COARSE TYPES OF TROPICAL MATROID POLYTOPES

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Abstract. Describing the combinatorial structure of the tropical complex $\mathcal{C}$ of a tropical matroid polytope, we obtain a formula for the coarse types of the maximal cells of $\mathcal{C}$. Due to the connection between tropical complexes and resolutions of monomial ideals, this yields the generators for the corresponding coarse type ideal introduced in [7]. Furthermore, a complete description of the minimal tropical halfspaces of the uniform tropical matroid polytopes, i.e. the tropical hypersimplices, is given.

1. Introduction

Tropical matroid polytopes have been introduced in [4] as the tropical convex hull of the cocircuits, or dually, of the bases of a matroid. The arrangement of finitely many points V in the tropical torus $\mathbb{T}^d$ has a natural decomposition $\mathcal{C}_V$ of $\mathbb{T}^d$ into (ordinary) polytopes, the tropical complex, equipped with a (fine) type T, which encodes the relative position to the generating points. The *coarse types* only count the cardinalities of T. In [5], Develin and Sturmfels showed that the bounded cells of $\mathcal{C}_V$ yield the tropical convex hull of V, which is dual to the regular subdivision Σ of a product of two simplices (or equivalently—due to the Cayley Trick—to the regular mixed subdivisions of a dilated simplex). The authors of [3] and [7] use the connection of the cellular structure of $\mathcal{C}_V$ or rather of Σ to minimal cellular resolutions of certain monomial ideals to provide an algorithm for determining the facial structure of the bounded subcomplex of $\mathcal{C}_V$. A main result of [7] says that the labeled complex $\mathcal{C}_V$ supports a minimal cellular resolution of the ideal I generated by monomials corresponding to the set of all (coarse) types.

The main theme of this paper is the study of the tropical complex of tropical convex polytopes associated with matroids arising from graphs—the *tropical matroid polytopes*. Recall that a *matroid* M is a finite collection $\mathcal{F}$ of subsets of $[n] = 1, 2, \ldots, n$, called *independent sets*, such that three properties are satisfied: (i) $\emptyset \in \mathcal{F}$, (ii) if $X \in \mathcal{F}$ and $Y \subseteq X$ then $Y \in \mathcal{F}$, (iii) if $U, V \in \mathcal{F}$ and $|U| = |V| + 1$ there exists $x \in U \setminus V$ such that $V \cup x \in \mathcal{F}$. The last one is also called the *exchange property*. The maximal independent sets are the *bases* of M. A matroid can also be defined by specifying its *non-bases*, i.e. the subsets of E with cardinality k that are not bases. For more details on matroids see the survey of Oxley [15] and the books of White([16], [17], [18]). An important class of matroids are the graphic or cycle matroids proven to be regular, that is, they are representable over every field. A *graphic matroid* is associated with a simple undirected graph G by letting E be the set of edges of G and taking as the bases the edges of the spanning forests. Matroid polytopes were first studied in connection with optimization and linear programming, introduced by Jack Edmonds [8]. A nice polytopal characterization for a matroid polytope was given by Gelfand et al. [10] stating that each of its edges is a parallel translate of $e_i - e_j$ for some i and j.

In the case of tropical matroid polytopes the coarse types display the number $b_{I,J}$ of bases B of the associated matroid with subsets I, J, where all elements of I but none of J are contained in B.

Theorem 1. *Let $\mathcal{C}$ be the tropical complex of a tropical matroid polytope with $d+1$ elements and rank k. The set of all coarse types of the maximal cells arising in $\mathcal{C}$ is given by the tuples $(t_1,\ldots,t_{d+1})$ with*

$$t_j \;=\; \begin{cases} b_{\{i_1\},\emptyset} + b_{\emptyset,\{i_1,i_2,\ldots,i_{d'+1}\}} & \text{if } j = i_1\,, \\ b_{\{i_l\},\{i_1,\ldots,i_{l-1}\}} & \text{if } j = i_l \in \{i_2,\ldots i_{d'+1}\}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

where $d' \in [d-k+1]$ and $\{i_1,i_2,\ldots,i_{d'+1}\}$ is a sequence of elements such that $[d+1]\setminus\{i_1,i_2,\ldots,i_{d'}\}$ contains a basis of the associated matroid.

Subsequently, we relate our combinatorial result to commutative algebra. For the coarse type $\mathbf{t}(p)$ of p and $x^{\mathbf{t}(p)} = {x_1}^{\mathbf{t}(p)_1}{x_2}^{\mathbf{t}(p)_2}\cdots{x_{d+1}}^{\mathbf{t}(p)_{d+1}}$ the monomial ideal

$$I = \langle x^{\mathbf{t}(p)} \colon p \in \mathbb{T}^d \rangle \subset \mathbb{K}[x_1,\ldots,x_{d+1}]$$

is called the *coarse type ideal*. In [7], Corollary 3.5, it was shown that I is generated by the monomials, which are assigned to the coarse types of the inclusion-maximal cells of the tropical complex. As a direct consequence of Theorem 3.6 in [7], we obtain the generators of I.

Corollary 2. *The coarse type ideal I for the tropical complex of a tropical matroid polytope with $d+1$ elements and rank k is equal to*

$$\langle x_{i_1}^{t_{i_1}} x_{i_2}^{t_{i_2}} \cdots x_{i_{d'+1}}^{t_{i_{d'+1}}} : [d+1]\setminus\{i_1,\ldots,i_{d'}\} \text{ contains a basis } \rangle$$

where $(t_{i_1},t_{i_2},\ldots,t_{i_{d'+1}}) = \big(b_{\{i_1\},\emptyset} + b_{\emptyset,\{i_1,i_2,\ldots,i_{d'+1}\}}, b_{\{i_2\},\{i_1\}},\ldots,b_{\{i_{d'+1}\},\{i_1,\ldots,i_{d'}\}}\big)$.

Furthermore, we apply these results to the special case of uniform matroids, introduced and studied in [11]. We close this work by stating the minimal tropical halfspaces containing a uniform tropical matroid polytope by using the characterization of Proposition 1 in [9].

2. Basics of tropical convexity

We start with collecting basic facts about tropical convexity and fixing the notation. Defining *tropical addition* by $x \oplus y := \min(x,y)$ and *tropical multiplication* by $x \odot y := x+y$ yields the *tropical semi-ring* $(\mathbb{R},\oplus,\odot)$. Component-wise tropical addition and *tropical scalar multiplication*

$$\lambda \odot (\xi_0,\ldots,\xi_d) := (\lambda\odot\xi_1,\ldots,\lambda\odot\xi_d) = (\lambda+\xi_0,\ldots,\lambda+\xi_d)$$

equips $\mathbb{R}^{d+1}$ with a semi-module structure. For $x,y\in\mathbb{R}^{d+1}$ the set

$$[x,y]_{\text{trop}} := \{(\lambda\odot x)\oplus(\mu\odot y) \mid \lambda,\mu\in\mathbb{R}\}$$

defines the *tropical line segment* between x and y. A subset of $\mathbb{R}^{d+1}$ is *tropically convex* if it contains the tropical line segment between any two of its points. A direct computation shows that if $S\subset\mathbb{R}^{d+1}$ is tropically convex then S is closed under tropical scalar multiplication. This leads to the definition of the *tropical torus* as the quotient semi-module

$$\mathbb{T}^d := \mathbb{R}^{d+1}/(\mathbb{R}\odot(1,\ldots,1)).$$

Note that $\mathbb{T}^d$ was called "tropical projective space" in [5], [11], [6], and [14]. Tropical convexity gives rise to the hull operator tconv. A *tropical polytope* is the tropical convex hull of finitely many points in $\mathbb{T}^d$.

Like an ordinary polytope each tropical polytope P has a unique set of generators which is minimal with respect to inclusion; these are the *tropical vertices* of P.

There are several natural ways to choose a representative coordinate vector for a point in $\mathbb{T}^d$. For instance, in the coset $x+(\mathbb{R}\odot(1,\ldots,1))$ there is a unique vector $c(x)\in\mathbb{R}^{d+1}$ with non-negative coordinates such that at least one of them is zero; we refer to $c(x)$ as the *canonical coordinates* of $x\in\mathbb{T}^d$. Moreover, in the same coset there is also a unique vector $(\xi_0,\ldots,\xi_d)$ such that $\xi_0 = 0$. Hence, the map

$$c_0 : \mathbb{T}^d \to \mathbb{R}^d, (\xi_1,\ldots,\xi_{d+1}) \mapsto (\xi_2-\xi_1,\ldots,\xi_{d+1}-\xi_1)$$

is a bijection. Often we will identify $\mathbb{T}^d$ with $\mathbb{R}^d$ via this map.

