Conformal Killing vector fields and Rellich type identities on Riemannian manifolds, II

Yuri Bozhkov and Enzo Mitidieri

Abstract. We propose a general Noetherian approach to Rellich integral identities. Using this method we obtain a higher order Rellich type identity involving the polyharmonic operator on Riemannian manifolds admitting homothetic transformations. Then we prove a biharmonic Rellich identity in a more general context. We also establish a nonexistence result for semilinear systems involving biharmonic operators.

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1. Introduction

It is well-known that integral identities play an important role in the theory of functions and differential equations. Basically there are two kinds of such identities which are most frequently used, namely, Pohozaev type identities [16, 17] and Rellich type identities [19]. An important observation to be made is that the Pohozaev identities are satisfied by solutions of Dirichlet boundary problems while the Rellich identities concern functions which belong to certain functions spaces without any reference to other relations which they may satisfy like differential equations or boundary conditions. For this reason, as it has been pointed out in [6], the Rellich's Identity is an 'important tool for obtaining, among other things, a priori bounds of solutions for semilinear Hamiltonian elliptic systems [7], nonexistence results [10, 11] and sharp Hardy type inequalities [13].

The main purpose of this paper is to propose an unified approach to both Rellich and Pohozaev type identities. We have initiated this research with the paper [4] in which we devised and developed a Noetherian approach to Pohozaev's identities whose essential point is that the latter can be obtained from the Noether's identity [8, 9] after integration and application

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of the divergence theorem taking into account the corresponding equations (or systems) and boundary conditions. In this procedure one chooses critical values of the involved parameters. (See [5] on the last point.) In a subsequent work [6] we have applied this method to some semilinear partial differential equations and systems on Riemannian manifolds. For this purpose we have employed conformal Killing vector fields, which are related to Noether symmetries of critical differential equations (see [3]). In fact, this generalizes the original idea of Pohozaev [16] who has made use of the radial vector filed $X = r\frac{\partial}{\partial r}$ on $\mathbb{R}^n$. We have also obtained in [6] a Rellich type identity on manifolds following the argument in [10, 11].

The corner stone of this work is the observation that the Rellich type identities for functions on Riemannian manifolds can be generated by integration of the Noether's Identity for appropriate differential functions (see below for the latter notions). E. g., the Rellich's Identity for a function [19] or for a pair of functions [10, 11] can be obtained in this way using a well-known Lagrangian and the radial vector field determining a dilation in $\mathbb{R}^n$. That is, both the Pohozaev's and the Rellich's identities come from the Noether's Identity. We claim that there is a certain kind of interplay between the integral identities of Pohozaev-Rellich type, Hardy-Sobolev Inequalities, Liouville type theorems, existence of conformal Killing vector fields and divergence symmetries of nonlinear Poisson equations on Riemannian manifolds. These connections will be studied in more details elsewhere.

The present paper is a natural continuation of [6]. Nevertheless, it can be read independently of [6].

To begin with, let M be an oriented Riemannian manifold of dimension $n \geq 3$ endowed with a metric $g = (g_{ij})$. We assume that M has a boundary ∂M of class C^∞. The local coordinates of M will be denoted by $x = (x^1, ..., x^n)$. We denote by dV and dS the volume and surface measures with respect to the metric g, and by ν - the outward unit vector normal to ∂M.

Now we introduce further notation and state the Noether's Identity which is the main ingredient of the proposed method. For more details see [8, 9, 15].

We consider a collection of smooth functions $u^\alpha(x)$, $\alpha = 1, 2, ..., m$, defined on the manifold M. For an integer number $k \geq 1$ we let $u_{(k)}$ denote the set of all partial derivatives of $u^\alpha(x)$, up to order k. That is,

$$u_{(k)} = \left\{ u^\alpha_{i_1 \dots i_s} \mid \alpha = 1, 2, ..., m,\ s = 1, ..., k,\ i_1, ..., i_s = 1, ..., n \right\},$$

where

$$u^\alpha_{i_1 \dots i_s} = \frac{\partial^s u^\alpha}{\partial x^{i_1} ... \partial x^{i_s}}$$

and $u = u_{(0)} = (u^1, ..., u^m)$.

Following Olver [15] we introduce the notion of differential function.

Definition 1.1. A smooth function of x, u and derivatives of u up to some finite, but unspecified order, is called differential function.

The vector space of all differential functions of all orders is denoted by $\mathcal{A}$.

We recall that the total derivative operator

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + ... + u_{ii_1i_2...i_l}^\alpha \frac{\partial}{\partial u_{i_1i_2...i_l}^\alpha} + ...$$

Above and throughout this paper we shall suppose summation from 1 to n over repeated Latin indices and from 1 to m over repeated Greek indices.

Given n differential functions $\xi^i = \xi^i(x, u, u_{(1)}, ...) \in \mathcal{A}$ and m differential functions $\eta^\alpha = \eta^\alpha(x, u, u_{(1)}, ...) \in \mathcal{A}$, let

$$\begin{aligned} \eta_i^{(1)\alpha} &= D_i\eta^\alpha - (D_i\xi^j)u_j^\alpha, \quad i = 1, 2, ..., n; \\ \eta_{i_1i_2...i_l}^{(l)\alpha} &= D_{i_l}\eta_{i_1i_2...i_{l-1}}^{(l-1)\alpha} - (D_{i_l}\xi^j)u_{i_1i_2...i_{l-1}j}^\alpha \\ &= D_{i_1}D_{i_2}...D_{i_l}Q^\alpha + \xi^i u_{i_1i_2...i_ki}^\alpha, \end{aligned}$$

where $i_l = 1, 2, ..., n$ for $l = 2, 3, ..., k$, $k = 2, 3, ...$ and $Q^\alpha = \eta^\alpha - \xi^i u_i^\alpha$ are the Lie characteristic functions.

Further one associates to ξ^i and η^α the following partial differential operators acting on $\mathcal{A}$:

- The operator:

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} \eta_{i_1i_2...i_s}^{(s)\alpha} \frac{\partial}{\partial u_{i_1i_2...i_s}^\alpha};$$

- The Euler operator $E = (E_1, ..., E_m)$, where E_α is defined by

$$E_\alpha = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1}D_{i_2}...D_{i_s} \frac{\partial}{\partial u_{i_1i_2...i_s}^\alpha};$$

- The Noether operator $N = (N^1, ..., N^n)$, where:

$$\begin{aligned} N^i = \xi^i + Q^\alpha \left[\frac{\partial}{\partial u_i^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1}D_{i_2}...D_{i_s} \frac{\partial}{\partial u_{i_1i_2...i_si}^\alpha} \right] \\ + \sum_{r=1}^{\infty} D_{j_1}D_{j_2}...D_{j_r}Q^\alpha \left[\frac{\partial}{\partial u_{j_1j_2...j_ri}^\alpha} \right. \\ \left. + \sum_{s=1}^{\infty} (-1)^s D_{l_1}D_{l_2}...D_{l_s} \frac{\partial}{\partial u_{l_1l_2...l_sj_1j_2...j_ri}^\alpha} \right]. \end{aligned}$$

These operators are related by the Noether's Identity [8, 9]:

$$X + D_i\xi^i = Q^\alpha E_\alpha + D_i N^i. \tag{1.1}$$

The identity (1.1) was explicitly stated for the first time in the work [8] by Ibragimov who named it in honour of E. Noether. As Ibragimov has pointed out it is clear that this identity makes the proof of the Noether Theorem [14, 15] purely algebraic and very simple.

