Then $\hat\omega$ is exact, indeed, in suitable formal coordinates (x, y) we can write*

$$\hat\omega = d\Big(\frac{\hat f}{x^n y^m}\Big)$$

for some $n, m \in \mathbb N$ and some formal function $\hat f \in \hat{\mathcal O}_2$.

Proof. Since the polar set of $\hat\omega$ consists of two transverse formal curves, we can find formal coordinates (x, y) such that this polar set corresponds to the coordinates axes. We write $\hat\omega = (\hat P/x^{n+1}y^{m+1})dx + (\hat Q/x^{n+1}y^{m+1})dy$ where $\hat P$, $\hat Q \in \mathbb C[[x, y]]$. We can write $\hat P = \sum\limits_{\nu=0}^{\infty} P_\nu$ and $\hat Q = \sum\limits_{\nu=0}^{\infty} Q_\nu$ in terms of homogeneous polynomials P_ν, Q_ν of degree $\nu - n - m$. Then $\hat\omega = \sum\limits_{\nu=0}^{\infty} (P_\nu/x^{n+1}y^{m+1})dx + (Q_\nu/x^{n+1}y^{m+1})dy = \sum\limits_{\nu=-n-m}^{\infty} \omega_\nu$ where $\omega_\nu = (P_\nu/x^{n+1}y^{m+1})dx + (Q_\nu/x^{n+1}y^{m+1})dy$ is a homogeneous rational one-form of degree $\nu - n - m - 2$. Then $d\hat\omega = \sum\limits_{\nu=-n-m-2}^{\infty} d\omega_\nu$ where each one-form $d\omega_\nu$ is homogeneous of degree $\nu - 1$. Therefore, since $\hat\omega$ is closed we have $0 = d\hat\omega = \sum\limits_{\nu=-n-m}^{\infty} d\omega_\nu$ and then $d\omega_\nu = 0, \forall \nu \geq -n - m$. Since $\hat\omega$ has no residues, the same holds for ω_ν. Moreover, because each form ω_ν is of the form $\omega_\nu = P_\nu/x^{n+1}y^{m+1})dx + (Q_\nu/x^{n+1}y^{m+1})dy$, we conclude from the Integration lemma that $\omega_\nu = d(\frac{f_\nu}{x^n y^m})$ for some homogeneous polynomial f_ν of degree ν. Thus $\hat\omega = d(\sum\limits_{\nu=0}^{\infty} f_\nu/x^n y^m) = d(\hat f/x^n y^m)$ where $\hat f = \sum\limits_{\nu=0}^{\infty} f_\nu \in \hat{\mathcal O}_2$.

□

As a consequence we obtain the following particular case of Proposition 6:

Proposition 7. *Let $\hat\omega$ be a closed formal meromorphic one-form in two complex variables and assume that the polar set of $\hat\omega$ consists of two transverse formal curves. Then $\hat\omega$ writes in suitable formal coordinates (x, y) as*

$$\hat\omega = \lambda \frac{dx}{x} + \mu \frac{dy}{y} + d\Big(\frac{\hat f}{x^n y^m}\Big)$$

for some λ, $\mu \in \mathbb{C}$, some $n, m \in \mathbb{N}$ and some formal function $\hat{f} \in \hat{\mathcal{O}}_2$.

Proof. As in the proof of Lemma 3 we choose formal coordinates (x, y) such that the polar set of $\hat{\omega}$ corresponds to the coordinate axes. Denote by $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$ the residue of $\hat{\omega}$ at the x-axis and y-axis respectively. Then $\hat{\theta} = \hat{\omega} - \lambda \frac{dx}{x} + \mu \frac{dy}{y}$ is a closed formal meromorphic one-form with polar set contained in the coordinate axes and zero residues. By Lemma 3 we can write $\hat{\theta} = d(\frac{\hat{f}}{x^n y^m})$ for some formal function $\hat{f} \in \hat{\mathcal{O}}_2$. □

An improvement of the above proposition is the following:

Lemma 4. *Let $\hat{\omega}_j$, $j = 1, 2$ be linearly independent closed formal meromorphic one-forms in two variables with polar sets along two transverse formal curves. Then there are formal coordinates (x, y) such that each $\hat{\omega}_j$ writes:*

$$\hat{\omega}_j = a_j \frac{dx}{x} + b_j \frac{dy}{y} + d(\frac{c_j}{x^{n_j} y^{m_j}}) \tag{1}$$

for some constant $a_j, b_j, c_j \in \mathbb{C}$ and some $n_j, m_j \in \mathbb{N}$.

Proof. By Proposition 7 we can write $\hat{\omega}_j = a_j \frac{dx}{x} + b_j \frac{dy}{y} + d(\frac{\hat{f}_j}{x^{n_j} y^{m_j}})$, where $a_j, b_j \in \mathbb{C}$; $n_j, m_j \in \mathbb{N}$ and $\hat{f}_j \in \hat{\mathcal{O}}_k$. Let us write $n_1 = n, m_1 = m$ and $n_2 = p, m_2 = q$. We take a model of formal change of coordinates $\hat{\phi} = (xu, yv)$, where we want that $\hat{\phi}^*(a_1 \frac{dx}{x} + b_1 \frac{dy}{y} + d(\frac{c_1}{x^n y^m})) = \hat{\omega}_1$ and $\hat{\phi}^*(a_2 \frac{dx}{x} + b_2 \frac{dy}{y} + d(\frac{c_2}{x^p y^q})) = \hat{\omega}_2$ implies $a_1 \frac{du}{u} + b_1 \frac{dv}{v} = d(\frac{1}{x^n y^m}.(\hat{f}_1 - \frac{c_1}{u^n v^m}))$ and $a_2 \frac{du}{u} + b_2 \frac{dv}{v} = d(\frac{1}{x^p y^q}.(\hat{f}_2 - \frac{c_2}{u^p v^q}))$ then,

$$(a_1 \ln u + b_1 \ln v) x^n y^m - \hat{f}_1 + \frac{c_1}{u^n v^m} + k_1 x^n y^m = 0$$

and

$$(a_2 \ln u + b_2 \ln v) x^p y^q - \hat{f}_2 + \frac{c_2}{u^p v^q} + k_2 x^p y^q = 0$$

Now define a formal meromorphic function $\hat{R}$ by

$$\hat{R}(x, y, u, v) = (\hat{R}_1(x, y, u, v), \hat{R}_2(x, y, u, v)$$

where

$$\hat{R}_1(x, y, u, v) = a_1 \ln u + b_1 \ln v) x^n y^m - \hat{f}_1 + \frac{c_1}{u^n v^m} + k_1 x^n y^m$$

and

$$\hat{R}_2(x, y, u, v) = a_2 \ln u + b_2 \ln v) x^p y^q - \hat{f}_2 + \frac{c_2}{u^p v^q} + k_2 x^p y^q.$$

We have $\hat{R}(0, 0, u, v) = (0, 0)$, so that if $c_1 = \hat{f}_1(0)$ and $c_2 = \hat{f}_2(0)$ we have $\frac{1}{u^n(0) v^m(0)} = 1$ and $\frac{1}{u^p(0) v^q(0)} = 1$ and as $\mathrm{Det}(J_2 R(0, (u, v)) = (nq - mp) u^{k+n+1} v^{m+q+1} \neq 0$, if (m, n) and (p, q) are linearly independent, from the formal version of the Implicit function theorem we obtain a unique solution (u, v). □

Remark 5. Let $G \leq \hat{\mathrm{Diff}}_1(\mathbb{C}^2, 0)$ be a subgroup. Given a closed meromorphic 1-form $\hat{\omega}$ such that $\hat{\omega}$ is invariant by G, if $\hat{\omega}$ is conjugated to a 1-form $\hat{\alpha}$ by a diffeomorphism $\hat{h}$, then the 1-form $\hat{\alpha}$ is invariant by the group $\hat{h}^{-1} \circ \hat{g} \circ \hat{h}$. As the groups G and $\hat{h}^{-1} \circ \hat{g} \circ \hat{h}$ have similar algebraic proprieties, there is no loss of generality in assuming that the forms are as in the normal form of Lemma 4.