The *tropical hyperplane* $\mathcal{H}_a$ defined by the *tropical linear form* $a = (\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{R}^{d+1}$ is the set of points $(\xi_1, \dots, \xi_{d+1}) \in \mathbb{T}^d$ such that the minimum

$$(\alpha_1 \odot \xi_1) \oplus \cdots \oplus (\alpha_{d+1} \odot \xi_{d+1})$$

is attained at least twice. For $d = 3$ the tropical hyperplane is shown in Figure 1(b). The complement of a tropical hyperplane in $\mathbb{T}^d$ has exactly $d+1$ connected components, each of which is an *open sector*. A *closed sector* is the topological closure of an open sector. The set

$$S_k := \big\{(\xi_1, \dots, \xi_{d+1}) \in \mathbb{T}^d \mid \xi_k = 0 \text{ and } \xi_i > 0 \text{ for } i \neq k\big\},$$

for $1 \leq k \leq d+1$, is the *k-th open sector* of the tropical hyperplane $\mathcal{Z}$ in $\mathbb{T}^d$ defined by the zero tropical linear form. Its closure is

$$\bar{S}_k := \big\{(\xi_1, \dots, \xi_{d+1}) \in \mathbb{T}^d \mid \xi_k = 0 \text{ and } \xi_i \geq 0 \text{ for } i \neq k\big\}.$$

We also use the notation $\bar{S}_I := \bigcup\{\bar{S}_i \mid i \in I\}$ for any set $I \subset [d+1] := \{1, \dots, d+1\}$.

If $a = (\alpha_1, \dots, \alpha_{d+1})$ is an arbitrary tropical linear form then the translates $-a + S_k$ for $1 \leq k \leq d+1$ are the open sectors of the tropical hyperplane $\mathcal{H}_a$. The point $-a$ is the unique point contained in all closed sectors of $\mathcal{H}_a$, and it is called the *apex* of $\mathcal{H}_a$. For each $I \subset [d+1]$ with $1 \leq \#I \leq d$ the set $-a + \bar{S}_I$ is the *closed tropical halfspace* of $\mathcal{H}_a$ of type I. A tropical halfspace $H(-a, I)$ can also be written in the form

$$\begin{aligned} H(-a,I) &= \{x \in \mathbb{T}^d \mid \text{ the minimum of } \bigoplus_{i=1}^{d+1} \alpha_i \odot \xi_i \text{ is attained} \\ &\qquad \text{at a coordinate } i \in I\} \\ &= \{x \in \mathbb{T}^d \mid \bigoplus_{i \in I} (\alpha_i \odot \xi_i) \leq \bigoplus_{j \in J} (\alpha_j \odot \xi_j)\} \end{aligned}$$

where I and J are disjoint subsets of $[d+1]$ and $I \cup J = [d+1]$. The tropical polytopes in $\mathbb{T}^d$ are exactly the bounded intersections of finitely many closed tropical halfspaces; see [9] and [11].

We concentrate on the combinatorial structure of tropical polytopes. Let $V := (v_1, \dots, v_n)$ be a sequence of points in $\mathbb{T}^d$. The *(fine) type* of $x \in \mathbb{T}^d$ with respect to V is the ordered $(d+1)$-tuple $\text{type}_V(x) := (T_1, \dots, T_{d+1})$ where

$$T_k := \{i \in \{1, \dots, n\} \mid v_i \in x + \bar{S}_k\}.$$

For a given type $\mathcal{T}$ with respect to V the set

$$X_V^\circ(\mathcal{T}) := \big\{x \in \mathbb{T}^d \mid \text{type}_V(x) = \mathcal{T}\big\}$$

is a relatively open subset of $\mathbb{T}^d$ and is called the *cell* of type $\mathcal{T}$ with respect to V. The set $X_V^\circ(\mathcal{T})$ as well as its topological closure are tropically and ordinary convex; in [13], these were called *polytropes*. With respect to inclusion the types with respect to V form a partially ordered set. The intersection of two cells $X_V(\mathcal{S})$ and $X_V(\mathcal{T})$ with type $\mathcal{S}$ and $\mathcal{T}$ is equal to the polyhedron $X_V(\mathcal{S} \cup \mathcal{T})$. The collection of all (closed) cells induces a polyhedral subdivision $\mathcal{C}_V$ of $\mathbb{T}^d$. A min-tropical polytope $P = \text{tconv}(V)$ is the union of cells in the bounded subcomplex $\mathcal{B}_V$ of $\mathcal{C}_V$ induced by the arrangement $\mathcal{A}_V$ of max-tropical hyperplanes with apices $v \in V$. A cell of $\mathcal{C}_V$ is unbounded if and only if one of its type components is the empty set. The type of x equals the union of the types of the (maximal) cells that contain x in their closure. The dimension of a cell X_T can be calculated as the number of the connected components of the undirected graph $G = \big(\{1, 2, \dots, d+1\}, \{(j,k) \mid T_j \cap T_k \neq \emptyset\}\big)$ minus one. The zero-dimensional cells are called pseudovertices of P.

Replacing the (fine) type entries $T_k \subseteq [n]$ for $k \in [d+1]$ of a point $p \in \mathbb{T}^d$ by their cardinalities $t_k := |T_k|$ we get the *coarse type* $t_V(p) = (t_1, \dots, t_{d+1}) \in \mathbb{N}^{d+1}$ of p. A coarse type entry t_k displays how many generating points lie in the k-th closed sector $p + \overline{S_k}$. In [7], the authors associate the tropical complex of a tropical polytope with a monomial ideal, the coarse type ideal

$$I := \langle {x_1}^{t_1} {x_2}^{t_2} \cdots {x_{d+1}}^{t_{d+1}} : p \in \mathbb{T}^d \rangle \subset \mathbb{K}[x_1, \dots, x_{d+1}].$$

By Corollary 3.5 of [7], I is generated by the monomials assigned to the coarse types of the inclusion-maximal cells of the tropical complex. The tropical complex $\mathcal{C}_V$ gives rise to minimal cellular resolutions of I.

Theorem 3 ([7], Theorem 3.6)**.** *The labeled complex $\mathcal{C}_V$ supports a minimal cellular resolution of the ideal I generated by monomials corresponding to the set of all (coarse) types.*

Considering cellular resolutions of monomial ideals, introduced in [1] and [2], is a natural technique to construct resolutions of monomial ideals using labeled cellular complexes and provide an important interface between topological constructions, combinatorics and algebraic ideas. The authors of [3] and [7] use this to give an algorithm for determining the facial structure of a tropical complex. More precisely, they associate a squarefree monomial ideal I with a tropical polytope and calculate a minimal cellular resolution of I, where the i-th syzygies of I are encoded by the i-dimensional faces of a polyhedral complex.

A tropical halfspace is called *minimal* for a tropical polytope P if it is minimal with respect to inclusion among all tropical halfspaces containing P. Consider a tropical halfspace $H(a, I) \subset \mathbb{T}^d$ with $I \subset [d+1]$ and apex $a \in \mathbb{T}^d$, and a tropical polytope $P = \operatorname{tconv}\{v_1, \dots, v_n\} \subseteq \mathbb{T}^d$. To show that $H(a, I)$ is minimal for P, it suffices to prove, by Proposition 1 of [9], that the following three criteria hold for the type $(T_1, T_2, \dots, T_{d+1}) = \operatorname{type}_V(a)$ of the apex a:

(i) $\bigcup_{i \in I} T_i = [n]$,

(ii) for each $j \in I^C$ there exists an $i \in I$ such that $T_i \cap T_j \neq \emptyset$,

(iii) for each $i \in I$ there exists $j \in I^C$ such that $T_i \cap T_j \not\subset \bigcup_{k \in I \setminus \{i\}} T_k$.