Now we outline the proposed Noetherian approach to Rellich's Identity. Let u be a vector-valued smooth function and $P = (P_1, ..., P_m)$ - a linear or nonlinear partial differential operator of an arbitrary even order $2k$. Let $L \in \mathcal{A}$ be a differential function of order k such that

$$Pu = E(L),$$

where E is the Euler operator (see above). The function L will play the role of a 'Lagrangian'. Then one writes the Noether's Identity corresponding to L and a suitable operator X

$$XL + L\, D_i\xi^i = (\eta^\alpha - \xi^i u_i^\alpha)P_\alpha u + D_i(N^i L),$$

integrates and applies the divergence theorem:

$$\int_M [XL + (D_i\xi^i)L]dV = \int_M (\eta^\alpha - \xi^i u_i^\alpha)P_\alpha u\, dV + \int_{\partial M} (N^i L)\nu_i dS. \quad (1.2)$$

The identity (1.2) is the most general form of the Rellich's Identity. We emphasize that in (1.2) one has the freedom to chose the operator X (that is, the differential functions ξ^i and η^α) as well as the Lagrangian L depending on the specific research objective. In this way various Rellich type identities can be obtained, one of which is established in the following

Theorem 1.2. *Let $u, v \in C^2(\bar{M})$ be two given functions, $h = h^i(x)\frac{\partial}{\partial x^i}$ a $C^1(\bar{M})$ vector field, and $F = F(x, u_{(1)}, v_{(1)}) \in \mathcal{A}$. Then the following identity holds:*

$$\begin{aligned}\int_M \{divF_p\ (h, F_q)dV \quad &+ \quad divF_q\ (h, F_p)\}dV = -\int_M \mathcal{L}_h g_{ik}\ F_{u_i}F_{v_j}dV \\ &- \quad \int_M h_j\{F_{u_i}\nabla_i F_{v_j} + F_{v_i}\nabla_i F_{u_j}\}dV \qquad (1.3) \\ &+ \quad \int_{\partial M} \{(F_p, \nu)(h, F_q) + (F_p, \nu)(h, F_q)\}dS,\end{aligned}$$

where

$$F_p = \left(\frac{\partial F}{\partial u_1}(x, u_1, ...u_n), ..., \frac{\partial F}{\partial u_n}(x, u_1, ...u_n)\right),$$
$$F_q = \left(\frac{\partial F}{\partial v_1}(x, v_1, ...v_n), ..., \frac{\partial F}{\partial v_n}(x, u_1, ...u_n)\right),$$

∇^i is the covariant derivative corresponding to the Levi-Civita connection, uniquely determined by g, and $\mathcal{L}_h g_{ij}$ is the Lie derivative of the metric g_{ij} with respect to the vector field h.

In the Euclidean case, the identity (1.3) coincides with the main identity in [11], up to an integration by parts.

One more time we observe that the smooth vector field h can be chosen in an arbitrary way which provides a wide variety of possibilities for applications, even in the Euclidean case!

However, we shall not proceed further in this generality. Our next purpose is to make use of conformal Killing vector fields in order to obtain

Riemannian analogs of some Rellich type integral identities established in [11]. We recall that a nonisometric conformal Killing vector field on M is a vector field $h = h^i \frac{\partial}{\partial x^i}$ which satisfies

$$\nabla^i h^j + \nabla^j h^i = \frac{2}{n}(div(h))\, g^{ij} = \mu\, g^{ij},\ \mu \neq 0. \tag{1.4}$$

Here $div(h) = \nabla_i h^i$ is the covariant divergence operator and g^{ij} is the inverse matrix of g_{ij}.

We would like to point out that, in fact, the conformal Killing vector field generalizes the dilations in $\mathbb{R}^n$, determined by $x^i \frac{\partial}{\partial x^i}$, and enables us to follow in the general manifold case some of the arguments successfully used in the Euclidean case. To show this, for example, we obtain a higher order Rellich type identity involving the polyharmonic operator on Riemannian manifolds admitting proper homothetic transformations, that is, transformations whose infinitesimal generators h satisfy (1.4) with $\mu = const \neq 0$. It is easy to see that in this case one can assume $div(h) = n$ without loss of generality. (Note that $h = x^i \frac{\partial}{\partial x^i} = r\frac{\partial}{\partial r}$, corresponding to a dilational transformation in $\mathbb{R}^n$, satisfies the last relation.)

Theorem 1.3. *Let $u, v \in C^{4m}(\bar{M})$ be two given functions and let $h = h^i(x)\frac{\partial}{\partial x^i}$ be a $C^1(\bar{M})$ contravariant conformal Killing vector field such that $div(h) = n$.*

Then the following identity holds:

$$\begin{aligned}
R_{2m}(u,v) &= (4m-n)\int_M \Delta_g^{2m} u\, v\, dV + \int_{\partial M} \Delta_g^m u \Delta_g^m v (h,\nu)\, dS \\
&+ (4m-n)\sum_{l=0}^{m-1}\int_{\partial M} \Delta_g^{m+l} u\, (\nabla(\Delta_g^{m-l-1} v),\nu)\, dS \\
&- (4m-n)\sum_{l=0}^{m-1}\int_{\partial M} \Delta_g^{m-l-1} v\, (\nabla(\Delta_g^{m+l} u),\nu)\, dS \\
&+ \sum_{l=0}^{m-1}\int_{\partial M} (2l\, \Delta_g^l v + h^k \nabla_k(\Delta_g^l v))\, (\nabla(\Delta_g^{2m-1-l} u),\nu)\, dS \\
&- \sum_{l=0}^{m-1}\int_{\partial M} (2l\, (\nabla(\Delta_g^l v),\nu) + \nabla^i h^k \nabla_k(\Delta_g^l v)\nu_i \\
&\qquad + h^k \nabla^i \nabla_k(\Delta_g^l v)\nu_i)\, (\Delta_g^{2m-1-l} u)\, dS \\
&+ \sum_{l=0}^{m-1}\int_{\partial M} (2l\, \Delta_g^l u + h^k \nabla_k(\Delta_g^l u))\, (\nabla(\Delta_g^{2m-1-l} v),\nu)\, dS \\
&- \sum_{l=0}^{m-1}\int_{\partial M} (2l\, (\nabla(\Delta_g^l u),\nu) + \nabla^i h^k \nabla_k(\Delta_g^l u)\nu_i \\
&\qquad + h^k \nabla^i \nabla_k(\Delta_g^l u)\nu_i)\, (\Delta_g^{2m-1-l} v)\, dS.
\end{aligned} \tag{1.5}$$

where

$$R_{2m}(u,v) = \int_M \{\Delta_g^{2m} u\ (h, \nabla v) + \Delta_g^{2m} v\ (h, \nabla u)\} dV$$

and Δ_g is the Laplace-Beltrami operator corresponding to the metric g.

A similar identity involving odd powers of Δ_g is also valid. We omit it in order not to increase the volume of this paper.