The second part of the proof of Theorem **B** follows from the following proposition:

Proposition 8. *Let $G < \hat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ be a subgroup with two transverse separatrices, if there are linearly independent closed formal meromorphic formal one-forms $\hat{\omega}_j$, $(j = 1, 2)$ which are invariant by G then G is abelian. Indeed, G is formally conjugate to a group of diffeomorphisms generated by any of the following types:*

(a) $\hat{g}(x, y) = (x\frac{(1+\frac{k_2}{c_2}x^p y^q)^{\frac{m}{D}}}{(1+\frac{k_1}{c_1}x^n y^m)^{\frac{q}{D}}}, y\frac{(1+\frac{k_1}{c_1}x^n y^m)^{\frac{p}{D}}}{(1+\frac{k_2}{c_2}x^p y^q)^{\frac{n}{D}}})$

(b) $\hat{g}(x, y) = (ax, \frac{a^{-\frac{n}{m}}y}{(1+kx^n y^m)^{\frac{1}{m}}})$

(c) $\hat{g}(x, y) = (\frac{b^{-\frac{m}{n}}x}{(1+kx^n y^m)^{\frac{1}{n}}}, by)$

Proof. A diffeomorphism $\hat{g} \in G$ writes $\hat{g}(x, y) = (xu, yv)$ where $u, v \in \hat{\mathcal{O}}_2$ satisfy $\hat{u}(0) \neq 0, \hat{v}(0) \neq 0$. From equation (1) and $\hat{g}^*(\hat{\omega}_j) = \hat{\omega}_j$ we obtain:

$$a_1 \ln u + b_1 \ln v = \frac{c_1}{x^n y^m}.(1 - \frac{1}{u^n v^m}) + k_1 \quad \text{and} \quad a_2 \ln u + b_2 \ln v = \frac{c_2}{x^p y^q}.(1 - \frac{1}{u^p v^q}) + k_2$$

If $a_1 = a_2 = b_1 = b_2 = 0$ then $u^n v^m = \frac{1}{1+\frac{k_1}{c_1}x^n y^m}$ and $u^p v^q = \frac{1}{1+\frac{k_2}{c_2}x^p y^q}$, as (m, n) and (p, q) must be linearly independent we have

$$\hat{g}(x, y) = (x\frac{(1 + \frac{k_2}{c_2}x^p y^q)^{\frac{m}{D}}}{(1 + \frac{k_1}{c_1}x^n y^m)^{\frac{q}{D}}}, y\frac{(1 + \frac{k_1}{c_1}x^n y^m)^{\frac{p}{D}}}{(1 + \frac{k_2}{c_2}x^p y^q)^{\frac{n}{D}}})$$

with $D = nq - pm$. The group G therefore has just linear diffeomorphisms as above and is an abelian group.

Assume now that the left side of equality is holomorphic we have, $\frac{1}{u^n v^m} = 1$ and $\frac{1}{u^p v^q} = 1$ when (m, n) and (p, q) are linearly independent, we have that u and v are constant, so that G is linear therefore G is abelian.

A similar argumentation with the other possible cases gives the forms: $\hat{g}(x, y) = (ax, \frac{a^{-\frac{n}{m}}y}{(1+kx^n y^m)^{\frac{1}{m}}})$, $\hat{g}(x, y) = (\frac{b^{-\frac{m}{n}}x}{(1+kx^n y^m)^{\frac{1}{n}}}, by)$ and $\hat{g}(x, y) = (\frac{x}{(1+k_1 x^n)^{\frac{1}{n}}}, \frac{y}{(1+k_2 y^m)^{\frac{1}{m}}})$. In particular, on each case, G is abelian. □

Remark 6 (holomorphic case)**.** If G is invariant by two linearly independent closed formal one-forms (without poles) then $G = \{\mathrm{Id}\}$.

Proof. Let $\hat{g} \in G$ and write $\hat{\omega}_j = d\hat{f}_j$, $(j = 1, 2)$ left invariant by G. Take $\hat{\Phi} = (\hat{f}_1, \hat{f}_2)$, as $d\hat{f}_1$ and $d\hat{f}_2$ are linearly independent in this neighborhood of the origin, we have that $\hat{\Phi}$ is a formal diffeomorphism. Therefore we may assume that $\hat{\omega}_1 = dx$ and $\hat{\omega}_2 = dy$, *i.e.*, $(\hat{\Phi} \circ \hat{g} \circ \hat{\Phi}^{-1})^*(dx) = dx$ and $(\hat{\Phi} \circ \hat{g} \circ \hat{\Phi}^{-1})^*(dy) = dy$, because $(\hat{\Phi} \circ \hat{g} \circ \hat{\Phi}^{-1})^*(dx) = (\hat{\Phi}^{-1})^* \circ \hat{g}^* \circ \hat{\Phi}^*(dx) = (\hat{\Phi}^{-1})^* \circ \hat{g}^*(d\hat{f}_1) = (\hat{\Phi}^{-1})^*(d\hat{f}_1) = (\hat{\Phi}^{-1})^* \circ \hat{\Phi}^*(dx) = dx$, therefore $\hat{\Phi} \circ \hat{g} \circ \hat{\Phi}^{-1} = \{\mathrm{Id}\}$ and consequently $G = \{\mathrm{Id}\}$. □

6 Metabelian groups

Now we study metabelian groups in $\hat{\mathrm{Diff}}(\mathbb{C}^2,0)$, that is, subgroups $G < \hat{\mathrm{Diff}}(\mathbb{C}^2,0)$ such that the group of commutators $G^{(1)} = [G,G]$ is abelian. Let G be such a metabelian subgroup. Then, the derivative group $DG < \mathrm{GL}(\mathbb{C},2)$ is also metabelian but not necessarily abelian. For instance, take G as the linear subgroup of 2×2 triangular superior matrices. Then G is not abelian but $G^{(1)}$ is abelian.

Now if the group DG is abelian then $G^{(1)}$ is flat, which is a very useful property. For this reason, in our statements below, we require that DG is abelian.

Lemma 5. *Let $G < \hat{\mathrm{Diff}}(\mathbb{C}^2,0)$ be a subgroup with DG abelian. Suppose that, there are two linearly independent vector fields $\hat{\xi}$ and $\hat{\zeta}$, projectively invariant by G and such that $[\hat{\xi},\hat{\zeta}]=0$. Then G is metabelian.*

Proof. Since $\hat{\xi}$ and $\hat{\zeta}$ are projectively invariant by G, for each $\hat{g}\in G$ there are constants $a_{\hat{g}}, b_{\hat{g}} \in \mathbb{C}$ such that $\hat{g}^*\hat{\xi} = a_{\hat{g}}\hat{\xi}$ and $\hat{g}^*\hat{\zeta} = b_{\hat{g}}\hat{\zeta}$. Given now a flat element $\hat{h}\in G_1$ we have $a_h = 1$ and $b_h = 1$, so that $\hat{h}^*\hat{\xi} = \hat{\xi}$ and $\hat{h}^*\hat{\zeta} = \hat{\zeta}$. This implies that G_1 is abelian. Since DG is abelian, we have that $[G,G] < G_1$, so that $[G,G]$ is abelian. $\square$

Proof of Theorem 2. Let $G < \hat{\mathrm{Diff}}(\mathbb{C}^2,0)$ be a metabelian subgroup. Since DG is abelian by Proposition 3 we have two cases:

Case 1. $[G,G] \le \langle \exp(N\hat{\xi}) \mid N \text{ is a rational funtion}, \hat{\xi}(N) = 0\rangle$ and $\hat{f} = \exp(\hat{\xi}) \in [G,G]$. Then for all $\hat{g}\in G$, $[\hat{g},\hat{f}] \in [G,G]$ so that, there is a rational function $\widetilde{N}$ such that $[\hat{g},\hat{f}] = \exp(\widetilde{N}\hat{\xi})$ then $\hat{g}\circ\exp(\hat{\xi})\circ\hat{g}^{-1} = \exp(\widetilde{N}\hat{\xi})\circ\exp(\hat{\xi}) = \exp((\widetilde{N}+1)\hat{\xi})$ therefore $\hat{g}^*(\hat{\xi}) = N\hat{\xi}$

Case 2. $[G,G] \le \langle \exp(s\hat{\xi})\circ\exp(t\hat{\zeta}) \mid s,t\in\mathbb{C}^*\rangle$, take $\hat{f} = \exp(\hat{\xi})$. Then for all $\hat{g}\in G$, $[\hat{g},\hat{f}]\in[G,G]$ so that, there are $\widetilde{s_1}$ and t_1 such that $[\hat{g},\hat{f}] = \exp(\widetilde{s_1}\hat{\xi})\circ\exp(t_1\hat{\zeta})$ then $\hat{g}\circ\exp(\hat{\xi})\circ\hat{g}^{-1} = \exp(\widetilde{s_1}\hat{\xi}+t_1\hat{\zeta})\circ\exp(\hat{\xi}) = \exp(s_1\hat{\xi}+t_1\hat{\zeta})$. Therefore $\hat{g}^*(\hat{\xi}) = s_1\hat{\xi}+t_1\hat{\zeta}$. Analogously we have $\hat{g}^*(\hat{\zeta}) = s_2\hat{\xi}+t_2\hat{\zeta}$. Let us now construct the formal closed meromorphic one-forms $\hat{\omega}_j$, $j=1,2$. We can write $\hat{\xi}_j = A_j\frac{\partial}{\partial x} + B_j\frac{\partial}{\partial y}$ then

$$\begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix} \begin{bmatrix} A_1(z) & A_2(z) \\ B_1(z) & B_2(z) \end{bmatrix} = \begin{bmatrix} s_1A_1(\hat{g})+t_1A_2(\hat{g}) & s_2A_1(\hat{g})+t_2A_2(\hat{g}) \\ s_1B_1(\hat{g})+t_1B_2(\hat{g}) & s_2B_1(\hat{g})+t_2B_2(\hat{g}) \end{bmatrix}$$

taking transposes, we have:

$$\begin{bmatrix} A_1(z) & B_1(z) \\ A_2(z) & B_2(z) \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} \end{bmatrix} = \begin{bmatrix} s_1A_1(\hat{g})+t_1A_2(\hat{g}) & s_1B_1(\hat{g})+t_1B_2(\hat{g}) \\ s_2A_1(\hat{g})+t_2A_2(\hat{g}) & s_2B_1(\hat{g})+t_2B_2(\hat{g}) \end{bmatrix}$$

thus

$$\begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} \end{bmatrix} \begin{bmatrix} s_1A_1(\hat{g})+t_1A_2(\hat{g}) & s_1B_1(\hat{g})+t_1B_2(\hat{g}) \\ s_2A_1(\hat{g})+t_2A_2(\hat{g}) & s_2B_1(\hat{g})+t_2B_2(\hat{g}) \end{bmatrix}^{-1} = \begin{bmatrix} A_1(z) & B_1(z) \\ A_2(z) & B_2(z) \end{bmatrix}^{-1}$$

so that we can take:

$$\begin{bmatrix} C_1(z) & C_2(z) \\ D_1(z) & D_2(z) \end{bmatrix} = \begin{bmatrix} s_1A_1(z)+t_1A_2(z) & s_1B_1(z)+t_1B_2(z) \\ s_2A_1(z)+t_2A_2(z) & s_2B_1(z)+t_2B_2(z) \end{bmatrix}^{-1}$$

and define $\hat{\omega}_j = C_j dx + D_j dy$. Then $\hat{g}^*(\hat{\omega}_1) = \hat{g}^*(\frac{1}{rQ(z)}.(s_2B_1 + t_2B_2, -(s_1B_1 + t_1B_2)) = \frac{1}{Q(z)}(B_2, -A_2) = s_1\hat{\omega}_1 + s_2\hat{\omega}_2$, where $Q(x) = C_1(z).D_2(z) - C_2(z).D_1(z)$ and $r = s_1t_2 - s_2t_1$. Analogously $\hat{g}^*(\hat{\omega}_2) = t_1\hat{\omega}_1 + t_2\hat{\omega}_2$, clearly $\hat{\omega}_1$ and $\hat{\omega}_2$ are linearly independent. It remains to show that the $\hat{\omega}_j$ are closed forms, i,e. $\frac{\partial D_j}{\partial x} - \frac{\partial C_j}{\partial y} = 0$. Since $[\hat{\xi}_1, \hat{\xi}_2] = 0$ then

$$\begin{aligned}
\frac{\partial A_2}{\partial x}A_1 + \frac{\partial A_2}{\partial y}B_1 &= \frac{\partial A_1}{\partial x}A_2 + \frac{\partial A_1}{\partial y}B_2 \\
\frac{\partial B_2}{\partial x}A_1 + \frac{\partial B_2}{\partial y}B_1 &= \frac{\partial B_1}{\partial x}A_2 + \frac{\partial B_1}{\partial y}B_2
\end{aligned}$$

Thus ousting the value of C_j and D_j and using the above equations we can conclude. □

Next we study the possible normal forms of groups as in the conclusion of Theorem 2.

Remark 7 (groups leaving invariant a linear system of closed forms)**.** Let $G \leq \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ be a subgroup of formal diffeomorphisms of two variables, that preserves the coordinate axes $(x = 0)$ and $(y = 0)$. Suppose that we have

$$\hat{g}^*(\hat{\omega}_j) = a_j\hat{\omega}_1 + b_j\hat{\omega}_2, \ \forall \hat{g} \in G. \tag{2}$$

where $a_j, b_j \in \mathbb{C}^*$ and $\hat{\omega}_j$ is a closed formal meromorphic one-form. A diffeomorphism $\hat{g} \in G$ writes $\hat{g}(x, y) = (xu, yv)$ where $u, v \in \hat{\mathcal{O}}_2$ We have the following possibilities for $\hat{\omega}_1, \hat{\omega}_2$ in suitable formal coordinates:

(1) (**simple poles case**) If both forms have simple poles along the coordinate axes we can write $\hat{\omega}_1 = \alpha_1\frac{dx}{x} + \beta_1\frac{dy}{y}$ and $\hat{\omega}_2 = \alpha_2\frac{dx}{x} + \beta_2\frac{dy}{y}$. From equation (2) we get

$$\alpha_1\frac{dx}{x} + \beta_1\frac{dy}{y} + \alpha_1\frac{du}{u} + \beta_1\frac{dv}{v} = \hat{g}^*(\hat{\omega}_j) = (a_1\alpha_1 + b_1\alpha_2)\frac{dx}{x} + (a_1\beta_1 + b_1\beta_2)\frac{dy}{y}$$

In matrix form we have:

$$\begin{bmatrix}\alpha_1 & \beta_1 \\ \alpha_2 & \beta_2\end{bmatrix}\begin{bmatrix}\frac{dx}{x} \\ \frac{dy}{y}\end{bmatrix} + \begin{bmatrix}\alpha_1 & \beta_1 \\ \alpha_2 & \beta_2\end{bmatrix}\begin{bmatrix}\frac{du}{u} \\ \frac{dv}{v}\end{bmatrix} = \begin{bmatrix}a_1 & b_1 \\ a_2 & b_2\end{bmatrix}\begin{bmatrix}\alpha_1 & \beta_1 \\ \alpha_2 & \beta_2\end{bmatrix}\begin{bmatrix}\frac{dx}{x} \\ \frac{dy}{y}\end{bmatrix}$$

Comparing the poles we obtain:

$$\begin{bmatrix}a_1 & b_1 \\ a_2 & b_2\end{bmatrix} = \mathrm{Id} \quad \text{and} \quad \begin{bmatrix}\frac{du}{u} \\ \frac{dv}{v}\end{bmatrix} = 0$$

Thus $\hat{g}(x, y) = (xu_0, yv_0)$ with u_0 and v_0 constant. In this case the group G is linear.