Here, we denote the complement of a set $I \subseteq [d+1]$ as $I^C = [d+1] \setminus I$.
Obvious minimal tropical halfspaces of a tropical polytope $P = \operatorname{tconv}(V) \subseteq \mathbb{T}^d$ are its cornered halfspaces, see [12]. The *k-th corner* of P is defined as

$$c_k(V) := (-v_{1,k}) \odot v_1 \oplus (-v_{2,k}) \odot v_2 \oplus \cdots \oplus (-v_{n,k}) \odot v_n.$$

The tropical halfspace $H_k := c_k(V) + \overline{S_k}$ is called the *k-th cornered tropical halfspace* of P and the intersection of all $d+1$ cornered halfspaces is the *cornered hull* of P.

3. Tropical Matroid Polytopes

The tropical matroid polytope of a matroid $\mathcal{M}$ is defined in [4] as the tropical convex hull of the negative incidence vectors of the bases of $\mathcal{M}$. In this paper, we restrict ourselves to matroids arising from graphs.

The *graphic matroid* of a simple undirected graph $G = (V, E)$ is $\mathcal{M}(G) = (E, \mathcal{I} = \{F \subseteq E\colon F \text{ is acyclic}\})$. While the forests of G form the system of independent sets of $\mathcal{M}(G)$ its bases are the spanning forests. We will assume that G is connected, so the bases of $\mathcal{M}(G)$ are the spanning trees of G. Furthermore, we exclude bridges, i.e. edges whose deletion increases the number of connected components of G, leading to elements that are contained in every basis. Let $d+1$ be the number of elements and n be the number of bases of $\mathcal{M} := \mathcal{M}(G)$ and $\mathcal{B} := \{B_1, \dots, B_n\}$ its bases. It follows from the exchange property of matroids that all bases of $\mathcal{M}$ have the same number of elements, which is called the *rank* of $\mathcal{M}$. Consider the 0/1-matrix $M \in \mathbb{R}^{(d+1)\times n}$ with rows indexed by the elements of the ground set E and columns indexed by the bases of $\mathcal{M}$ which has a 0 in entry (i, j) if the i-th element is in the j-th basis. The *tropical matroid polytope* P of $\mathcal{M}$ is the tropical convex hull of the columns of M. Let

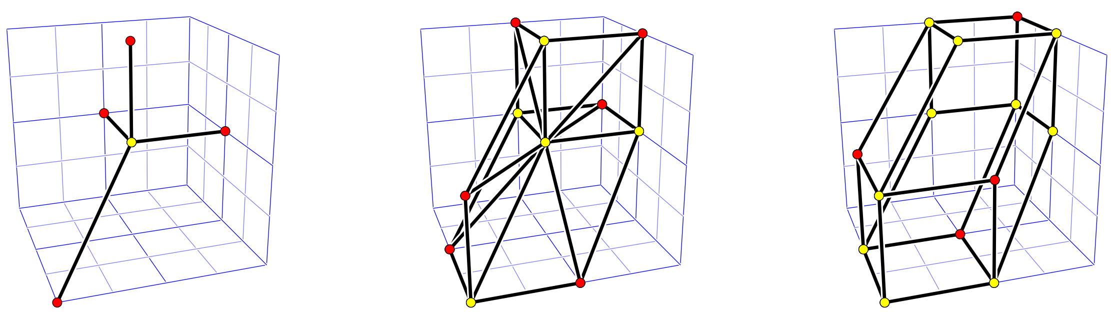
$$V = \left\{ -e_B := \sum_{i \in B} -e_i \middle| B \in \mathcal{B} \right\} \tag{1}$$

be the set of generators of P. It turns out that these are just the tropical vertices of P, see Lemma 8. If the underlying matroid has rank k, then the canonical coordinate vectors of V have exactly k zeros and $d+1-k$ ones and will be denoted as v_{B_i} or for short v_i if the corresponding basis is $B_i \in \mathcal{B}$. Note that with $\oplus$ as max instead of min the generators of a tropical matroid polytope are the positive incidence vectors of the bases of the corresponding matroid. Throughout this paper we write $\mathcal{P}_{k,d}$ for the set of all tropical matroid polytopes arising from a graphic matroid with $d+1$ elements and rank k.

Example 4. *The tropical hypersimplex Δ_k^d in $\mathbb{T}^d$ studied in [11] is a tropical matroid polytope of a uniform matroid of rank k with $d+1$ elements and $\binom{d+1}{k}$ bases. It is defined as the tropical convex hull of all points $-e_I := \sum_{i\in I} -e_i$ where e_i is the i-th unit vector of $\mathbb{R}^{d+1}$ and I is a k-element subset of $[d+1]$. The tropical vertices of Δ_k^d are*

$$\operatorname{Vert}(\Delta_k^d) = \left\{ -e_I \middle| I \in \binom{[d+1]}{k} \right\} \text{ for all } k > 0.$$

In [11], it is shown that $\Delta_{k+1}^d \subsetneq \Delta_k^d$ implying that the first tropical hypersimplex contains all other tropical hypersimplices in $\mathbb{T}^d$. The first tropical hypersimplex $\Delta^d = \Delta_1^d$ in $\mathbb{T}^d$ is the d-dimensional tropical standard simplex which is also a polytrope. Clearly, we have for a tropical matroid polytope $P \in \mathcal{P}_{k,d}$ the chain $P \subseteq \Delta_k^d \subsetneq \cdots \subsetneq \Delta_1^d = \Delta^d$. For $d = 3$ the three tropical hypersimplices are shown in Figure 1.



(a) $k = 3$ (b) $k = 2$ (c) $k = 1$

FIGURE 1. The three 3-dimensional tropical hypersimplices with $\Delta_3^3 \subset \Delta_2^3 \subset \Delta_1^3$.

The origin $\mathbf{0} \in \mathbb{T}^d$ and its fine type are crucial for the calculation of the fine and the coarse types of the maximal cells in the cell complex of P.

Lemma 5. *A tropical matroid polytope $P \in \mathcal{P}_{k,d}$ with generators V contains the origin $\mathbf{0} \in \mathbb{T}^d$. Its type is $\operatorname{type}_V(\mathbf{0}) = (T_1^{(0)}, T_2^{(0)}, \ldots, T_{d+1}^{(0)})$ with $T_i^{(0)} = \{j \mid i \in B_j\}$.*

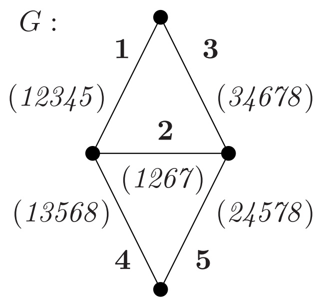
Proof. By Proposition 3 of [5] about the shape of a tropical line segment, the only pseudovertex of the tropical line segment between two distinct 0-1-vectors u and v in $\mathbb{T}^d$ is the point w with $w_l = 0$ if $u_l = 0$ or $v_l = 0$ and $w_l = 1$ otherwise. Since every element of E is contained in any basis of $\mathcal{M}(G)$ (apply any spanning-tree-greedy-algorithm for the connected components of G starting from this element) and by using the previous argument, the origin must be contained in P.

An index j is contained in the i-th type coordinate $T_i^{(0)}$ if $v_{j,i} = \min\{v_{j,1}, v_{j,2}, \ldots, v_{j,d+1}\}$, which is satisfied by all indices $i \in B_j$. □

The i-th type entry $T_i^{(0)}$ of $\mathbf{0}$ contains all bases of $\mathcal{M}$ with element i, and $|T_i^{(0)}|$ is the number of bases of $\mathcal{M}$ containing i.

Now it is time to introduce our running example.

Example 6. *The graphical matroid given by the following graph G has $d+1=5$ elements (edges with bold indices), rank $k=3$, $n=8$ bases $B_1=\{\mathbf{1},\mathbf{2},\mathbf{4}\}$, $B_2=\{\mathbf{1},\mathbf{2},\mathbf{5}\}$, $B_3=\{\mathbf{1},\mathbf{3},\mathbf{4}\}$, $B_4=\{\mathbf{1},\mathbf{3},\mathbf{5}\}$, $B_5=\{\mathbf{1},\mathbf{4},\mathbf{5}\}$, $B_6=\{\mathbf{2},\mathbf{3},\mathbf{4}\}$, $B_7=\{\mathbf{2},\mathbf{3},\mathbf{5}\}$, $B_8=\{\mathbf{3},\mathbf{4},\mathbf{5}\}$ and the non-bases $\{\mathbf{1},\mathbf{2},\mathbf{3}\}$, $\{\mathbf{2},\mathbf{4},\mathbf{5}\}$.*



Let P be the corresponding tropical matroid polytope with its generators

$$V = \{v_{B_1},\dots,v_{B_8}\} = \left\{ \begin{pmatrix}0\\0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix} \right\}.$$

The type of the origin $\mathbf{0}$ of P is $(12345, 1267, 34678, 13568, 24578)$ where the i-th type entry contains all bases using the edge i (italic edge attributes).