Let $u = v$. From (1.5) we immediately obtain

Corollary 1.4. *Let $u \in C^{4m}(\bar{M})$ be a given function and let $h = h^i(x)\frac{\partial}{\partial x^i}$ be a $C^1(\bar{M})$ contravariant conformal Killing vector field such that $div(h) = n$.*

Then the following identity holds:

$$\int_M \Delta_g^{2m} u(h, \nabla u)\ dV = \frac{4m-n}{2} \int_M \Delta_g^{2m} u\ u\ dV + \frac{1}{2} \int_{\partial M} (\Delta_g^m u)^2 (h, \nu)\ dS$$

$$\begin{aligned}
&+ \quad \frac{4m-n}{2} \sum_{l=0}^{m-1} \int_{\partial M} \Delta_g^{m+l} u\ (\nabla(\Delta_g^{m-l-1} u), \nu)\ dS \\
&- \quad \frac{4m-n}{2} \sum_{l=0}^{m-1} \int_{\partial M} \Delta_g^{m-l-1} u\ (\nabla(\Delta_g^{m+l} u), \nu)\ dS \\
&+ \quad \sum_{l=0}^{m-1} \int_{\partial M} (2l\ \Delta_g^l u + h^k \nabla_k (\Delta_g^l u))\ (\nabla(\Delta_g^{2m-1-l} u), \nu)\ dS \\
&- \quad \sum_{l=0}^{m-1} \int_{\partial M} (2l\ (\nabla(\Delta_g^l u), \nu) + \nabla^i h^k \nabla_k (\Delta_g^l u) \nu_i \\
&\qquad + h^k \nabla^i \nabla_k (\Delta_g^l u) \nu_i)\ (\Delta_g^{2m-1-l} u)\ dS.
\end{aligned} \tag{1.6}$$

The identity (1.6) in the Euclidean case with $h = x^i \frac{\partial}{\partial x^i}$ is similar to an identity obtained in [11] and used in [12] to establish a nonexistence result for positive radial solutions of semilinear polyharmonic equations in $\mathbb{R}^n$.

Further we prove a biharmonic Rellich identity in a more general context.

Theorem 1.5. *Let $u, v \in C^4(\bar{M})$ be two given functions and let $h = h^i(x)\frac{\partial}{\partial x^i}$ be a $C^1(\bar{M})$ conformal Killing vector field satisfying (1.4). Then the following identity holds:*

$$\int_M \{\Delta_g^2 u\ (h, \nabla v) dV + \Delta_g^2 v\ (h, \nabla u)\} dV = \frac{4-n}{2} \int_M \mu\ \Delta_g u \Delta_g v dV$$

$$
\begin{aligned}
&+ \quad \frac{1}{n-1}\int_M (\mathcal{L}_h R + \mu R)(u\Delta_g v + v\Delta_g u)dV \\
&- \quad \int_M (u(\nabla\mu, \nabla(\Delta_g v)) + v(\nabla\mu, \nabla(\Delta_g u)))dV \\
&+ \quad \int_{\partial M} (\Delta_g u \Delta_g v(h,\nu) + (u\Delta_g v + v\Delta_g u)(\nabla\mu, \nu) \\
&+ \quad (h, \nabla v)\nabla^i(\Delta_g u)\nu_i - \Delta_g u(\nabla^i h^k . v_k \nu_i + h^k \nabla_k \nabla^i v . \nu_i) \\
&+ \quad (h, \nabla u)\nabla^i(\Delta_g v)\nu_i - \Delta_g v(\nabla^i h^k . u_k \nu_i + h^k \nabla_k \nabla^i u . \nu_i))dS,
\end{aligned} \tag{1.7}
$$

where R is the scalar curvature of M and $\mathcal{L}_h R$ is its Lie derivative with respect to the vector field h.

Setting $u = v$ in (1.7) we immediately obtain

Corollary 1.6. *Let $u \in C^4(\bar{M})$ be a given function and let $h = h^i(x)\frac{\partial}{\partial x^i}$ be a $C^1(\bar{M})$ conformal Killing vector field satisfying (1.4). Then the following identity holds:*

$$
\begin{aligned}
\int_M \Delta_g^2 u \, (h, \nabla u)dV &= \frac{4-n}{4}\int_M \mu \, (\Delta_g u)^2 dV - \int_M u(\nabla\mu, \nabla(\Delta_g u))dV \\
&+ \quad \frac{1}{n-1}\int_M (\mathcal{L}_h R + \mu R)u\Delta_g u \, dV \\
&+ \quad \int_{\partial M} (\frac{1}{2}(\Delta_g u)^2(h,\nu) + u\Delta_g u(\nabla\mu, \nu) \\
&+ \quad (h, \nabla u)\nabla^i(\Delta_g u)\nu_i - \Delta_g u(\nabla^i h^k . u_k \nu_i + h^k \nabla_k \nabla^i u . \nu_i))dS.
\end{aligned} \tag{1.8}
$$

This paper is organized as follows. In section 2 we prove Theorem 1.2. Then, aiming to show how the proposed approach works, we re-establish in section 3 two known integral identities obtained in [6] and [11]. The proofs of theorems 1.3 and 1.5 are presented in sections 4 and 5 respectively. In section 6 we establish a nonexistence result for systems of two biharmonic equations on Riemannian manifolds.

For further applications, e. g. Hardy and Caffarelli-Kohn-Nirenberg type inequalities, see [6] and [2].

2. A general Rellich type identity

In this section we prove Theorem 1.2.

In the Noether's identity (1.1) we take $m = 2$, $u^1 = u$ and $u^2 = v$. Consider the vector field

$$X = \xi^i(x)\frac{\partial}{\partial x_i} + \eta(x,u,v,u_{(1)},v_{(1)})\frac{\partial}{\partial u} + \phi(x,u,v,u_{(1)},v_{(1)})\frac{\partial}{\partial v} \tag{2.1}$$

where $\xi^i, \eta^1 = \eta, \eta^2 = \phi \in \mathcal{A}$. Then the Noether's Identity (1.1) applied to $L = L(x,u,v,u_{(1)},v_{(1)})$, another differential function, reads:

$$\begin{aligned} X^{(1)}L + LD_i\xi^i &= E_u(L)(\eta - \xi^j u_j) + E_v(L)(\phi - \xi^j u_j) \\ &+ D_i\Big[\xi^i L + \frac{\partial L}{\partial u_i}(\eta - \xi^j u_j) + \frac{\partial L}{\partial v_i}(\phi - \xi^j v_j)\Big], \end{aligned} \tag{2.2}$$

where

$$D_i = \frac{\partial}{\partial x_i} + u_i\,\frac{\partial}{\partial u} + v_i\,\frac{\partial}{\partial v} + u_{ij}\,\frac{\partial}{\partial u_j} + v_{ij}\,\frac{\partial}{\partial v_j}...$$

$$+u_{ii_1i_2...i_l}\,\frac{\partial}{\partial u_{i_1i_2...i_l}} + v_{ii_1i_2...i_l}\,\frac{\partial}{\partial v_{i_1i_2...i_l}} + ...$$

is the total derivative operator, $E = (E_u, E_v)$ is the Euler operator with components

$$E_u = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}D_{i_2}...D_{i_s}\frac{\partial}{\partial u_{i_1i_2...i_s}}$$

and

$$E_v = \frac{\partial}{\partial v} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}D_{i_2}...D_{i_s}\frac{\partial}{\partial v_{i_1i_2...i_s}},$$

and $X^{(1)}$ is the first order prolongation of X given by

$$X^{(1)} = X + \eta_i^{(1)}\frac{\partial}{\partial u_i} + \phi_i^{(1)}\frac{\partial}{\partial v_i}.$$