(2) (**Pure polar case**) Assume now that $\hat{\omega}_j$ has poles of order higher than one and no residues. We can write $\hat{\omega}_1 = d(\frac{1}{x^ny^m})$ and $\hat{\omega}_2 = d(\frac{1}{x^py^q})$. Given now a diffeomorphism $\hat{g}(x, y) = (xu, yv)$ in G from equation (2) we have:

$$d(\frac{1}{x^ny^mu^nv^m}) = a_1d(\frac{1}{x^ny^m}) + b_1d(\frac{1}{x^py^q}) \quad \text{and} \quad d(\frac{1}{x^py^qu^pv^q}) = a_2d(\frac{1}{x^ny^m}) + b_2d(\frac{1}{x^py^q})$$

Thus

$$\frac{1}{x^ny^mu^nv^m} = \frac{a_1}{x^ny^m} + \frac{b_1}{x^py^q} + k_1 \quad \text{and} \quad \frac{1}{x^py^qu^pv^q} = \frac{a_2}{x^ny^m} + \frac{b_2}{x^py^q}$$

now

$$\frac{1}{u^nv^m} = a_1 + b_1x^{n-p}y^{m-q} + k_1x^ny^m \quad \text{and} \quad \frac{1}{u^pv^q} = a_2 + b_2x^{p-n}y^{q-m} + k_2x^py^q$$

thus

$$u = \frac{(a_2 + b_2x^{p-n}y^{q-m} + k_2x^py^q)^{\frac{m}{D}}}{(a_1 + b_1x^{n-p}y^{m-q} + k_1x^ny^m)^{\frac{q}{D}}} \quad \text{and} \quad u = \frac{(a_1 + b_1x^{n-p}y^{m-q} + k_1x^ny^m)^{\frac{p}{D}}}{(a_2 + b_2x^{p-n}y^{q-m} + k_2x^py^q)^{\frac{n}{D}}}$$

where $D = qn - pm$, therefore we have:

$$\hat{g}(x,y) = \left(x.\frac{(a_2 + b_2x^{p-n}y^{q-m} + k_2x^py^q)^{\frac{m}{D}}}{(a_1 + b_1x^{n-p}y^{m-q} + k_1x^ny^m)^{\frac{q}{D}}} \, , \; y.\frac{(a_1 + b_1x^{n-p}y^{m-q} + k_1x^ny^m)^{\frac{p}{D}}}{(a_2 + b_2x^{p-n}y^{q-m} + k_2x^py^q)^{\frac{n}{D}}} \right)$$

Other mixed cases are studied in the same way.

The following example contradicts Corollary 4.2 in [1].

Example 1. An example of a flat group $G < \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$, which is solvable but not metabelian is $G =< (\hat{h}(x), \hat{a}(x) + \hat{b}(x)y); \hat{h} \in H >$, where $H < \hat{\text{Diff}}_1(\mathbb{C}, 0)$ is any metabelian flat subgroup, $\hat{a}(x) \in \mathbb{C}[[x]]$ has order greater than 2 and $\hat{b}(x) \in \mathbb{C}[[x]]$ is a unit, $\hat{b}(0) = 1$.

7 Dicritic groups with abelian commutators

Unlike the one-dimensional case two commuting flat diffeomorphisms may have different orders of tangency to the identity: take $\hat{f} = \exp(\hat{\xi})$ and $\hat{g} = \exp(\hat{\zeta})$, where the vector fields $\hat{\xi}$ and $\hat{\zeta}$ are given as in (1) above. This is the main reason why we do not have an equivalence between the concepts of metabelian, quasi-abelian and solvable groups in dimension $n \geq 2$. From now on we shall take a closer look at this issue. Firstly, in this section, we investigate the characterization of quasi-abelian groups. For this we shall refer to the following concepts, which are two main notions in this paper.

Definition 1 (Dicritic and regular dicritic vector fields and diffeomorphisms)**.** A diffeomorphism $\hat{f} \in \hat{\text{Diff}}_{r+1}(\mathbb{C}^n, 0)$ is called *dicritic* if $\hat{f}(z) = z + \hat{f}_{r+1}(z) + \hat{f}_{r+2}(z) + \cdots$, where $\hat{f}_{r+1}(z) = f(z)z$ and f is a homogeneous polynomial of degree r. A formal vector field $\hat{\xi} \in \hat{\mathfrak{X}}_{k+1}(\mathbb{C}^n, 0)$, $k \geq 1$ is called *dicritic* if $\hat{\xi} = f(z)\vec{R} + (p^{(1)}_{k+2} + \cdots)\frac{\partial}{\partial z_1} + \cdots + (p^{(n)}_{k+2} + \cdots)\frac{\partial}{\partial z_n}$ where f is a homogeneous polynomial of degree k and $\vec{R} = z_1\frac{\partial}{\partial z_1} + \cdots + z_n\frac{\partial}{\partial z_n}$; $\hat{\xi}$ is called *regular dicritic* if $\hat{\xi}$ is dicritic and there are $i_0, j_0 \in \{1, \ldots, n\}$, such that $\gcd(f, z_{j_0}p^{(i_0)}_{k+2} - z_{i_0}p^{(j_0)}_{k+2}) = 1$. This implies that 0 is an isolated singularity of $\hat{\xi}$. The map $\hat{f}$ is called *regular dicritic* if there is $\hat{\xi}$ regular dicritic such that $\hat{f} = \exp(\hat{\xi})$.

Now we pave the way to Theorem **C**. For the first part we shall need some lemmas below.

Lemma 6. *Let $\hat{\xi} = f(z)\vec{R}$ and $\hat{\zeta} = g(z)\vec{R}$, where $\vec{R}$ is the radial vector field and f and g are homogeneous polynomials of degree k e s respectively. Then $[\hat{\xi}, \hat{\zeta}] = 0$ if and only if $k = s$.*

Proof. Trivial, because $[\hat{\xi},\hat{\zeta}]=(k-s)f(z)g(z)\vec{R}$. $\square$

Lemma 7. *Let $\hat{\xi}\in\widehat{\mathfrak{X}}_{k+1}(\mathbb{C}^n,0)$ a dicritic vector field. For any vector field $\hat{\zeta}$ with order greater than 2, such that $[\hat{\xi},\hat{\zeta}]=0$, we have that $\hat{\zeta}$ is a dicritic vector field with order $k+1$.*

Proof. Suppose that $\hat{\zeta}$ has order $r\geq 2$, thus:

$$\hat{\xi}=f(z)\overrightarrow{R}+(p_{k+2}^{(1)}+\cdots)\frac{\partial}{\partial z_1}+\cdots+(p_{k+2}^{(n)}+\cdots)\frac{\partial}{\partial z_n}$$

$$\hat{\zeta}=(q_r^{(1)}+\cdots)\frac{\partial}{\partial z_1}+\cdots+(q_r^{(n)}+\cdots)\frac{\partial}{\partial z_n}$$

Now, the term of lower order of $[\hat{\xi},\hat{\zeta}]$ is:

$$[f\overrightarrow{R},q_r^{(1)}\frac{\partial}{\partial z_1}+\cdots+q_r^{(n)}\frac{\partial}{\partial z_n}]=(r-1)f.(q_r^{(1)}\frac{\partial}{\partial z_1}+\cdots+q_r^{(n)}\frac{\partial}{\partial z_n})-(q_r^{(1)}\frac{\partial f}{\partial z_1}+\cdots+q_r^{(n)}\frac{\partial f}{\partial z_n}).\overrightarrow{R}$$