In the next lemma we will show that the tropical standard simplex Δ^d is the cornered hull of all tropical matroid polytopes in $\mathcal{P}_{k,d}$.

Lemma 7. *The cornered hull of a tropical matroid polytope $P \in \mathcal{P}_{k,d}$ with generators V is the d-dimensional tropical standard simplex Δ^d. The i-th corner of P is the vector e_i. The type of e_i with respect to V is $\operatorname{type}_V(e_i) = (T_1,\dots,T_{d+1})$ with*

$$T_j = \begin{cases} [d+1] & \text{if } j=i\,,\\ \{l \mid j\in B_l \text{ and } i\notin B_l\} & \text{otherwise}\,.\end{cases}$$

Proof. For $B\in\mathcal{B}$ the i-th (canonical) coordinate of v_B is

$$v_{B,i} = \begin{cases} 0 & \text{if } i\in B\,,\\ 1 & \text{otherwise}\,.\end{cases}$$

The j-th coordinate of the i-th corner $c_i(V)$ of P is

$$c_i(V)_j = \min_{J\in\mathcal{B}}(v_{J,j}-v_{J,i}) = \begin{cases} 0 & \text{if } i=j\,,\\ -1 & \text{otherwise}\,.\end{cases}$$

In canonical coordinates we get $c_i(V)=e_i$, which at the same time is the i-th apex vertex of the tropical standard simplex Δ^d. The type of e_i is $\operatorname{type}_V(e_i)=(T_1,T_2,\dots,T_{d+1})$, where some index l is contained in the j-th coordinate T_j for $j\neq i$ if $v_{l,j}=\min\{v_{l,1},v_{l,2},\dots,v_{l,i}-1,\dots,v_{l,d+1}\}$. This is satisfied by all bases $B_l\in\mathcal{B}$ with $j\in B_l$ and $i\notin B_l$. For $j=i$ all indices $l\in[d+1]$ are contained in T_i since the right hand side of $v_{l,i}-1=\min\{v_{l,1},v_{l,2},\dots,v_{l,i}-1,\dots,v_{l,d+1}\}$ is smaller or equal than the left hand side in every case. □

Besides the point $\mathbf{0}$, the other pseudovertices of a tropical matroid polytope correspond to unions of its bases.

Lemma 8. *The pseudovertices of $P\in\mathcal{P}_{k,d}$ are*

$$\operatorname{PV}(P) = \left\{ -e_J \middle| J = \bigcup_{i\in I} B_i \text{ for some } I\subseteq[n] \right\}.$$

The pseudovertices of the first tropical hypersimplex are

$$\operatorname{PV}(\Delta^d) = \left\{ -e_J \middle| J \in \bigcup_{j=1}^{d} \binom{[d+1]}{j} \right\}.$$

Let $(T_1^{(0)},\dots,T_{d+1}^{(0)})$ be the type of the pseudovertex $\mathbf{0}$ with respect to V and consider a point $-e_J\in\operatorname{PV}(P)$. If the complement J^C of J is equal to $\{i_1,\dots,i_r\}$, then the type $(T_1,\dots,T_{d+1})$

of $-e_J$ with respect to V is given by

$$T_j \;=\; \begin{cases} T_j^{(0)} \setminus (T_{i_1}^{(0)} \cup \cdots \cup T_{i_r}^{(0)}) & \text{if } j \in J\,, \\ T_j^{(0)} \cup ({T_{i_1}^{(0)}}^C \cap \cdots \cap {T_{i_r}^{(0)}}^C) & \text{otherwise}\,. \end{cases}$$

Proof. Consider the point $v_J := c(-e_J) = e_{J^C}$ with canonical coordinates

$$v_{J,i} = \begin{cases} 0 & \text{if } i \in J\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

and $\text{type}_V(v_J) = (T_1, \ldots, T_{d+1})$.

Since the union of the elements of one or more bases of $\mathcal{M}$ consists of at least k elements, the index set J has at least k elements and thus we have $r \leq d-k+1$ for the cardinality r of J^C. We can assume that $J^C = \{1, 2, \ldots, r\}$. Then some index l occurs in the j-th coordinate T_j if and only if

$$\begin{aligned} v_{l,j} - v_{J,j} &= \min\{v_{l,1}-1, \ldots, v_{l,r}-1, v_{l,r+1}, \ldots, v_{l,d+1}\} \\ &= \min\{v_{l,1}-1, \ldots, v_{l,r}-1\} \in \{-1, 0\}. \end{aligned} \tag{2}$$

For $j \in J$ the left hand side of equation (2) is $v_{l,j} - 0 \in \{0, 1\}$. If $j \in B_l$, we get $v_{l,j} - v_{J,j} = 0-0$ and this is minimal in (2) if the coordinates $v_{l,i}$ are equal to one for all $i \in J^C$, i.e. $i \notin B_l$. If $j \notin B_l$, we get $v_{l,j} - v_{J,j} = 1 \notin \{-1, 0\}$. Therefore, T_j is equal to $\{(l \mid j \in B_l) \wedge (i \notin B_l \text{ for all } i \in J^C)\} = T_j^{(0)} \setminus (T_{i_1}^{(0)} \cup \cdots \cup T_{i_r}^{(0)})$.

For $j \in J^C$ the left hand side is $v_{l,j} - 1 \in \{0, -1\}$. If $j \in B_l$, we get $v_{l,j} - v_{J,j} = -1 = \min\{v_{l,1}-1, \ldots, v_{l,j}-1, \ldots, v_{l,r}-1\}$. If $j \notin B_l$, we get $v_{l,j} - v_{J,j} = 1-1 = 0$ and this is minimal in (2) if the coordinates $v_{l,i}$ are equal to one for all $i \in J^C$, i.e. $i \notin B_l$. Therefore, T_j is equal to $\{l \mid j \in B_l \textbf{ or } (i \notin B_l \text{ for all } i \in J^C)\} = T_j^{(0)} \cup ({T_{i_1}^{(0)}}^C \cap \cdots \cap {T_{i_r}^{(0)}}^C)$.

If $r = d-k+1$, the pseudovertex $v := c(-e_J)$ is a generator of P. Each of its type entries contains the index, which is assigned to a basis $B \in \mathcal{B}$. Since B is the only basis with $i_1, \ldots, i_{d-k+1} \notin B$, its index is the only element of $T_j = T_j^{(0)} \setminus (T_{i_1}^{(0)} \cup \cdots \cup T_{i_{d-k+1}}^{(0)})$ for $j \in B$. For this reason, the generators as defined in (1) are exactly the tropical vertices of P.

Now we consider the other points of $\text{PV}(V)$, i.e. $r < d-k+1$. The intersection of two type entries $T_{j_1} \cap T_{j_2}$ is equal to

$$T_{j_1} \cap T_{j_2} \;=\; \begin{cases} (T_{j_1}^{(0)} \cap T_{j_2}^{(0)}) \setminus (T_{i_1}^{(0)} \cup \cdots \cup T_{i_r}^{(0)}) & \text{if } j_1, j_2 \in J\,, \\ (T_{j_1}^{(0)} \cap T_{j_2}^{(0)}) \cup ({T_{i_1}^{(0)}}^C \cap \cdots \cap {T_{i_r}^{(0)}}^C) & \text{otherwise}\,. \end{cases} \tag{3}$$

In the first case of 3, $T_{j_1} \cap T_{j_2}$ consists of at least one tropical vertex v_l with $v_{l,j_1} = v_{l,j_2} = 0$ and $v_{l,i} = 1$ for all $i \in J^C$. In the second case there are even more tropical vertices allowed and $T_{j_1} \cap T_{j_2} \neq \emptyset$. Hence, Proposition 17 of [5] tells us that the cell X_T has dimension 0, i.e. the given points really are pseudovertices of P. For $J = \bigcup_{i \in I} B_i$ and $J' = \bigcup_{i \in I'} B_i$ with $I \neq I' \subseteq [n]$ the tropical line segment between v_J and $v_{J'}$ is the concatenation of the two ordinary line segments $[v_J, v_{J \cup \tilde{J}}]$ and $[v_{J \cup \tilde{J}}, v_{J'}]$. The point $v_{J \cup \tilde{J}}$ is again a point of $\text{PV}(P)$. Therefore, there are no other pseudovertices as the given points in $\text{PV}(P)$.