Now we shall choose special forms of the differential functions involved in (2.2). Namely, let $F = F(x,u_{(1)},v_{(1)}) \in \mathcal{A}$ and $h = h^i(x)\frac{\partial}{\partial x^i}$ be a smooth vector filed. Then we set $L = F$, $\xi^i = 0$,

$$\eta = (h, F_q) = h_j F_{v_j} = g_{ij}h^i F_{v_j}, \quad \phi = (h, F_p) = h_j F_{u_j} = g_{ij}h^i F_{u_j}.$$

The differential operator X assumes the form,

$$X = (h, F_q)\frac{\partial}{\partial u} + (h, F_p)\frac{\partial}{\partial v},$$

and the General Prolongation Formula ([15], p. 113) immediately gives its first order prolongation:

$$X^{(1)} = X + D_i(h, F_q)\,\frac{\partial}{\partial u_i} + D_i(h, F_p)\,\frac{\partial}{\partial v_i}.$$

We substitute this data into the Noether's Identity (2.2). The left-hand side of (2.2) is

$$\begin{aligned} X^{(1)}L + LD_i\xi^i &= D_i(h, F_q)F_{u_i} + D_i(h, F_p)F_{v_i} \\ &= (F_p, \nabla(h, F_q)) + (F_q, \nabla(h, F_p)) \end{aligned} \tag{2.3}$$

while its right-hand side is given by,

$$\begin{aligned} E_u(L)(h, F_q) &+ E_v(L)(h, F_p) + D_i[F_{u_i}(h, F_q) + F_{v_i}(h, F_p)] \\ &= -D_iF_{u_i}(h, F_q) - D_iF_{v_i}(h, F_p) \\ &\quad +D_i[F_{u_i}(h, F_q) + F_{v_i}(h, F_p)] \\ &= -div\ F_p\ (h, F_q) - div\ F_q\ (h, F_p) \\ &\quad +D_i[F_{u_i}(h, F_q) + F_{v_i}(h, F_p)]. \end{aligned} \tag{2.4}$$

From (2.2), (2.3) and (2.4) it follows that

$$\begin{aligned} div\ F_p\ (h, F_q) + div\ F_q\ (h, F_p) &= -(F_p, \nabla(h, F_q)) - (F_q, \nabla(h, F_p)) \\ &+ D_i[F_{u_i}(h, F_q) + F_{v_i}(h, F_p)]. \end{aligned} \tag{2.5}$$

Actually, the identity (2.5) itself is obvious! Nevertheless its usefulness comes from the fact that it is directly obtained from the Noether's Identity, providing in this way a general procedure to get Rellich type identities, as it can be seen below.

We observe that the identity (2.5) holds if we replace the partial derivatives D_i by the covariant derivatives ∇_i corresponding to the Levi-Civita connection of M. In fact, we could have done this in the Noether's Identity from the beginning of this procedure noting that the function L contains only the first derivatives of u and v. We shall come back to this point later.

Differentiating in (2.5) and interchanging some of the indices we obtain:

$$\begin{aligned} divF_p\ (h, F_q)dV &+ divF_q\ (h, F_p) = -\{\nabla_i h_j + \nabla_j h_i\}F_{u_i}F_{v_j} \\ &- h_j\{F_{u_i}\nabla_i F_{v_j} + F_{v_i}\nabla_i F_{u_j}\} \\ &+ D_i[F_{u_i}(h, F_q) + F_{v_i}(h, F_p)]. \end{aligned} \tag{2.6}$$

We observe that in (2.6) the Lie derivative of the metric with respect to the vector field h appears, namely

$$\mathcal{L}_h g_{ij} = \nabla_i h_j + \nabla_j h_i.$$

Thus

$$\begin{aligned} divF_p\ (h,F_q)dV &+ divF_q\ (h,F_p) = -\mathcal{L}_h g_{ij}\ F_{u_i}F_{v_j} \\ &- h_j\{F_{u_i}\nabla_i F_{v_j} + F_{v_i}\nabla_i F_{u_j}\} \qquad (2.7) \\ &+ D_i[F_{u_i}(h,F_q) + F_{v_i}(h,F_p)]. \end{aligned}$$

Then by integrating the identity (2.7) and applying the divergence theorem, we get (1.3).

3. A Rellich type identity involving the Laplace operator

In this section we illustrate the proposed Noetherian approach to Rellich type identities recovering two known integral identities obtained in [6] and [11].

For this purpose let us now suppose that h in the preceding section is a conformal Killing vector field:

$$\mathcal{L}_h g_{ij} = \nabla_i h_j + \nabla_j h_i = \frac{2}{n}(div\ h)g_{ij}.$$

Hence and from (1.3):

$$\begin{aligned} \int_M \{divF_p\ (h,F_q)dV &+ divF_q\ (h,F_p)\}dV = -\frac{2}{n}\int_M div\ h\ g_{ij}F_{u_i}F_{v_j}dV \\ &- \int_M h_j\{F_{u_i}\nabla_i F_{v_j} + F_{v_i}\nabla_i F_{u_j}\}dV \qquad (3.1) \\ &+ \int_{\partial M}\{(F_p,\nu)(h,F_q) + (F_p,\nu)(h,F_q)\}dS. \end{aligned}$$

Let

$$F = g^{ij}(u_i u_j + v_i v_j)/2.$$

Then

$$F_{u_i} = g^{is}u_s = u^i, \quad F_{v_k} = g^{ks}v_s = v^k, \quad (h,F_p) = h^k u_k, \quad (h,F_q) = h^k v_k,$$

$$divF_p = \nabla_i u^i = \Delta_g u, \quad divF_q = \nabla_i v^i = \Delta_g v.$$

With this choice of F the identity (3.1) yields

$$\begin{aligned} \int_M \{\Delta_g u\ (h,\nabla v)dV &+ \Delta_g v\ (h,\nabla u)\}dV = -\frac{2}{n}\int_M div\ h\ (\nabla u,\nabla v)dV \\ &- \int_M h_j(u^i\nabla_i v^j + v^i\nabla_i u^j)dV \qquad (3.2) \\ &+ \int_{\partial M}\{(F_p,\nu)(h,F_q) + (F_p,\nu)(h,F_q)\}dS. \end{aligned}$$

Further we integrate by parts in the term of the second line of (3.2) taking into account the fact that the second covariant derivatives of a function commute:

$$-\int_M h_j(u^i\nabla_i v^j \quad + \quad v^i\nabla_i u^j)dV = -\int_M h^k\nabla_k(\nabla u,\nabla v)dV$$
$$= \quad \int_M div\ h\ (\nabla u,\nabla v)dV - \int_{\partial M}(\nabla u,\nabla v)(h,\nu)dS. \tag{3.3}$$

Substituting (3.3) into (3.2) we obtain

$$\int_M\{\Delta_g u\ (h,\nabla v)dV + \Delta_g v\ (h,\nabla u)\}dV$$
$$= \quad \frac{n-2}{n}\int_M div\ h\ (\nabla u,\nabla v)dV \tag{3.4}$$
$$+ \quad \int_{\partial M}\{\frac{\partial u}{\partial\nu}(h,\nabla v) + \frac{\partial v}{\partial\nu}(h,\nabla u) - (\nabla u,\nabla v)(h,\nu)\}dS.$$

Clearly (3.4) is the identity (17) of [6]. Moreover, if $M=\Omega$ is a bounded domain in $\mathbb{R}^n$, $g_{ij}=\delta_{ij}-$the Euclidean metric and $h=x^i\frac{\partial}{\partial x^i}$, then we obtain the Rellich type identity (2.5) established in [11], pp. 128-129.