As $[\hat{\xi},\hat{\zeta}]=0$, we have that $(r-1)f.q_r^{(j)}=(\nabla f.Q_r)z_j$, onde $Q_r=(q_r^{(1)},\ldots,q_r^{(n)})$. Thus $(r-1)f.q_r^{(1)}z_j=(r-1)f.q_r^{(j)}z_1$ and as $q_r^{(1)}\neq 0$, we have $q_r^{(j)}=\frac{q_r^{(1)}}{z_1}.z_j=g.z_j$, for $j=1,\ldots,n$. So the 1-Jet of $\hat{\zeta}$ is $g\overrightarrow{R}$, therefore $\hat{\zeta}$ is dicritic vector field and by the previous lemma $\hat{\zeta}$ have order $k+1$. $\square$

Lemma 8. *Let $\hat{\xi},\hat{\zeta}\in\hat{\mathfrak{X}}_k(\mathbb{C}^n,0)$, $k\geq 2$. Suppose that $\hat{\xi}$ is regular dicritic and $\hat{\zeta}$ is dicritic. If $[\hat{\xi},\hat{\zeta}]=0$, then there is $c\in\mathbb{C}\setminus\{0\}$ such that $\hat{\zeta}=c.\hat{\xi}$.*

Proof. Since $\hat{\xi}$ and $\hat{\zeta}$ are dicritic, then

$$\hat{\xi}=f(z)\vec{R}+(p_{k+2}^{(1)}+\cdots)\frac{\partial}{\partial z_1}+\cdots+(p_{k+2}^{(n)}+\cdots)\frac{\partial}{\partial z_n}$$

$$\hat{\zeta}=g(z)\vec{R}+(q_{k+2}^{(1)}+\cdots)\frac{\partial}{\partial z_1}+\cdots+(q_{k+2}^{(n)}+\cdots)\frac{\partial}{\partial z_n}$$

We have $[f\vec{R},g\vec{R}]=0$, by Lemma 6. Since $[\hat{\xi},\hat{\zeta}]=0$, then the $2k+2$-jet to lie bracket is

$$[g\vec{R},p_{k+2}^{(1)}\frac{\partial}{\partial z_1}+\cdots+p_{k+2}^{(n)}\frac{\partial}{\partial z_n}]-[f\vec{R},q_{k+2}^{(1)}\frac{\partial}{\partial z_1}+\cdots+q_{k+2}^{(n)}\frac{\partial}{\partial z_n}]=0$$

Now note that

$$[f\vec{R},q_{k+2}^{(1)}\frac{\partial}{\partial z_1}+\cdots+q_{k+2}^{(n)}\frac{\partial}{\partial z_n}]=(k+1)f.(q_{k+2}^{(1)}\frac{\partial}{\partial z_1}+\cdots+q_{k+2}^{(n)}\frac{\partial}{\partial z_n})-(q_{k+2}^{(1)}\frac{\partial f}{\partial z_1}+\cdots+q_{k+2}^{(n)}\frac{\partial f}{\partial z_n}).\vec{R}$$

Then we have

$$(k+1)(f.q_{k+2}^{i}-g.p_{k+2}^{i})=z_i(\nabla f.Q_{k+2}-\nabla g.P_{k+2})$$

for $i\in\{1,\ldots,n\}$, thus

$$\frac{f.q_{k+2}^{i_0}-g.p_{k+2}^{i_0}}{z_{i_0}}=\frac{f.q_{k+2}^{j_0}-g.p_{k+2}^{j_0}}{z_{j_0}}$$

or equivalently, $f.(z_{j_0}q^{(i_0)}_{k+2}-z_{i_0}q^{(j_0)}_{k+2}) = g.(z_{j_0}p^{(i_0)}_{k+2}-z_{i_0}p^{(j_0)}_{k+2})$. But by hypothesis $\gcd(f, z_{j_0}p^{(i_0)}_{k+2}-z_{i_0}p^{(j_0)}_{k+2}) = 1$, then $f \mid g$. As f and g has the same degree $g = c.f$ were $c \in \mathbb{C}^*$. Thus the $2k+2$- jet of Lie bracket is:

$$[f\overrightarrow{R}, (q^{(1)}_{k+2} - cp^{(1)}_{k+2})\frac{\partial}{\partial z_1} + \cdots + (q^{(n)}_{k+2} - cp^{(n)}_{k+2})\frac{\partial}{\partial z_n}] = 0$$

Using the same argument of the previous lemma we have

$$(q^{(1)}_{k+2} - cp^{(1)}_{k+2})z_j = (q^{(j)}_{k+2} - cp^{(j)}_{k+2})z_1$$

so, $q^{(1)}_{k+2} - cp^{(1)}_{k+2} = 0$, in consequence $q^{(j)}_{k+2} - cp^{(j)}_{k+2} = 0$, for all $j = 1, \ldots, n$, or

$$(q^{(1)}_{k+2} - cp^{(1)}_{k+2})\frac{\partial}{\partial z_1} + \cdots + (q^{(n)}_{k+2} - cp^{(n)}_{k+2})\frac{\partial}{\partial z_n} = \frac{(q^{(1)}_{k+2} - cp^{(1)}_{k+2})}{z_1}\overrightarrow{R}$$

but this latter does not occur, because by the Lemma 6 we have $\frac{(q^{(1)}_{k+2}-cp^{(1)}_{k+2})}{z_1}$ has degree k, and this is impossible because $q^{(1)}_{k+2} - cp^{(1)}_{k+2}$ has degree $k+2$. Then, we have $q^{(j)}_{k+2} = cp^{(j)}_{k+2}$, $\forall j \in \{1, \ldots, n\}$.
Finally suppose that $Q_{k+j} = cP_{k+j}$ for $j = 1, \ldots, i$, the $(2k+i+1)$-jet of lie bracket is

$$[g\vec{R}, p^{(1)}_{k+i+1}\frac{\partial}{\partial z_1} + \cdots + p^{(n)}_{k+i+1}\frac{\partial}{\partial z_n}] - [f\vec{R}, q^{(1)}_{k+i+1}\frac{\partial}{\partial z_1} + \cdots + q^{(n)}_{k+i+1}\frac{\partial}{\partial z_n}] = 0$$

by the supposed the following sum is symmetric

$$\sum_{j=2}^{i}[p^{(1)}_{k+j}\frac{\partial}{\partial z_1} + \cdots + p^{(n)}_{k+j}\frac{\partial}{\partial z_n}, q^{(1)}_{k+i+2-j}\frac{\partial}{\partial z_1} + \cdots + q^{(n)}_{k+i+2-j}\frac{\partial}{\partial z_n}] = 0$$

Then similarly to the case $k+2$ we have that $Q_{k+j+1} = cP_{k+j+1}$ therefore $\hat{\zeta} = c\hat{\xi}$ □

Remark 8. We cannot exclude the regularity condition in the previous lemma, since the two vector fields of item 2.(d) in Remark 4 are dicritic and commute, but they are not regular dicritic and are not linearly dependent.

The following proposition is found in [1].

Proposition 9. *Let $\hat{f} \in \hat{\text{Diff}}_{r+1}(\mathbb{C}^n, 0)$ and $\hat{g} \in \hat{\text{Diff}}_{s+1}(\mathbb{C}^n, 0)$. Suppose that $\hat{f}$ is dicritic and $\hat{f}(\hat{g}(z)) = \hat{g}(\hat{f}(z))$. Then $r = s$ and G is also dicritic.*

The following proposition is the main tool in the proof of Theorem **C**.