Now we consider the tropical standard simplex Δ^d. If the tropical vertex $v_l := v_{B_l}$, $B_l \in \binom{[d+1]}{1}$, of Δ^d is given by the vector $v_{B_l} = -e_l$ $(l = 1, \ldots, d+1)$, then the type of the origin $\mathbf{0}$ with respect to Δ^d is $T^{\mathbf{0}} = (1, 2, \ldots, d+1)$. Therefore, this is an interior point of Δ^d. Let v_J with $J \in \bigcup_{j=1}^{d} \binom{[d+1]}{j}$ be any pseudovertex of Δ^d. Since for $i \in J$ and $i \notin J$, we have $v_{i,i} - v_{J,i} = 0 = \min\{v_{l,1}-1, \ldots, v_{l,r}-1, v_{l,r+1}, \ldots, v_{l,i}, \ldots, v_{l,d+1}\}$ and $v_{i,i} - v_{J,i} = -1 = \min\{v_{l,1}-1, \ldots, v_{l,i}-1, \ldots, v_{l,r}-1\}$, respectively, it follows that the index i is contained in the i-th entry of T for all $i = 1, \ldots, d+1$, i.e. $T^{\mathbf{0}} \subset T$. Hence, Δ^d is a polytrope. □

Let $v_J = \sum_{i \in J} -e_i = -e_J$ be a pseudovertex of P with $J = \bigcup_{i \in I} B_i$ for $I \subseteq [n]$. If the complement J^C of J is equal to $\{i_1, i_2, \ldots, i_r\}$ with $r \leq d-k+1$, we will denote v_J as $e_{i_1, i_2, \ldots, i_r}$

and its type with respect to P as

$$\text{type}_V(v_J) = T(v_J) = \big(T_1(v_J), \ldots, T_{d+1}(v_J)\big).$$

Because of the previous lemma, the i-th entry of $T(v_J)$ contains all bases using edge $i \in J$ that are possible after deleting the edges of J^C in the corresponding graph G or, equivalently, all bases that are possible after (re-)inserting edge $i \in J^C$ into $(V(G), E(G) \setminus \{J^C\})$.

We call a sequence of pseudovertices $e_\emptyset, e_{i_1}, e_{i_1,i_2}, \ldots, e_{i_1,i_2,\ldots,i_{d-k+1}}$, or rather the set $\{i_1, \ldots, i_{d-k+1}\} \subset [d+1]$, *valid* if the edge set $E \setminus \{i_1, \ldots, i_{d-k+1}\}$ contains a spanning tree of the underlying graph G. The first point $e_\emptyset = \mathbf{0}$ is assigned to the total edge set E of G. Then we delete edge after edge such that the graph is still connected until the edge set forms a connected graph without cycles. So the last point of a valid sequence is the tropical vertex v_B of P with $B = [d+1] \setminus \{i_1, i_2, \ldots, i_{d-k+1}\}$.

It turns out that the pseudovertices of the valid sequences and subsequences of them play a major role in the calculation of the maximal bounded und unbounded cells of P.

Lemma 9. *The maximal bounded cells of $P \in \mathcal{P}_{k,d}$ are of dimension $d-k+1$. They form the tropical convex hull of the pseudovertices of a valid sequence $\mathbf{0}, e_{i_1}, e_{i_1,i_2}, \ldots, e_{i_1,i_2,\ldots,i_{d-k+1}}$, where the last pseudovertex is a tropical vertex v_B according to the basis $B = [d+1] \setminus \{i_1, i_2, \ldots, i_{d-k+1}\} \in \mathcal{B}$ of $\mathcal{M}$.*

Let $T^{(0)} = (T_1^{(0)}, \ldots, T_{d+1}^{(0)})$ be the type of the pseudovertex $\mathbf{0}$ with respect to P. Then the type $T = (T_1, \ldots, T_{d+1})$ of the interior of the bounded cell $X_T = \text{tconv}(\mathbf{0}, e_{i_1}, e_{i_1,i_2}, \ldots, e_{i_1,i_2,\ldots,i_{d-k+1}})$ is given by $T_{i_1} = T_{i_1}^{(0)}, T_{i_2} = T_{i_2}^{(0)} \setminus T_{i_1}^{(0)}, \ldots, T_{i_{d-k+1}} = T_{i_{d-k+1}}^{(0)} \setminus (T_{i_1}^{(0)} \cup T_{i_2}^{(0)} \cup T_{i_{d-k}}^{(0)})$ and $T_j = T_j^{(0)} \setminus (T_{i_1}^{(0)} \cup T_{i_2}^{(0)} \cup T_{i_{d-k+1}}^{(0)})$ for all $j \in B$.

Proof. First, we will show that this sequence really defines a bounded cell of P, i.e. $T_j \neq \emptyset$ for all $j \in [d+1]$. So consider the type entry at some coordinate $i_j \in B^C$

$$\begin{aligned}
T_{i_j} &= T_{i_j}(\mathbf{0}) \cap T_{i_j}(e_{i_1}) \cap \ldots \cap \\
&\quad\; T_{i_j}(e_{i_1,\ldots,i_{j-1}}) \cap \\
&\quad\; T_{i_j}(e_{i_1,\ldots,i_j}) \cap \ldots \cap \\
&\quad\; T_{i_j}(e_{i_1,\ldots,i_{d-k+1}}) \\
&= \{l \mid i_j \in B_l\} \cap \{l \mid i_j \in B_l \text{ and } i_1 \notin B_l\} \cap \ldots \cap \\
&\quad\; \{l \mid i_j \in B_l \text{ and } (i_1, \ldots, i_{j-1} \notin B_l)\} \cap \\
&\quad\; \{l \mid i_j \in B_l \text{ or } (i_1, \ldots, i_j \notin B_l)\} \cap \ldots \cap \\
&\quad\; \{l \mid i_j \in B_l \text{ or } (i_1, \ldots, i_{d-k+1} \notin B_l)\} \\
&= \{l \mid i_j \in B_l \text{ and } (i_1, \ldots, i_{j-1} \notin B_l)\} \\
&= T_{i_j}^{(0)} \setminus (T_{i_1}^{(0)} \cup \ldots \cup T_{i_{j-1}}^{(0)}).
\end{aligned}$$

The cardinality of $T_{i_j} = T_{i_j}^{(0)} \cap T_{i_1}^{(0)\,C} \cap \ldots \cap T_{i_{j-1}}^{(0)\,C}$ is equal to the number of tropical vertices v of P with $v_{i_j} = 0$ and $v_{i_1} = \ldots = v_{i_{j-1}} = 1$ (in canonical coordinates) respectively to the number of bases B with $i_j \in B$ and $i_1, \ldots, i_{j-1} \notin B$, which is greater than 0 since we consider only valid sequences. So every type coordinate T_{i_j} contains at least one entry. In the case of uniform matroids we have the choice of $d+1-j$ *free* coordinates from which $k-1$ must be equal to 0, i.e. the cardinality of T_{i_j} is equal to $\binom{d+1-j}{k-1}$.

Analogously, the other type entries $T_j = T_j^{(0)} \setminus (T_{i_1}^{(0)} \cup T_{i_2}^{(0)} \cup T_{i_{d-k+1}}^{(0)}) = \{v_B\}$ for $j \in B$ and their cardinality $|T_j| = 1$ can be verified. Furthermore, we have $T_1 \cup \cdots \cup T_{d+1} = [n]$, because $T_1^{(0)} \cup \cdots \cup T_{d+1}^{(0)} = [n]$. Since no type entry of T is empty, the cell X_T is bounded. More precisely, $T_{i_1}, \ldots, T_{i_{d-k+1}}$ is a partition of the indices of $\text{Vert}(P) \setminus \{v_B\}$, and the other type coordinates each contain the index of the tropical vertex v_B; we call this a *pre-partition.* By Proposition 17 in [5], the dimension of X_T is $d-k+1$.