Before concluding this section we would like to observe that in [6] the identity (3.4) is obtained following the argument in [11] while here it is obtained by applying the proposed Noetherian approach to Rellich identities.

4. A higher order Rellich type identity

In this section we prove Theorem 1.3, namely, we obtain a Rellich type identity involving the polyharmonic operator Δ_g^k on a Riemannian manifold (M^n,g), where $k\geq 2$ is an even number. The case $k-$odd can be treated in a similar way.

As in the preceding section, we take $u^1=u$, $u^2=v$, $\xi^i=0$, $\eta^1=\eta=(h,\nabla v)$, $\eta^2=\phi=(h,\nabla v)$ in the Noether's Identity (1.1) which is applied to the differential function

$$L=\frac{1}{2}(\Delta_g^m u)^2+\frac{1}{2}(\Delta_g^m v)^2,$$

where $m=k/2\geq 1$. Then the Noether's Identity reads

$$X^{(k)}L = E_u(L)\eta + E_v(L)\phi + \nabla_i w^i, \tag{4.1}$$

where $X^{(k)}$ is the $k-$th order prolongation of

$$X=\eta\frac{\partial}{\partial u}+\phi\frac{\partial}{\partial v},$$

(E_u,E_v) is the Euler operator and $w^i=N^i(L)$, $(N=(N^1,...,N^n)$ being the Noether operator) will be explicitly calculated later.

Calculations similar to those presented in [1] lead to

$$X^{(k)}L = (\Delta_g^m u)(\Delta_g^m \eta) + (\Delta_g^m v)(\Delta_g^m \phi) \tag{4.2}$$

and

$$E_u(L)\eta + E_v(L)\phi = \Delta_g^{2m} u.\eta + \Delta_g^{2m} v.\phi. \tag{4.3}$$

Thus (4.1), (4.2) and (4.3) imply the following.

Proposition 4.1. *If* $u, v \in C^{4m}(M)$, *then*

$$(\Delta_g^m u)(\Delta_g^m \eta) + (\Delta_g^m v)(\Delta_g^m \phi) = \Delta_g^{2m} u.\eta + \Delta_g^{2m} v.\phi + \nabla_i w^i, \tag{4.4}$$

where $\eta = (h, \nabla v)$, $\phi = (h, \nabla v)$ *and*

$$\begin{aligned} w^i \;&=\; -\sum_{s=0}^{m-1} \Delta_g^s \eta \; \nabla^i(\Delta_g^{2m-1-s} u) + \sum_{s=0}^{m-1} \nabla^i(\Delta_g^s \eta) \; \Delta_g^{2m-1-s} u \\ &- \sum_{s=0}^{m-1} \Delta_g^s \phi \; \nabla^i(\Delta_g^{2m-1-s} v) + \sum_{s=0}^{m-1} \nabla^i(\Delta_g^s \phi) \; \Delta_g^{2m-1-s} v. \end{aligned} \tag{4.5}$$

Now we would like to comment on an important point. To write (4.4) we have substituted in the Noether's Identity (1.1) the partial derivatives D_i by the covariant derivatives ∇_i. In this way we have actually used a *covariant* Noether's Identity which up to our knowledge has not been previously established and used in the literature. Therefore it requires a rigorous proof. However such a proof is very lengthy since it contains a lot of technical details, in particular, details regarding its application to the polyharmonic Lagrangian L (as in [1]) as well as careful commutations of the covariant derivatives which appear during this procedure. For these reasons we shall not present such a proof here merely pointing out that such a problem (viz. study of covariant Noether's Identity) will be treated elsewhere. Nevertheless, with (4.4) at hand, we can prove it directly using a simple alternative argument.

Proof of Proposition 4.1. We calculate the divergence

$$\begin{aligned}
\nabla_i w^i = & - \sum_{s=0}^{m-1} \nabla_i \Delta_g^s \eta \, \nabla^i (\Delta_g^{2m-1-s} u) - \sum_{s=0}^{m-1} \Delta_g^s \eta \, (\Delta_g^{2m-s} u) \\
& + \sum_{s=0}^{m-1} \Delta_g^{s+1} \eta \, \Delta_g^{2m-1-s} u + \sum_{s=0}^{m-1} \nabla^i \Delta_g^s \eta \, \nabla_i (\Delta_g^{2m-1-s} u) \\
& - \sum_{s=0}^{m-1} \nabla_i \Delta_g^s \phi \, \nabla^i (\Delta_g^{2m-1-s} v) - \sum_{s=0}^{m-1} \Delta_g^s \phi \, (\Delta_g^{2m-s} v) \\
& + \sum_{s=0}^{m-1} \Delta_g^{s+1} \phi \, \Delta_g^{2m-1-s} v + \sum_{s=0}^{m-1} \nabla^i \Delta_g^s \phi \, \nabla_i (\Delta_g^{2m-1-s} v) \\
= & - \sum_{s=1}^{m-1} \Delta_g^s \eta \, (\Delta_g^{2m-s} u) - \eta \Delta_g^{2m} u + \sum_{s=1}^{m} \Delta_g^s \eta \, \Delta_g^{2m-s} u \\
& - \sum_{s=1}^{m-1} \Delta_g^s \phi \, (\Delta_g^{2m-s} v) - \phi \Delta_g^{2m} v + \sum_{s=1}^{m} \Delta_g^s \phi \, \Delta_g^{2m-s} v \\
= & - \sum_{s=1}^{m-1} \Delta_g^s \eta \, (\Delta_g^{2m-s} u) - \eta \Delta_g^{2m} u + \sum_{s=1}^{m-1} \Delta_g^s \eta \, \Delta_g^{2m-s} u + \Delta_g^m \eta \Delta_g^m u \\
& - \sum_{s=1}^{m-1} \Delta_g^s \phi \, (\Delta_g^{2m-s} v) - \phi \Delta_g^{2m} v + \sum_{s=1}^{m-1} \Delta_g^s \phi \, \Delta_g^{2m-s} v + \Delta_g^m \phi \Delta_g^m v \\
= & - \eta \Delta_g^{2m} u - \phi \Delta_g^{2m} v + \Delta_g^m \eta \Delta_g^m u + \Delta_g^m \phi \Delta_g^m v,
\end{aligned}$$

which completes the proof.

The next step is to calculate $\Delta_g^m \eta$ and $\Delta_g^m \phi$ appearing in (4.4). This is done in a sequence of lemmas and propositions.