Proposition 10. *Let $G < \hat{\text{Diff}}_1(\mathbb{C}^n, 0)$ be a subgroup of diffeomorphisms tangent to the identity and $\hat{f} \in \hat{\text{Diff}}_1(\mathbb{C}^n, 0)$ a regular dicritic diffeomorphism. If $\hat{f}$ commutes with $\hat{g}$ then $G \le \langle \exp(t\hat{X}) \mid t \in \mathbb{C}\rangle$, where $\hat{f} = \exp(\hat{\xi})$. In particular, G is abelian.*

Proof. Let $\hat{g} \in G$ be a diffeomorphism, from Proposition 9, $\hat{g}$ is a dicritic diffeomorphism of same order than $\hat{f}$, we say $k+1$. From the exponential bijection there is $\hat{\zeta}$ such that $\exp(\hat{\zeta}) = \hat{g}$. Then

$$\hat{\zeta} = g(z)\vec{R} + (q^{(1)}_{k+2} + \cdots)\frac{\partial}{\partial z_1} + \cdots + (q^{(n)}_{k+2} + \cdots)\frac{\partial}{\partial z_n}$$

and thus $\hat{\zeta}$ is dicritic. Since $\hat{f}$ commutes with $\hat{g}$ it commutes with $\hat{g}(z) = \exp(\hat{\zeta})(z)$ and from Lemma 1, $\hat{f}$ commutes with $\exp(t\hat{\zeta})(z)$ for all $t \in \mathbb{C}$. Similarly, for each t we have that $\exp(t\hat{\zeta})(z)$ commutes with $\exp(s\hat{\xi})(z)$ and thus $[\hat{\xi}, \hat{\zeta}] = 0$. From Lemma 8, there is $r \in \mathbb{C}$ such that $\hat{\zeta} = r\hat{\xi}$. Consequently $\hat{g}(z) = \exp(r\hat{\xi})(z)$ and therefore $G \leq \langle \exp(t\hat{\xi}) \mid t \in \mathbb{C} \rangle$. $\square$

Proof of Theorem **C**. Let $G \leq \hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ be a subgroup with a regular dicritic diffeomorphism $\hat{f} = \exp(\hat{\xi}) \in G$. First, suppose that G is quasi-abelian. From Proposition 10, we have $G_1 \leq \langle \exp(t\hat{\xi}) \mid t \in \mathbb{C} \rangle$. Let $\hat{g} \in G$, as $\hat{f} \in G_1$ then $[\hat{f}, \hat{g}] \in G_1$. Thus there is $t_{\hat{g}} \in \mathbb{C}^*$ such that $[\hat{f}, \hat{g}] = \exp(t_{\hat{g}}\hat{X})$, then $\hat{g} \circ \hat{f} \circ \hat{g}^{-1} \circ \hat{f}^{-1} = \exp(t_{\hat{g}}\hat{\xi})$ so that

$$\begin{aligned} \hat{g} \circ \exp(\hat{\xi}) \circ \hat{g}^{-1} &= \exp(t_{\hat{g}}\hat{X}) \circ \exp(\hat{\xi}) = \exp((t_{\hat{g}} + 1)\hat{\xi}) \\ &= \exp(c_{\hat{g}}\hat{\xi}) \end{aligned}$$

from the same argument used in the proof of Lemma 1, we have $\forall s \in \mathbb{C}$, $\hat{g} \circ \exp(s\hat{\xi}) \circ \hat{g}^{-1} = \exp(sc_{\hat{g}}\hat{\xi})$. Therefore $\hat{g}^*\hat{X} = c_{\hat{g}}\hat{\xi}$, $\forall \hat{g} \in G$. Conversely, suppose that $\forall \hat{g} \in G$, $\exists c_{\hat{g}}$ such that $\hat{g}^*\hat{\xi} = c_{\hat{g}}\hat{\xi}$. We claim that $\forall \hat{g} \in G_1$, $c_{\hat{g}} = 1$. In fact, if $\hat{f}(z) = z + f(z)z + \cdots$ then $\exp(c_{\hat{g}}\hat{\xi})(z) = z + c_{\hat{g}}f(z)z + \cdots$. Thus if $\hat{g} \in G_1$, $\hat{g} \circ \hat{f}(z) = z + f(z)z + \hat{g}_{k+1}(z) + \cdots$ and $\exp(c_{\hat{g}}\hat{\xi})(z) \circ \hat{g}(z) = z + c_{\hat{g}}.f(z)z + \hat{g}_{k+1}(z) + \cdots$, then $c_{\hat{g}} = 1$. Consequently $\forall \hat{g} \in G_1$, $\hat{g}^*\hat{\xi} = \hat{\xi}$, *i.e.*, G_1 commutes with $\hat{f}$. From Proposition 10, G_1 is abelian, *i.e.*, G is quasi-abelian. $\square$

In the same way as Theorem **C** we have:

Proposition 11. *Let $G \leq \hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ be a subgroup with a regular dicritic diffeomorphism $\hat{f} = \exp(\hat{\xi}) \in G$, such that $DG \subset G$. The following conditions are equivalent:*

1. *G is abelian*
2. *DG is abelian and $\forall \hat{g} \in G$, $\hat{g}^*\hat{\xi} = \hat{\xi}$.*

Proof. It is immediate to verify that (1) $\Rightarrow$ (2). Let us now prove (2) $\Rightarrow$ (1). Since $DG \subset G$, for all $\hat{g} \in G$ we have that $\tilde{g} = D\hat{g}^{-1}(0) \circ \hat{g} \in G_1$. From (2) we have that G commutes with $\hat{f}$ then $D\hat{g}^{-1}(0) \circ \hat{g} = \exp(c_{\tilde{g}}X)$, therefore $G = D\hat{g}(0) \circ \exp(c_{\tilde{g}}X)$, $\forall \hat{g} \in G$. Now let $\hat{g}, \hat{h} \in G$ be diffeomorphisms, as $DG \subset G$ and from (2), we have that:

$$\hat{g} \circ \hat{h} = D\hat{g}(0) \circ \exp(c_{\tilde{g}}X) \circ D\hat{h}(0) \circ \exp(c_{\tilde{h}}X) = \hat{h} \circ \hat{g}$$

Therefore G is abelian. $\square$

8 Metabelian and solvable dicritic groups

Now we shall study metabelian subgroups of $\hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$. We strongly rely on the preceding argumentation. The main step is:

Proposition 12. *Let $G < \hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ be a subgroup with DG abelian, and $\hat{f} = \exp(\hat{\xi}) \in G$ a regular dicritic diffeomorphism. Then $\hat{\xi}$ is projectively invariant by G if, and only if, $\hat{f}$ commutes with $[G, G]$.*

Proof. Since DG is abelian the group of commutators of G is flat *i.e.*, $[G, G] \leq G_1$. If $\hat{f}$ commutes with $[G, G]$, from Proposition 10 we have that $[G, G] \leq \langle \exp(t\hat{\xi})/t \in \mathbb{C} \rangle$ and from the proof of Theorem 1 we know that $\hat{\xi}$ is projectively invariant by G. Assume now that the vector field $\hat{\xi}$ is projectively invariant by G. Once again, since $[G, G] \leq G_1$ we know that if $S \in [G, G]$ then $c_S = 1$. Thus $\forall S \in [G, G]$, $S^*\hat{\xi} = \hat{\xi}$, therefore $\hat{f} = \exp(\hat{\xi})$ commutes with $[G, G]$. $\square$

Proof of Theorem 3. Notice that if G is quasi-abelian and DG is abelian then G is metabelian. Therefore, Theorem 3 follows from Propositions 10 and 12. $\square$

The following is a partial converse of (2) in Theorem 3.