Removing one pseudovertex $e_{i_1,\ldots,i_r}$ with $r \in [d-k+1]$ from a valid sequence, we obtain $T_{i_{r+1}} = T^{(0)}_{i_{r+1}} \setminus (T^{(0)}_{i_1} \cup \cdots \cup T^{(0)}_{i_{r-1}})$ and $T_{i_r} \cap T_{i_{r+1}} \neq \emptyset$. This yields a bounded cell with lower dimension than $d-k+1$.

Adding a pseudovertex e_J to X_T, $J \neq B$ with $J^C = \{j_1,\ldots,j_r\}$ $(1 \leq r \leq d-k+1)$ and $(j_1,\ldots,j_l) \neq (i_1,\ldots,i_l)$ for all $l = 1,\ldots,r$, we consider $T' = T \cap \mathrm{type}_P(e_J)$. To keep the status of a maximal bounded cell, the type of the cell still has to be a pre-partition of $[n]$ without empty type entries. There are three different cases (1)-(3).

(1) For $J^C \not\subseteq B^C$ and $J \cap B \neq \emptyset$, there is an index $j \in J \cap B$. We consider the j-th type entry of T' that is equal to $T_j \cap T^{(0)}_j(e_J) = T^{(0)}_j \cap {T^{(0)}_{i_1}}^C \cap \cdots \cap {T^{(0)}_{i_{d-k+1}}}^C \cap {T^{(0)}_{j_1}}^C \cap \cdots \cap {T^{(0)}_{j_r}}^C$. This is an empty set since there are no tropical vertices of P with $d-k+1+r$ entries equal to one. The cells with empty type entries are not bounded.

(2) For $J^C \not\subseteq B^C$ and $J \cap B = \emptyset$, we consider an index $j \in J \cap B^C$ that corresponds to a valid sequence with $i_t = j$, $t \in \{1,\ldots,d-k+1\}$. The j-th type entry of T' is equal to $T^{(0)}_j(e_J) \cap T_j = T^{(0)}_j \cap {T^{(0)}_{j_1}}^C \cap \cdots \cap {T^{(0)}_{j_r}}^C \cap {T^{(0)}}^C_{i_1} \cap \cdots \cap {T^{(0)}}^C_{i_{t-1}}$. Since $J^C \not\subseteq \{i_1,\ldots,i_{t-1}\}$, the cardinality of T'_j is less than $|T^{(0)}_j|$, and we get no valid partition of $[n]$.

(3) For $J^C \subset B^C$ we have $r < d+1-k$ (otherwise $J = B$). We choose the smallest index j such that $i_j \in J \cap B^C$. That means $i_1,\ldots,i_{j-1} \in J^C \subset B^C$. Since we have $(i_1,\ldots,i_l) \neq (j_1,\ldots j_l)$ for all $l = 1,\ldots,r$, we know that $(i_1,\ldots,i_{j-1}) \neq (j_1,\ldots,j_r)$ leading to $|T_{i_j}| = |T^{(0)}_{i_j} \cap {T^{(0)}_{i_1}}^C \cap \cdots \cap {T^{(0)}_{i_{j-1}}}^C| > |T'_{i_j}| = |T^{(0)}_{i_j} \cap {T^{(0)}_{j_1}}^C \cap \cdots \cap {T^{(0)}_{j_r}}^C|$. As in the other two cases this is no valid pre-partition of $[n]$.

In every case the adding of a pseudovertex from another sequence leads to unfeasible types of bounded cells.

Similarly, it is not difficult to see that removing a pseudovertex and adding a new one from another sequence leads to unfeasible types or lower dimensional bounded cells, i.e. mixing of valid sequences is not possible. Altogether, we get the desired maximal bounded cells of P. $\square$

There are $n \cdot (d+1-k)!$ maximal bounded cells of P since we have $(d+1-k)!$ possibilities to add edges to a spanning tree until we get the whole graph.

Example 10. *The tropical matroid polytope P from Example 6 is contained in the 4-dimensional tropical hyperplane with apex $\mathbf{0}$. It is shown in Figure 2 as the abstract graph of the vertices and edges of its bounded subcomplex. Its maximal bounded cells are ordinary simplices of dimension $d-k+1 = 2$, whose pseudovertices are the tropical vertices $V = \{v_{B_1},\ldots,v_{B_8}\}$ (dark), the origin $\mathbf{0}$ (the centered point) and the five corners $c_i = e_i$ (light). The four tropical vertices with indices 3, 4, 5 and 8 correspond to the bases that are possible after deleting edge 1 in the underlying graph and therefore adjacent to the point e_1. One valid sequence i_1, i_2 leading to a maxim bounded cell is for example the (tropical/ordinary) convex hull of $e_\emptyset = (0,0,0,0,0)$, $e_4 = (0,0,0,0,1)$ and $e_{4,2} = v_{B_1} = (0,0,1,0,1)$, i.e. $i_1 = 4$ and $i_2 = 2$, with interior cell type $(1,1,36,1,24578)$, representing the basis $B_1 = \{1,2,4\}$.*

All cells in the tropical complex $\mathcal{C}_V$, bounded or not, are pointed, i.e. they do not contain an affine line. So each cell of $\mathcal{C}_V$ must contain a bounded cell as an ordinary face.

We now state the main theorem about the coarse types of maximal cells in the cell complex of a tropical matroid polytope. Let $b_{I,J}$ denote the number of bases $B \in \mathcal{B}$ with $I \subseteq B$ and $J \subseteq B^C$.

Theorem 11. *Let $\mathcal{C}$ be the tropical complex induced by the tropical vertices of a tropical matroid polytope $P \in \mathcal{P}_{k,d}$. The set of all coarse types of the maximal cells arising in $\mathcal{C}$ is given by those tuples $(t_1,\ldots,t_{d+1})$ with*

$$t_j \;=\; \begin{cases} b_{\{i_1\},\emptyset} + b_{\emptyset,\{i_1,i_2,\ldots,i_{d'+1}\}} & \text{if } j = i_1\,, \\ b_{\{i_l\},\{i_1,\ldots,i_{l-1}\}} & \text{if } j = i_l \in \{i_2,\ldots i_{d'+1}\}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4}$$

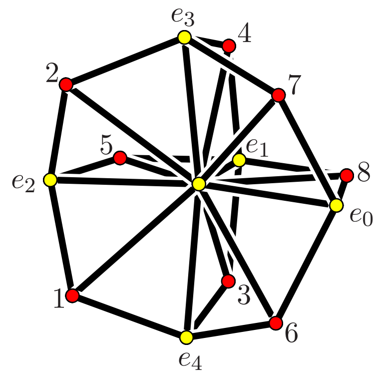


FIGURE 2. The abstract 1-skeleton of the bounded subcomplex of the tropical matroid polytope of Example 6.

where $e_{i_1}, \dots, e_{i_1,i_2,\dots,i_{d'}}$ form a subsequence of a valid sequence of P.

Proof. Depending on the maximal bounded (ordinary) face in the boundary, there are three types of maximal unbounded cells in $\mathcal{C}_V$.

The first one, X_T, contains a maximal bounded cell of dimension $d-k+1$, which is the tropical convex hull of the pseudovertices of a complete valid sequence $\mathbf{0}, e_{i_1}, e_{i_1,i_2}, \dots, e_{i_1,i_2,\dots,i_{d-k+1}}$ where $B^C = \{i_1, \dots, i_{d-k+1}\}$ is the complement of a basis of $\mathcal{M}$. To get full-dimensional we have the choice between $k-1$ of k free directions $-e_i$, $i \in B$. So let $-e_{j_1}^\infty, \dots, -e_{j_{k-1}}^\infty$ be the *extreme rays* of X_T, and $(T_1^{(0)}, \dots, T_{d+1}^{(0)})$ be the type of the pseudovertex $\mathbf{0}$ with respect to P. Then the type $T = (T_1, \dots, T_{d+1})$ of the interior of this unbounded cell X_T is given by the intersection of the types of its vertices and therefore $T_{i_1} = T_{i_1}^{(0)}, T_{i_2} = T_{i_2}^{(0)} \setminus T_{i_1}^{(0)}, \dots, T_{i_{d-k+1}} = T_{i_{d-k+1}}^{(0)} \setminus (T_{i_1}^{(0)} \cup T_{i_2}^{(0)} \cup T_{i_{d-k}}^{(0)})$, $T_i = T_i^{(0)} \setminus (T_{i_1}^{(0)} \cup T_{i_2}^{(0)} \cup T_{i_{d-k+1}}^{(0)})$ for $i \notin B^C \cup \{j_1, \dots, j_{k-1}\}$ and $T_{j_1} = \dots = T_{j_{k-1}} = \emptyset$. Choosing $d' = d-k+1$ and $i_{d'+1} = i$, we get the coarse type entries of equation (4).