Lemma 4.2. *If h is a conformal Killing vector field satisfying*

$$\nabla^i h^k + \nabla^k h^i = \mu g^{ij} = \frac{2}{n} \, div(h) \, g^{ik}, \tag{4.6}$$

then

$$\Delta_g h^k + R^k_{\ s} h^s = \frac{2-n}{2} g^{kj} \mu_j, \tag{4.7}$$

where $R^i_{\ j}$ is the Ricci tensor.

Proof. See [3]. □

In particular, if the function $\mu =$ constant, we have

$$\Delta_g h^k = -R^k_{\ s} h^s. \tag{4.8}$$

Lemma 4.3. *If h is a conformal Killing vector field satisfying (4.6), then for any $v \in C^{2l+2}(M)$ we have,*

$$2\nabla^i h^k \ \nabla_i \nabla_k (\Delta_g^l v) = \mu \ \Delta_g^{l+1} v. \tag{4.9}$$

Proof. The equality (4.9) is obtained by multiplying (4.6) by $\nabla_i \nabla_k (\Delta_g^l v)$ and changing some of the indices. □

Lemma 4.4. *For any $v \in C^{2l+3}(M)$ the following identity holds,*

$$\Delta_g (\nabla_k (\Delta_g^l v)) = \nabla_k (\Delta_g^{l+1} v) + R^s_{\ k} \nabla_s (\Delta_g^l v). \tag{4.10}$$

Proof. The equality (4.10) for $l = 0$ follows from the well-known commutation relation

$$(\nabla_k \nabla_l - \nabla_l \nabla_k) T_i = -R_{i\ kl}^{\ s} T_s$$

valid for any covariant field $T = (T_s)$. Here $R_{i\ kl}^{\ s}$ is the Riemann tensor. More generally, for a $p-$contravariant and $q-$covariant tensor

$$T = (T^{i_1 \ldots i_p}_{\qquad j_1 \ldots j_q})$$

we have the commutation formula

$$\begin{aligned} \nabla_i \nabla_k T^{i_1 \ldots i_p}_{\qquad j_1 \ldots j_q} &= \nabla_k \nabla_i T^{i_1 \ldots i_p}_{\qquad j_1 \ldots j_q} + T^{s i_2 \ldots i_p}_{\qquad j_1 \ldots j_q} R^{i_1}_{\ \ ski} + \ldots \\ &+ T^{i_1 \ldots i_{p-1} s}_{\qquad j_1 \ldots j_q} R^{i_p}_{\ \ ski} - T^{i_1 \ldots i_p}_{\qquad s j_2 \ldots j_q} R^s_{\ j_1 ki} \\ &- \ldots - T^{i_1 \ldots i_p}_{\qquad j_1 \ldots j_{q-1} s} R^s_{\ j_q ki}. \end{aligned} \tag{4.11}$$

Setting in (4.11) $p = q = l$ and

$$T^{i_1 \ldots i_l}_{\qquad j_1 \ldots j_l} = \nabla^{i_1} \nabla_{j_1} \ldots \nabla^{i_l} \nabla_{j_l} v$$

we obtain

$$\begin{aligned} \nabla_i \nabla_k (\nabla^{i_1} \nabla_{j_1} \ldots \nabla^{i_l} \nabla_{j_l} v) &= \nabla_k \nabla_i (\nabla^{i_1} \nabla_{j_1} \ldots \nabla^{i_l} \nabla_{j_l} v) \\ &+ \nabla^s \nabla_{j_1} (\nabla^{i_2} \nabla_{j_2} \ldots \nabla^{i_l} \nabla_{j_l} v) R^{i_1}_{\ \ ski} + \ldots \\ &+ \nabla^{i_1} \nabla_{j_1} \ldots \nabla^{i_{l-1}} \nabla_{j_{l-1}} (\nabla^s \nabla_{j_l} v) R^{i_l}_{\ \ ski} \\ &- \nabla^{i_1} \nabla_s (\nabla^{i_2} \nabla_{j_2} \ldots \nabla^{i_l} \nabla_{j_l} v) R^s_{\ j_1 ki} - \ldots \\ &- \nabla^{i_1} \nabla_{j_1} \ldots \nabla^{i_{l-1}} \nabla_{j_{l-1}} (\nabla^{i_l} \nabla_s v) R^s_{\ j_l ki}. \end{aligned} \tag{4.12}$$

Then we put $i_1 = j_1, \ldots, i_l = j_l$ in (4.12), sum up and cancel l pairs of terms involving the Riemann tensor. Thus

$$\nabla_i \nabla_k (\Delta_g^l v) = \nabla_k \nabla_i (\Delta_g^l v). \tag{4.13}$$

Further, we choose $p = l + 1$, $q = l$, substitute

$$T^{i i_1 \ldots i_l}_{\qquad j_1 \ldots j_l} = \nabla^i \nabla^{i_1} \nabla_{j_1} \ldots \nabla^{i_l} \nabla_{j_l} v$$

into (4.11), and sum up over $i_1 = j_1,...,i_l = j_l$. In this way we get:

$$\nabla_i \nabla_k \nabla^i (\Delta_g^l v) = \nabla_k \nabla_i \nabla^i (\Delta_g^l v) + \nabla^s (\Delta_g^l v) R^i{}_{ski}.$$

Hence

$$\nabla_i \nabla_k \nabla^i (\Delta_g^l v) = \nabla_k (\Delta_g^{l+1} v) + R^s{}_k \nabla_s (\Delta_g^l v). \tag{4.14}$$

Then from (4.13) and (4.14) it follows that

$$\begin{aligned} \Delta_g (\nabla_k (\Delta_g^l v)) &= \nabla_i \nabla^i \nabla_k (\Delta_g^l v) = \nabla_i \nabla_k \nabla^i (\Delta_g^l v) \\ &= \nabla_k (\Delta_g^{l+1} v) + R^s{}_k \nabla_s (\Delta_g^l v). \end{aligned}$$

This completes the proof of Lemma 4.4. □

Now let us suppose that M admits a conformal Killing vector field $h = h^i \frac{\partial}{\partial x^i}$ such that

$$\nabla^k h^s + \nabla^s h^k = c\, g^{ks} = \frac{2}{n}\, div(h)\, g^{ks},$$

where $c \neq 0$ is a constant. That is, we suppose that M admits a homothety which is not an infinitesimal isometry of M. In this case we may assume that $c = 2$ and hence

$$div(h) = n. \tag{4.15}$$

(Otherwise, since $c \neq 0$, we could consider $2h/c$ instead of h.)

Proposition 4.5. *Let $u, v \in C^{2l+1}(M)$ be given functions and let $h = h^i(x) \frac{\partial}{\partial x^i}$ be a $C^1(M)$ conformal Killing vector field such that $div(h) = n$. Then for any $l \in \mathbb{N}$ we have*

$$\Delta_g^l \eta = 2l\, \Delta_g^l v + h^k \nabla_k (\Delta_g^l v), \tag{4.16}$$

$$\Delta_g^l \phi = 2l\, \Delta_g^l u + h^k \nabla_k (\Delta_g^l u). \tag{4.17}$$

Proof. We shall only prove (4.16). We shall use an induction argument on l.