Proposition 13. *Let $G \leq \hat{\text{Diff}}(\mathbb{C}^n, 0)$ be a metabelian subgroup containing a regular dicritic diffeomorphism with order of tangency $k+1$. Suppose that DG is abelian and there is a linear diffeomorphism $\hat{h} \in G$, given by $\hat{h}(z) = \lambda z$, with $\lambda^k \neq 1$, $\lambda^{k+1} \neq 1$. Then there is a formal vector field $\hat{\xi} \in \hat{\mathfrak{X}}_j(\mathbb{C}^n, 0)$, $j \geq 2$, which is projectively invariant by G, i.e., such that $\forall \hat{g} \in G$, $\hat{g}^*\hat{\xi} = c_{\hat{g}}\hat{\xi}$, for some constant $c_{\hat{g}} \neq 0$.*

Proof. By hypothesis $[G, G]$ is an abelian subgroup of flat diffeomorphisms and we have that

$$[\hat{f}, \hat{h}](z) = z + \lambda(\lambda^k - 1)f(z)z + \lambda(\lambda^{k+1} - 1)P_{k+2}(z) + \cdots$$

thus $[\hat{f}, \hat{h}] = \exp(\hat{\xi})$, where

$$\hat{\xi} = \lambda(\lambda^k - 1)f(z)\vec{R} + (\lambda(\lambda^{k+1} - 1)p^{(1)}_{k+2} + \cdots)\frac{\partial}{\partial z_1} + \cdots + (\lambda(\lambda^{k+1} - 1)p^{(n)}_{k+2} + \cdots)\frac{\partial}{\partial z_n}$$

since $\hat{f}$ is regular dicritic and $\lambda^k \neq 1, \lambda^{k+1} \neq 1$, the above expression implies that $[\hat{f}, \hat{h}]$ is regular dicritic. According to Proposition 12 there is a projectively invariant formal vector field $\hat{\xi} = \exp([\hat{f}, \hat{h}])$. $\square$

Now we give an application of our results:

Corollary 4. *Let $G = \langle \hat{f}, \hat{h} \rangle \leq \hat{\text{Diff}}(\mathbb{C}^n, 0)$, where $\hat{f}$ is regular dicritic and $\hat{h}(z) = \lambda X$, $\lambda^k \neq 1$, $\lambda^{k+1} \neq 1$. The group G is metabelian if and only if $[\hat{f}, \hat{h}^2]$ and $[\hat{f}^2, \hat{h}]$ commute with $[\hat{f}, \hat{h}]$.*

Proof. Given two elements $\hat{\varphi}, \hat{\psi} \in \hat{\text{Diff}}(\mathbb{C}^2,)$ we shall write $\hat{\varphi}^*\hat{\psi} := \hat{\varphi} \circ \hat{\psi} \circ \hat{\varphi}$. If $G < \hat{\text{Diff}}(\mathbb{C}^2, 0)$ is metabelian, it is immediately seen that $[\hat{f}, \hat{h}^2]$ and $[\hat{f}^2, \hat{h}]$ commute with $[\hat{f}, \hat{h}]$. We prove the converse, in fact we have that $[\hat{f}, \hat{h}]$ is regular dicritic as noted above. Now, as $[\hat{f}, \hat{h}^2]$, $[\hat{f}^2, \hat{h}]$ commutes with $[\hat{f}, \hat{h}]$, we have $[\hat{f}^2, \hat{h}] = \exp(c\hat{\xi})$ and $[\hat{f}, \hat{h}^2] = \exp(r\hat{\xi})$, since $\hat{f}^*[\hat{f}, \hat{h}] \circ [\hat{f}, \hat{h}] = [\hat{f}^2, \hat{h}]$ and $[\hat{f}, \hat{h}] \circ \hat{h}^*[\hat{f}, \hat{h}] = [\hat{f}, \hat{h}^2]$ then

$$\exp(c\hat{\xi}) = [\hat{f}^2, \hat{h}] = \hat{f}^*[\hat{f}, \hat{h}] \circ [\hat{f}, \hat{h}] = \hat{f}^* \exp(\hat{\xi}) \circ \exp(\hat{\xi})$$

consequently, $\hat{f}^* \exp(\hat{\xi}) = \exp(c\hat{\xi}) \circ \exp(-\mathcal{X}) = \exp(\widetilde{c}\hat{\xi})$. Using the same argument in the proof of Lemma 1, $\hat{f}^*\hat{\xi} = \widetilde{c}\hat{\xi}$. Similarly $\hat{h}^*\hat{\xi} = \widetilde{r}\hat{\xi}$. Thus by Theorem 3 the group G is metabelian. $\square$

The next three lemmas will be used in the proof of Theorem 4.

Lemma 9. *Let $\hat{f} \in \hat{\text{Diff}}_{r+1}(\mathbb{C}^n, 0)$ and $\hat{g} \in \hat{\text{Diff}}_{s+1}(\mathbb{C}^n, 0)$ be formal diffeomorphisms. Then*

$$\hat{f}(\hat{g}(z)) - \hat{g}(\hat{f}(z)) = D\hat{f}_{r+1}(z)\hat{g}_{s+1}(z) - D\hat{g}_{s+1}\hat{f}_{r+1}(z) + O(|z|^{r+s+2})$$

so $[\hat{f}, \hat{g}] = \text{Id}$ or $[\hat{f}, \hat{g}] \in \hat{\text{Diff}}_p(\mathbb{C}^n, 0)$ with $p \geq r + s + 1$.

Proof. Let $\hat{f}(z) = z + \sum_{k=r}^{r+s} \hat{f}_{k+1}(z) + O(|z|^{r+s+2})$ and $\hat{g}(z) = z + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + O(|z|^{r+s+2})$ then:

$$\begin{aligned}
\hat{f}(\hat{g}(z)) &= z + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + O(|z|^{r+s+2}) + \\
&\quad + \sum_{k=r}^{r+s} \hat{f}_{k+1}(z + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + O(|z|^{r+s+2})) + O(|z|^{r+s+2}) \\
&= z + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + \sum_{k=r}^{r+s} (\hat{f}_{k+1}(z) + D\hat{f}_{k+1}\hat{g}_{s+1}(z) + O(|z|^{k+s+2})) \\
&\qquad + O(|z|^{r+s+2}) \\
&= z + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + \sum_{k=r}^{r+s} \hat{f}_{k+1}(z) + D\hat{f}_{r+1}(z)\hat{g}_{s+1}(z) + O(|z|^{r+s+2})
\end{aligned}$$

Similarly we have:

$$\hat{g}(\hat{f}(z)) = z + \sum_{k=r}^{r+s} \hat{f}_{k+1}(z) + \sum_{j=s}^{r+s} \hat{g}_{j+1}(z) + D\hat{g}_{s+1}(z)\hat{f}_{r+1}(z) + O(|z|^{r+s+2})$$

subtracting these two equalities we get the lemma. ∎

Lemma 10. *Let $\hat{f} \in \hat{\text{Diff}}_{r+1}(\mathbb{C}^n, 0)$ and $\hat{g} \in \hat{\text{Diff}}_{s+1}(\mathbb{C}^n, 0)$ be dicritic diffeomorphisms with $r \neq s$, given by*

$$\hat{f}(z) = z + f(z)z + \cdots \qquad \hat{g}(z) = z + g(z)z + \cdots$$

then $[\hat{f}, \hat{g}] \in \hat{\text{Diff}}_{s+r+1}(\mathbb{C}^n, 0)$ is dicritic and given by

$$[\hat{f}, \hat{g}](z) = z + (r - s)g(z)f(z)z + \cdots$$

Proof. From Lemma 9 the term of smaller order of $[\hat{f}, \hat{g}]$ is

$$\begin{aligned}
D\hat{f}_{r+1}(z)\hat{g}_{s+1}(z) - D\hat{g}_{s+1}\hat{f}_{r+1}(z) &= (f(z)I + (z_i\frac{\partial f}{\partial z_j}))\hat{g}_{s+1}(z) - D\hat{g}_{s+1}f(z)z \\
&= (f(z)I + (z_i\frac{\partial f}{\partial z_j}))\hat{g}_{s+1}(z) - (s+1)f(z)\hat{g}_{s+1} \\
&= (-sf(z)I + (z_i\frac{\partial f}{\partial z_j}))\hat{g}_{s+1}(z) \\
&= (-sf(z)I + (z_i\frac{\partial f}{\partial z_j}))g(z)z
\end{aligned}$$