The second type, X_T, of maximal unbounded cells contains a bounded cell of lower dimension $d' \in \{0, \dots, d-k\}$, which is the tropical convex hull of the pseudovertices of some subsequence $e_{i_1}, e_{i_1,i_2}, \dots, e_{i_1,i_2,\dots,i_{d'+1}}$. To get full-dimensional we still need the extreme rays $e_{i_1,i_2,\dots,i_{d'+1}} - e_l^\infty$ for all directions $l \notin \{i_1, \dots, i_{d'+1}\}$. Then the type $T = (T_1, \dots, T_{d+1})$ of the interior of this unbounded cell X_T is given by $T_{i_1} = T_{i_1}^{(0)} \cup ({T_{i_1}^{(0)}}^C \cap \dots \cap {T_{i_{d'+1}}^{(0)}}^C), T_{i_2} = T_{i_2}^{(0)} \setminus T_{i_1}^{(0)}, \dots, T_{i_{d'+1}} = T_{i_{d'+1}}^{(0)} \setminus (T_{i_1}^{(0)} \cup T_{i_2}^{(0)} \cup T_{i_{d'}}^{(0)})$, $T_j = \emptyset$ for $j \notin \{i_1, \dots, i_{d'+1}\}$ with the coarse type as given in equation (4).

The third and last type of maximal unbounded cells contains a bounded cell of dimension $d-k$ and is assigned to the non-bases of $\mathcal{M}$, i.e. to the subsets of E with cardinality k that are not bases. Let $i_1, \dots, i_{d-k+1}$ be the complement of a non-basis N and $i_1, \dots, i_{d-k}$ a valid subsequence. Then there is an unbounded cell X_T that is the tropical convex hull of the pseudovertices $\mathbf{0}, e_{i_1}, \dots, e_{i_{d-k}}$ and the extreme rays $\mathbf{0} - e_l^\infty$ for all directions $l \notin \{i_1, \dots, i_{d-k+1}\}$ and with type entries $T_{i_1} = T_{i_1}^{(0)}, T_{i_2} = T_{i_2}^{(0)} \setminus T_{i_1}^{(0)}, \dots, T_{i_{d-k+1}} = T_{i_{d-k+1}}^{(0)} \setminus (T_{i_1}^{(0)} \cup T_{i_2}^{(0)} \cup T_{i_{d-k}}^{(0)})$, $T_j = \emptyset$ for $j \notin \{i_1, \dots, i_{d-k+1}\}$. Choosing $d' = d-k$ and observing that $b_{\emptyset,\{i_1,i_2,\dots,i_{d'+1}\}} = 0$ for the non-basis $\{i_1, i_2, \dots, i_{d'+1}\}^C$ we get the desired result. □

Restricting ourselves to the uniform case, we get the following result.

Corollary 12. *The coarse types of the maximal cells in the tropical complex induced by the tropical vertices of the tropical hypersimplex Δ_k^d in $\mathbb{T}^d$ with $2 \le k < d+1$ are up to symmetry of*

$\mathrm{Sym}(d+1)$ *given by*

$$\left(\binom{d+1-\alpha}{k}+\binom{d}{k-1},\binom{d-1}{k-1},\ldots,\binom{d-(\alpha-1)}{k-1},\underbrace{0,\ldots,0}_{d+1-\alpha}\right)$$

where $0 \le \alpha \le d+2-k$ *correlates to the maximal dimension of a bounded cell of its boundary.*

Now we relate the combinatorial properties of the tropical complex $\mathcal{C}$ of a tropical matroid polytope to algebraic properties of a monomial ideal which is assigned to $\mathcal{C}$. As a direct consequence of Theorem 3 and Corollary 3.5 in [7], we can state the generators of the coarse type ideal

$$I = \langle x^{\mathbf{t}(p)} \colon p \in \mathbb{T}^d \rangle \subset \mathbb{K}[x_1, \ldots, x_{d+1}],$$

where $\mathbf{t}(p)$ is the coarse type of p and $x^{\mathbf{t}(p)} = {x_1}^{\mathbf{t}(p)_1}{x_2}^{\mathbf{t}(p)_2}\cdots {x_{d+1}}^{\mathbf{t}(p)_{d+1}}$.

Corollary 13. *The coarse type ideal* I *is equal to*

$$\langle x_{i_1}^{t_{i_1}} x_{i_2}^{t_{i_2}} \cdots x_{i_{d'+1}}^{t_{i_{d'+1}}} \colon [d+1] \setminus \{i_1, \ldots, i_{d'}\} \text{ contains a basis } \rangle$$

with $(t_{i_1}, t_{i_2}, \ldots, t_{i_{d'+1}}) = \big(b_{\{i_1\},\emptyset} + b_{\emptyset,\{i_1,i_2,\ldots,i_{d'+1}\}}, b_{\{i_2\},\{i_1\}}, \ldots, b_{\{i_{d'+1}\},\{i_1,\ldots,i_{d'}\}}\big)$.

Example 14. *The tropical complex* $\mathcal{C}$ *of the tropical matroid polytope of Example 6 has 73 maximal cells. There are five maximal cells for the case* $d' = 0$ *with* $t_{i_{d'+1}} = 8$ *and* $t_j = 0$ *for* $j \ne i_{d'+1}$, *and 48 for the case* $d' = 2$ *according to the 8 bases. Finally, there are 20 maximal cells for the case* $d' = 1$, *where* $[d+1] \setminus \{i_1\}$ *contains a basis, but* $[d+1] \setminus \{i_1, i_2\}$ *does not necessarily contain a basis.*

The coarse type ideal of $\mathcal{C}$ *is given by*

$$\begin{aligned} I = \langle\; & {x_1}^1{x_2}^2{x_3}^5, {x_1}^1{x_2}^5{x_3}^2, {x_1}^2{x_2}^1{x_3}^5, {x_1}^4{x_2}^1{x_3}^3, {x_1}^4{x_2}^3{x_3}^1, {x_1}^2{x_2}^5{x_3}^1, {x_2}^2{x_3}^6, {x_2}^6{x_3}^2,\\ & {x_2}^2{x_3}^5{x_4}^1, {x_2}^5{x_3}^2{x_4}^1, {x_1}^2{x_3}^6, {x_1}^5{x_3}^3, {x_1}^2{x_3}^5{x_4}^1, {x_1}^4{x_3}^3{x_4}^1, {x_3}^8, {x_3}^5{x_4}^3, {x_1}^8, {x_1}^5{x_2}^3,\\ & {x_1}^5{x_4}^3, {x_1}^4{x_3}^1{x_4}^3, {x_1}^4{x_2}^3{x_4}^1, {x_1}^4{x_2}^1{x_4}^3, {x_0}^2{x_4}^6, {x_4}^8, {x_0}^2{x_1}^1{x_4}^5, {x_0}^1{x_1}^2{x_4}^5, {x_1}^2{x_4}^6,\\ & {x_0}^1{x_2}^2{x_4}^5, {x_2}^2{x_4}^6, {x_0}^2{x_2}^1{x_4}^5, {x_1}^1{x_2}^2{x_4}^5, {x_1}^2{x_2}^1{x_4}^5, {x_0}^2{x_3}^1{x_4}^5, {x_3}^3{x_4}^5, {x_2}^2{x_3}^1{x_4}^5,\\ & {x_1}^2{x_3}^1{x_4}^5, {x_0}^1{x_2}^5{x_4}^2, {x_2}^6{x_4}^2, {x_2}^5{x_3}^1{x_4}^2, {x_1}^1{x_2}^5{x_4}^2, {x_0}^2{x_3}^6, {x_0}^1{x_2}^2{x_3}^5, {x_0}^2{x_2}^1{x_3}^5,\\ & {x_0}^2{x_1}^1{x_3}^5, {x_0}^1{x_1}^2{x_3}^5, {x_0}^2{x_3}^5{x_4}^1, {x_0}^1{x_2}^5{x_3}^2, {x_0}^3{x_2}^5, {x_2}^8, {x_0}^1{x_1}^2{x_2}^5, {x_1}^2{x_2}^6, {x_1}^2{x_2}^5{x_4}^1,\\ & {x_0}^1{x_1}^4{x_2}^3, {x_0}^6{x_4}^2, {x_0}^5{x_1}^1{x_4}^2, {x_0}^5{x_1}^2{x_4}^1, {x_0}^3{x_1}^4{x_4}^1, {x_0}^1{x_1}^4{x_4}^3, {x_0}^5{x_2}^1{x_4}^2, {x_0}^5{x_2}^3,\\ & {x_0}^5{x_1}^2{x_2}^1, {x_0}^3{x_1}^4{x_2}^1, {x_0}^5{x_3}^1{x_4}^2, {x_0}^5{x_2}^1{x_3}^2, {x_0}^5{x_3}^2{x_4}^1, {x_0}^6{x_3}^2, {x_0}^5{x_1}^1{x_3}^2, {x_0}^6{x_1}^2,\\ & {x_0}^3{x_1}^5, {x_0}^5{x_1}^2{x_3}^1, {x_0}^3{x_1}^4{x_3}^1, {x_0}^8, {x_0}^1{x_1}^4{x_3}^3 \,\rangle \subseteq R := \mathbb{R}[x_0, x_1, x_2, x_3, x_4] \end{aligned}$$