1.) Let $l = 1$. Differentiating two times $\eta = (h, \nabla v) = h^k v_k$ we obtain

$$\nabla^i \eta = \nabla^i h^k . v_k + h^k \nabla_k \nabla^i v,$$

$$\Delta_g \eta = \nabla_i \nabla^i \eta = \Delta_g h^k . v_k + 2 \nabla^i h^k . \nabla_i \nabla_k v + h^k \nabla_i \nabla_k \nabla^i v. \tag{4.18}$$

Then (4.16) follows from (4.8), (4.9) with $\mu = 2$, $l = 1$, (4.10) with $l = 0$ and (4.18).

2.) Now we suppose that (4.16) holds for some $l \in \mathbb{N}$. We have to prove that (4.16) holds for $l+1$. Differentiating two times (4.16) we have

$$\begin{aligned}
\Delta_g^{l+1}\eta &= \nabla_i\nabla^i(\Delta_g^l\eta) \\
&= \nabla_i[\,2l\,\nabla^i(\Delta_g^l v) + \nabla^i h^k.\nabla_k(\Delta_g^l v) + h^k\nabla^i\nabla_k(\Delta_g^l v)\,] \\
&= 2l\,(\Delta_g^{l+1} v) + \Delta_g h^k.\nabla_k(\Delta_g^l v) \\
&+ 2\nabla^i h^k.\nabla_i\nabla_k(\Delta_g^l v) + h^k\Delta_g\nabla_k(\Delta_g^l v) \\
&= 2l\,(\Delta_g^{l+1} v) - R^k_s h^s.\nabla_k(\Delta_g^l v) + 2\Delta_g^{l+1} v \\
&+ h^k\nabla_k(\Delta_g^{l+1} v) + R^s_k h^k.\nabla_s(\Delta_g^l v) \\
&= 2(l+1)\,(\Delta_g^{l+1} v) + h^k\nabla_k(\Delta_g^{l+1} v).
\end{aligned}$$

In the computations above we have used (4.8), (4.9) with $\mu = 2$, and (4.10). This completes the proof. □

Further, from (4.5), (4.16) and (4.17) we can express w^i as

$$\begin{aligned}
w^i = &- \sum_{l=0}^{m-1}(2l\,\Delta_g^l v + h^k\nabla_k(\Delta_g^l v))\,\nabla^i(\Delta_g^{2m-1-l}u) \\
&+ \sum_{l=0}^{m-1}(2l\,\nabla^i(\Delta_g^l v) + \nabla^i h^k\nabla_k(\Delta_g^l v) + h^k\nabla^i\nabla_k(\Delta_g^l v))\,(\Delta_g^{2m-1-l}u) \\
&- \sum_{l=0}^{m-1}(2l\,\Delta_g^l u + h^k\nabla_k(\Delta_g^l u))\,\nabla^i(\Delta_g^{2m-1-l}v) \\
&+ \sum_{l=0}^{m-1}(2l\,\nabla^i(\Delta_g^l u) + \nabla^i h^k\nabla_k(\Delta_g^l u) + h^k\nabla^i\nabla_k(\Delta_g^l u))\,(\Delta_g^{2m-1-l}v).
\end{aligned} \tag{4.19}$$

Proposition 4.6. *Let $u, v \in C^{4m}(\bar{M})$ be given functions and let $h = h^i(x)\frac{\partial}{\partial x^i}$ be a $C^1(\bar{M})$ conformal Killing vector field such that $div(h) = n$. Then the* *following identity holds:*

$$
\begin{aligned}
R_{2m}(u,v) &= \int_M \{\Delta_g^{2m} u \ (h, \nabla v) + \Delta_g^{2m} v \ (h, \nabla u)\} dV \\
&= (4m-n) \int_M \Delta_g^m u \Delta_g^m v \ dV \\
&+ \int_{\partial M} [\Delta_g^m u \Delta_g^m v (h,\nu) - (w,\nu)] dS,
\end{aligned} \tag{4.20}
$$

where $w = (w^i)$ *is given in (4.19).*

Proof. We substitute into (4.4) $\Delta_g^m \eta$ and $\Delta_g^m \phi$ from (4.16) and (4.17) respectively. In this way we obtain:

$$
\Delta_g^{2m} u \ (h, \nabla v) + \Delta_g^{2m} v \ (h, \nabla u) = 4m \ \Delta_g^m u \Delta_g^m v + h^k \nabla_k (\Delta_g^m u \Delta_g^m v) - \nabla_i w^i.
$$

Hence

$$
\begin{aligned}
R_{2m}(u,v) &= 4m \int_M \Delta_g^m u \Delta_g^m v \ dV + \int_M h^k \nabla_k (\Delta_g^m u \Delta_g^m v) \ dV \\
&- \int_{\partial M} (w,\nu) dS \\
&= 4m \int_M \Delta_g^m u \Delta_g^m v \ dV - \int_M div(h) (\Delta_g^m u \Delta_g^m v) \ dV \\
&+ \int_{\partial M} [\Delta_g^m u \Delta_g^m v (h,\nu) - (w,\nu)] dS,
\end{aligned}
$$

which implies (4.20) recalling that $div(h) = n$ (see (4.15)). □

After $2m$ integrations by parts, the first term in the right-hand side of (4.20) can be written in the following form.

Proposition 4.7. *Let* $u, v \in C^{4m}(\bar{M})$*, then the following identity holds:*

$$
\begin{aligned}
\int_M \Delta_g^m u \Delta_g^m v \ dV &= \int_M \Delta_g^{2m} u \ v \ dV + \sum_{l=0}^{m-1} \int_{\partial M} \Delta_g^{m+l} u (\nabla (\Delta_g^{m-l-1} v), \nu) dS \\
&- \sum_{l=0}^{m-1} \int_{\partial M} \Delta_g^{m-l-1} v \ (\nabla (\Delta_g^{m+l} u), \nu) \ dS.
\end{aligned} \tag{4.21}
$$

As a consequence, the identity (1.5) follows from (4.20) and (4.21). This completes the proof of Theorem 1.3.

5. A biharmonic Rellich type identity

In this section we prove Theorem 1.5. Let η and ϕ be as in the preceding section. However, let h be a conformal Killing vector field satisfying (4.6), where the function $\mu = 2\, div(h)/n$ is not necessarily a constant.

The identity (4.4) with $m = 1$ reads

$$\Delta_g u\ \Delta_g \eta + \Delta_g v\ \Delta_g \phi = \Delta_g^2 u.\eta + \Delta_g^2 v.\phi + \nabla_i w^i, \tag{5.1}$$

where

$$w^i = -\eta \nabla^i(\Delta_g u) + \nabla^i \eta \Delta_g u - \phi \nabla^i(\Delta_g v) + \nabla^i \phi \Delta_g v.$$

From (4.18), (4.7), (4.9) with $l = 0$ and (4.14) with $l = 0$ it follows that

$$\Delta_g \eta = \mu \Delta_g v + h^k \nabla_k(\Delta_g v) + \frac{2-n}{n} \mu^k \nabla_k v. \tag{5.2}$$

Analogously

$$\Delta_g \phi = \mu \Delta_g u + h^k \nabla_k(\Delta_g u) + \frac{2-n}{n} \mu^k \nabla_k u. \tag{5.3}$$