Then the i-th component of this is

$$-sf(z)g(z)z_i + g(z)z_i\nabla f(z)z = -sf(z)g(z)z_i + rf(z)g(z)z_i$$

thus

$$D\hat{f}_{r+1}(z)\hat{g}_{s+1}(z) - D\hat{g}_{s+1}\hat{f}_{r+1}(z) = (r - s)f(z)g(z)z$$

therefore $[\hat{f}, \hat{g}] \in \hat{\text{Diff}}_{s+r+1}(\mathbb{C}^n, 0)$ and this is dicritic, given by

$$\hat{h}(z) = z + (r - s)g(z)f(z)z + \cdots$$

∎

Lemma 11. *Let $G < \hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ be a solvable group, then G has no dicritic diffeomorphisms of different orders.*

Proof. Assume that there are $\hat{f}_1, \hat{f}_2 \in G$ dicritic diffeomorphisms of different orders, we say $p_1 + 1$ and $p_2 + 1$ respectively then by above lemma $\hat{f}_3 = [\hat{f}_1, \hat{f}_2]$ is dicritic of order $p_3 = p_1 + p_2 + 1 > p_2 + 1$, similarly, we have that $\hat{f}_4 = [\hat{f}_3, \hat{f}_2]$ is dicritic of order $p_4 = p_3 + p_2 > p_3$ and recurrently $\hat{f}_n = [\hat{f}_{n-1}, \hat{f}_{n-2}]$ is dicritic of order $p_n = p_{n-1} + p_{n-2} > p_{n-1}$, thus there is no $n \in \mathbb{N}$ such that $G^n = \{\mathrm{Id}\}$ and this contradicts the fact G is solvable. □

Proof of Theorem **D**. Let $G \leq \hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ be a subgroup of diffeomorphisms tangent to the identity containing a dicritic diffeomorphism $\hat{f}$ with order of tangency $k+1$. It is immediate to verify that (1) $\Rightarrow$ (2). Let us now prove (2) $\Rightarrow$ (3). Suppose that $\hat{f}(z) = z + f(z)z + \cdots$. Suppose by contradiction that there is $\hat{f}^{(1)} \in G$ with order of tangency $p_1 > k+1$ then obtain

$$\hat{f}^{(2)} = [\hat{f}^{(1)}, \hat{f}] = z + \hat{f}^{(2)}_{k_2} + \cdots .$$

we affirm that $\hat{f}^{(2)}_{k_2} \neq 0$ and thus $\hat{f}^{(2)}$ has order of tangent $k_2 = k + k_1 > k_1 + 1$. In fact, as the j-th coordinate of $\hat{f}^{(2)}_{k_2}$, is $(k_1 - 1)f.q^{(j)}_{k_1} - (\nabla f.Q_{k_1})z_j$, where $\hat{f}^{(1)} = z + Q_{k_1} + \ldots$ and $Q_{k_1} = (q^{(1)}_{k_1}, \ldots, q^{(n)}_{k_1})$, (in consequence $k_2 = k + k_1$), now if $\hat{f}^{(2)}_{k_2} = 0$, then $(k_1 - 1)f.q^{(j)}_{k_1} = (\nabla f.Q_{k_1})z_j$, for $j = 1, \ldots, n$. So following the same argument of Lemma 7 we have that $Q_{k_1} = (g.z_1, \ldots, g.z_n)$, with g homogeneous polinomial of degree p, thus Q_{k_1} has degree $k+1$, but this is impossible. Repeating this process we can define:

$$\hat{f}^{(n)} = [\hat{f}^{(n-1)}, \hat{f}] = z + \hat{f}^{(n)}_{k_n} + \cdots .$$

Analogously $\hat{f}^{(n)}_{k_n} \neq 0$ and thus $\hat{f}^{(n)}$ we has order of tangency $k_n = k + k_{n-1} > k_{n-1} + 1 > n$, thus there is no $n \in \mathbb{N}$, such that $\gamma_n(\Lambda) = \{Id\}$, what contradicts the fact that Λ is nilpotent. Therefore, we have $\Lambda \subseteq \widehat{\mathrm{Diff}}_{k+1}(\mathbb{C}^n, 0)$.
Now we prove (3) $\Rightarrow$ (1). From Lemma 9 we have that for $\hat{h}, \hat{g} \in G$, $[\hat{h}, \hat{g}] = \{\mathrm{Id}\}$ or $[\hat{h}, \hat{g}] \in \hat{\mathrm{Diff}}_{\ell}(\mathbb{C}^n, 0)$, $\ell \geq 2k+1$ thus $[\hat{h}, \hat{g}] = \{\mathrm{Id}\}$ and therefore G is abelian. □

Proof of Corollary 2. Let $G < \hat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ be a subgroup such that $[G, G]$ is a flat group containing a dicritic element $\hat{f} \in G$. Suppose that $[G, G]$ is nilpotent. Then, since by hypothesis the group $[G, G]$ contains a dicritic element, Theorem **D** implies that $[G, G]$ is abelian. Therefore G is metabelian. □

References

[1] Brochero Martinez, F.: *Groups of germs of analytic diffeomorphisms in* $(\mathbb{C}^2, 0)$, Journal of Dynamical and Control Systems, New York, V.**9** n.1 1–32 2003.

[2] Camacho, C., Lins Neto, A. and Sad, P.: *Foliations with algebraic limit sets*; Ann. of Math. 136 (1992), 429–446.

[3] Camacho C. and Scárdua, B.: *Holomorphic foliations with Liouvillian first integrals.* Ergodic Theory Dynam. Systems 21 (2001), no. 3, 717–756.

[4] Elizarov, P. M.; Il' Ilyashenko Yu. S.; Shcherbakov, A. A. Voronin, S. M, *" Finitely generated groups of germs of one-dimensional conformal mappings, and invariants for complex singular points of analytic foliations of complex plane"*, Nonlinear Stokes phenomena, Adv. Soviet Math. 14(1993) pag 57-105.

[5] Hartshorne, R.: *Ample subvarieties of algebraic varieties*; L.N.M., 156, 1970.

[6] Hironaka, H. and Matsumara, H.: *Formal functions and formal embeddings*; J. Math. Soc. Japan, Vol. 20, Nos.1-2, 1968.

[7] Il'yashenko Y.: *Topology of phase portraits of analytic differential equations on a complex projective plane. Trudy Sem. Petrovsk. No. 4 (1978), 83–136*

[8] Il'yasenko, Yuli; Yakovenko, Sergei, *Lectures on analytic differential equations*, Graduate studies in mathematics, V 86. American mathematical society.

[9] J.F. Mattei and R. Moussu; *Holonomie et intégrales premières.* Ann. Ec. Norm. Sup. 13 (1980), 469–523.

[10] Nakai, I.: *Separatrices for non-solvable dynamics on* $(\mathbb{C}, 0)$. Ann. Inst. Fourier (Grenoble) 44 (1994), no. 2, 569–599.

[11] Petrovski, I. and Landis, E.: On the number of limit cycles of the equation $dy/dx = P(x,y)/Q(x,y)$ where P and Q are polynomials of degree 2, Amer. Math. Society Trans., Series 2, 10 (1963) 125-176.

[12] Scárdua, B.; *Transversely affine and transversely projective foliations.* Ann. Sc. École Norm. Sup., 4^e série, t.**30**, 1997, p.169-204

[13] Scárdua, B.: *Integration of complex differential equations*, J. Dynam. Control Systems 5 (1999), 1–50.

[14] Scárdua, B.: *Differential Algebra and Liouvillian first integrals of foliations*, Journal of Pure and Applied Algebra (in press).

[15] A. A. Shcherbakov, *Dynamics of Local Groups of Conformal Mappings and Generic Properties of Differential Equations on* $\mathbb{C}^2$, Tr. Mat. Inst. im. V.A. Steklova, Ross. Akad. Nauk 254, 111–129 (2006) [Proc. Steklov Inst. Math. 254, 103–120 (2006)].

[16] M. Rosenlicht, *Some basic theorems on algebraic groups*, American Journal of Mathemathics 78, 401-443, 1956.

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