We obtain its minimal free resolution, which is induced by $\mathcal{C}$

$$\mathcal{F}_\bullet^{\mathcal{C}} \colon\ 0 \to R^{14} \to R^{78} \to R^{172} \to R^{180} \to R^{73} \to I \to 0,$$

where the exponents i *of the free graded* R*-modules* R^i *correspond to the entries of the* f*-vector* $f(\mathcal{C}) = (1, 14, 78, 172, 180, 73)$ *of* $\mathcal{C}$.

In (ordinary) convexity swapping between interior and exterior description of a polytope is a famous problem known as the *convex hull problem.* For a uniform matroid it is possible to indicate the minimal tropical halfspaces of its tropical matroid polytope.

Theorem 15. *The tropical hypersimplex* Δ_k^d *in* $\mathbb{T}^d$ *is the intersection of its cornered halfspaces and the tropical halfspaces* $H(\mathbf{0}, I)$, *where* I *is a* $(d-k+2)$*-element subset of* $[d+1]$.

Proof. For $k = 1$ the tropical standard simplex is a polytrope and coincides with its cornered hull. For $k \ge 2$ we want to verify the three conditions of Gaubert and Katz in Proposition 1 of [9].

Let $T = (T_1, \ldots, T_{d+1})$ be the type of the apex $\mathbf{0}$ of $H(\mathbf{0}, I)$. If a vertex $v \in \mathrm{Vert}(\Delta_k^d)$ appears in some type entry T_i, then the i-th (canonical) coordinate of v is equal to zero. Hence, exactly k entries of T contain the index of v. Since the cardinality of $I^C = [d+1] \setminus I$ is only $k-1$, every tropical vertex of Δ_k^d is contained in some sector $\overline{S_i}$ with $i \in I$, i.e. $\Delta_k^d \subseteq H(\mathbf{0}, I)$.

Consider the complement I^C of I. For all $i \in I^C$ there is a tropical vertex v with $v_i = 0$, i.e. $v \in T_i$. Since the cardinality of I^C is equal to $k-1$ and v has k entries equal to zero, there must be an index $j \in I$ such that $v_j = 0$. We can conclude that $T_i \cap T_j \neq \emptyset$.

The intersection $T_i \cap T_j$ is not empty for arbitrary $i, j \in [d+1]$, because its cardinality is equal to the number of tropical vertices v with $v_i = v_j = 0$, which is $\binom{d}{k-1}$ with $k > 1$. For $i \in I$ and $j \in I^C$, the set $T_i \cap T_j$ consists of all tropical vertices v with $v_i = 0$ and $v_j = 1$ (in canonical coordinates). On the other hand, the set $\bigcup_{k \in I \setminus \{i\}} T_k$ contains all tropical vertices v with $v_i = 1$. So we get $T_i \cap T_j \not\subset \bigcup_{k \in I \setminus \{i\}} T_k$.

Hence, we obtain that $H(\mathbf{0}, I)$ is a minimal tropical halfspace, and Δ_k^d is contained in the intersection of its cornered hull $\bigcap_{i \in [d+1]} H(e_i, \{i\})$ with $\bigcap_{I \in \binom{[d+1]}{d-k+2}} H(\mathbf{0}, I)$.

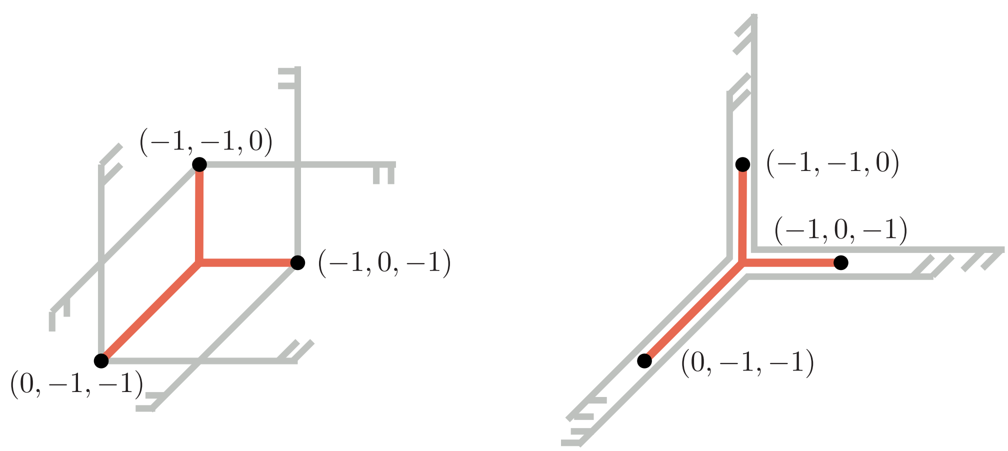
We still have to prove that the intersection of the given minimal tropical halfspaces is contained in Δ_k^d. Let us assume that there is a point $x \in \mathbb{T}^d \setminus \Delta_k^d$ with $\operatorname{type}_{\Delta_k^d}(x)_i = \emptyset$. Then for any tropical halfspace $H(\mathbf{0}, I)$, $I \in \binom{[d+1]}{d-k+2}$, with $i \in I^C$ we obtain $x \notin H(\mathbf{0}, I)$.

Consequently, the tropical hypersimplex Δ_k^d is the set of all points $x \in \mathbb{T}^d$ satisfying

$$\begin{aligned} \bigoplus_{i \in I} x_i &\leq \bigoplus_{j \in I^C} x_j \text{ for all } I \subseteq [d+1] \text{ with } |I| = d-k+2 \\ \text{and } (-1) \odot x_i &\leq \bigoplus_{j \neq i} x_j \text{ for all } i \in [d+1]. \end{aligned}$$

□

Example 16. *The second tropical hypersimplex Δ_2^3 in $\mathbb{T}^3$ is the intersection of the 4 cornered halfspaces $(c_i, \{i\})$ for $i = 1, \ldots, 4$ and the tropical halfspaces $(\mathbf{0}, \{1,2,3\})$, $(\mathbf{0}, \{1,2,4\})$, $(\mathbf{0}, \{1,3,4\})$ and $(\mathbf{0}, \{2,3,4\})$ with apex $\mathbf{0} \in \mathbb{T}^d$. The second tropical hypersimplex Δ_2^2 in $\mathbb{T}^2$ is the intersection of the three cornered halfspaces $(c_i, \{i\})$ for $i = 1, \ldots, 3$ and the tropical halfspaces $(\mathbf{0}, \{1,2\})$, $(\mathbf{0}, \{1,3\})$ and $(\mathbf{0}, \{2,3\})$ with apex $\mathbf{0} \in \mathbb{T}^2$, see Figure 3.*



(a) (b)

Figure 3. The tropical hypersimplex Δ_2^2 (dark) is given as the intersection of its cornered halfspaces (light coloured in Figure 3(a)) and the minimal tropical halfspaces $(\mathbf{0}, \{1,2\})$, $(\mathbf{0}, \{1,3\})$ (light coloured in Figure 3(b)).

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