Then, substituting (5.2) and (5.3) into (5.1) and integrating by parts, after some work, we obtain that

$$\begin{aligned}
\int_M \{\Delta_g^2 u\ (h, \nabla v) dV \quad &+ \quad \Delta_g^2 v\ (h, \nabla u)\} dV = \frac{4-n}{2} \int_M \mu\ \Delta_g u \Delta_g v dV \\
&- \quad \int_M \Delta_g \mu (u \Delta_g v + v \Delta_g u) dV \\
&- \quad \int_M (u(\nabla \mu, \nabla(\Delta_g v)) + v(\nabla \mu, \nabla(\Delta_g u)) dV \\
&+ \quad \int_{\partial M} (\Delta_g u \Delta_g v (h, \nu) + (u \Delta_g v + v \Delta_g u)(\nabla \mu, \nu) \\
&+ \quad (h, \nabla v) \nabla^i(\Delta_g u) \nu_i + (h, \nabla u) \nabla^i(\Delta_g v) \nu_i \\
&- \quad \Delta_g u (\nabla^i h^k . v_k \nu_i + h^k \nabla_k \nabla^i v . \nu_i) \\
&- \quad \Delta_g v (\nabla^i h^k . u_k \nu_i + h^k \nabla_k \nabla^i u . \nu_i)) dS.
\end{aligned} \tag{5.4}$$

But if h satisfies (4.6), then

$$\Delta_g \mu = -\frac{1}{n-1} (\mathcal{L}_h R + \mu R), \tag{5.5}$$

where $\mathcal{L}_h$ is the Lie derivative with respect to the vector field h. See [20]. From (5.4) and (5.5) we get (1.7).

We observe that if the conformal factor $\mu = 2$, that is, if $div(h) = n$ (see (4.16)), then, after two integrations by parts, the identity (1.7) is a particular case of (1.5) with $m = 1$.

6. A nonexistence result for a higher order semilinear Hamiltonian elliptic system

In this section, we bound ourselves to point out a very simple application of the identities proved in this paper. However, as mentioned in the introduction, the interested reader can easily realize the huge number of possible applications of these identities to other related problems in a different context.

Consider on (M, g) the following nonlinear system of two biharmonic equations

$$\begin{cases} \Delta_g^2 u = \dfrac{\partial G}{\partial v}, \\ \Delta_g^2 v = \dfrac{\partial G}{\partial u}, \end{cases} \tag{6.1}$$

with Navier boundary conditions on ∂M

$$u = v = \Delta u = \Delta v = 0. \tag{6.2}$$

Theorem 6.1. *Suppose that (M, g) admits a $C^1(\bar{M})$ contravariant conformal Killing vector field h such that $div(h) = n$ and $(h, \nu) > 0$ on ∂M. Let $G = G(s,t) \in C^1(\mathbb{R}^2)$ satisfy the conditions*

(1) $G(0,0) = \dfrac{\partial G}{\partial s}(0,0) = \dfrac{\partial G}{\partial t}(0,0) = 0,$*;*

(2) *if* $s, t \geq 0$, *then* $\dfrac{\partial G}{\partial s}(s,t) \geq 0$ *and* $\dfrac{\partial G}{\partial t}(s,t) \geq 0$*;*

(3) *there exist constants $c \geq n/(n-4)$ and $a \in (0,1)$ such that for any $s \in \mathbb{R}^1$ and $t \in \mathbb{R}^1$:*

$$cH(s,t) \leq as\frac{\partial G}{\partial s}(s,t) + (1-a)t\frac{\partial G}{\partial t}(s,t). \tag{6.3}$$

Then there is no nontrivial classical solution (that is $C^4(M) \cap C^3(\bar{M})$) of the Hamiltonian system (6.1) with Navier boundary conditions.

Proof. By (1.5) with $m = 2$ and $l = 0$, we obtain:

$$\int_M \{\Delta_g^2 u\,(h, \nabla v) + \Delta_g^2 v\,(h, \nabla u)\} dV = (4-n)\int_M \Delta_g^2 u\, v dV + A, \tag{6.4}$$

where

$$A = A(u,v) = A(v,u) = \int_{\partial M} h^k u_k \nabla^i(\Delta_g v)\nu_i dS + \int_{\partial M} h^k v_k \nabla^i(\Delta_g u)\nu_i dS.$$

On the other hand, multiplying the first equation in (6.1) by av, the second - by $(1-a)u$, adding and integrating by parts, taking into account the boundary conditions (6.2), we get that

$$\int_M \Delta_g^2 u\, v dV = \int_M \Delta_g u \Delta_g v dV = \int_M (auG_u + (1-a)vG_v) dV. \tag{6.5}$$

Then (6.4) and (6.5) imply

$$\int_M \{\Delta_g^2 u\ (h, \nabla v) + \Delta_g^2 v\ (h, \nabla u)\} dV = (4-n)\int_M (auG_u + (1-a)vG_v)dV + A. \tag{6.6}$$

Integrating by parts in the left-hand side of (6.6) and using $div(h) = n$ we obtain that

$$n\int_M G(u,v)dV = (n-4)\int_M (auG_u + (1-a)vG_v)dV - A. \tag{6.7}$$

The conditions on G and the maximum principle imply that

$$\frac{\partial u}{\partial \nu} < 0, \frac{\partial v}{\partial \nu} < 0, \frac{\partial(\Delta_g u)}{\partial \nu} > 0, \frac{\partial(\Delta_g v)}{\partial \nu} > 0$$

on ∂M. Hence

$$0 > \frac{\partial u}{\partial \nu}\frac{\partial(\Delta_g v)}{\partial \nu} = g^{ik}u_k\nu_i g^{js}(\Delta_g v)_s\nu_j = g^{ik}g^{js}u_j\nu_i(\Delta_g v)_s\nu_k$$

$$= g^{js}u_j(\Delta_g v)_s = (\nabla u, \nabla(\Delta_g v)),$$

that is

$$(\nabla u, \nabla(\Delta_g v)) < 0 \tag{6.8}$$

on ∂M. Above we have used the fact that on the boundary

$$u_k\nu_j = u_j\nu_k \tag{6.9}$$

for (6.2) (see [11]) and also $|\nu|^2 = g^{ik}\nu_i\nu_i = 1$. Analogously

$$(\nabla v, \nabla(\Delta_g u)) < 0 \tag{6.10}$$

on ∂M. Further

$$A = \int_{\partial M} (\nabla u, \nabla(\Delta_g v))(h,\nu)dS + \int_{\partial M} (\nabla v, \nabla(\Delta_g u))(h,\nu)dS. \tag{6.11}$$

Then (6.8), (6.10), (6.11) and $(h,\nu) > 0$ imply $-A > 0$. Then from (6.7) it follows that

$$\frac{n}{n-4}\int_M G(u,v)dV > (n-4)\int_M (auG_u + (1-a)vG_v)dV$$

which contradicts (6.3). This completes the proof. □

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Yuri Bozhkov
Instituto de Matemática, Estatistica e Computação Científica - IMECC
Universidade Estadual de Campinas - UNICAMP
Rua Sérgio Buarque de Holanda, 651
13083-859 - Campinas - SP
Brasil
e-mail: bozhkov@ime.unicamp.br

Enzo Mitidieri
Dipartimento di Matematica e Informatica
Università degli Studi di Trieste
Via Valerio 12/1, 34127 Trieste
Italia
e-mail: mitidier@units.it