ORDINARY VARIETIES AND THE COMPARISON BETWEEN MULTIPLIER IDEALS AND TEST IDEALS II

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Abstract. We consider the following conjecture: if X is a smooth n-dimensional projective variety over a field k of characteristic zero, then there is a dense set of reductions X_s to positive characteristic such that the action of the Frobenius morphism on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective. We also consider the conjecture relating the multiplier ideals of an ideal $\mathfrak{a}$ on a nonsingular variety in characteristic zero, and the test ideals of the reductions of $\mathfrak{a}$ to positive characteristic. We prove that the latter conjecture implies the former one.

1. Introduction

This note is motivated by the joint paper with V. Srinivas [MS], aimed at understanding the following conjecture relating invariants of singularities in characteristic zero with corresponding invariants in positive characteristic. For a discussion of the notions involved, see below.

Conjecture 1.1. *Let Y be a nonsingular variety over an algebraically closed field k of characteristic zero, and $\mathfrak{a}$ a nonzero ideal on Y. Given any model Y_A and $\mathfrak{a}_A$ for Y and $\mathfrak{a}$ over a subring A of k, finitely generated over $\mathbf{Z}$, there is a dense set of closed points $S \subset \operatorname{Spec} A$ such that*

$$\mathcal{J}(Y, \mathfrak{a}^{\lambda})_s = \tau(Y_s, \mathfrak{a}_s^{\lambda}) \tag{1}$$

for every $\lambda \in \mathbf{R}_{\geq 0}$ and every $s \in S$.

In the conjecture, we denote by Y_s the fiber of Y_A over $s \in S$, and $\mathfrak{a}_s$ is the ideal on Y_s induced by $\mathfrak{a}_A$. The ideals $\mathcal{J}(Y, \mathfrak{a}^{\lambda})$ are the multiplier ideals of $\mathfrak{a}$. These are fundamental invariants of the singularities of $\mathfrak{a}$, that have seen a lot of recent applications due to their appearance in vanishing theorems (see [Laz, Chapter 9]). The ideals $\tau(Y_s, \mathfrak{a}_s^{\lambda})$ are the (generalized) test ideals of Hara and Yoshida [HY], defined in positive characteristic using the Frobenius morphism. The above conjecture asserts therefore that for a dense set of closed points, we have the equality between the test ideals of $\mathfrak{a}$ and the reductions of the multiplier ideals of $\mathfrak{a}$ for *all* exponents. We note that it is shown in [HY] that if $\lambda \in \mathbf{R}_{\geq 0}$ is fixed, then the equality in (1) holds for every s is an open subset of the closed points in Spec A.

The following conjecture was proposed in [MS].

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Conjecture 1.2. *Let X be a smooth, irreducible n-dimensional projective variety defined over an algebraically closed field k of characteristic zero. If X_A is a model of X defined over a subring A of k, finitely generated over $\mathbf{Z}$, then there is a dense set of closed points $S \subseteq \operatorname{Spec} A$ such that the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective for every $s \in S$.*

It is expected, in fact, that there is a set S as in Conjecture 1.2 such that X_s is ordinary in the sense of Bloch and Kato [BK] for every $s \in S$. In particular, this would imply that the action of the Frobenius on each cohomology group $H^i(X_s, \mathcal{O}_{X_s})$ is bijective (see [MS, Remark 5.1]). The main result of [MS] was that Conjecture 1.2 implies Conjecture 1.1. In this note we show that the converse is true:

Theorem 1.3. *If Conjecture 1.1 holds, then so does Conjecture 1.2.*

The following is an outline of the proof. Given a variety X as in Conjecture 1.2, we embed it in a projective space $\mathbf{P}_k^N$ such that $r := N - n \geq n + 1$, and the ideal $\mathfrak{a} \subseteq k[x_0, \ldots, x_N]$ defining X is generated by quadrics. In this case it is easy to compute the multiplier ideals $\mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda)$ for $\lambda < r$, and in particular we see that $(x_0, \ldots, x_N)^{2r-N-1} \subseteq \mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda)$ for every $\lambda < r$. It follows from a general property of multiplier ideals that if $g_1, \ldots, g_r$ are general linear combinations of a system of generators of $\mathfrak{a}$, and if $h = g_1 \cdots g_r$, then $\mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda) = \mathcal{J}(\mathbf{A}_k^{N+1}, h^{\lambda/r})$ for every $\lambda < r$. In this case, Conjecture 1.1 implies that for a dense set of closed points $s \in \operatorname{Spec} A$, the ideal $(x_0, \ldots, x_N)^{2r-N-1}$ is contained in $\tau(\mathbf{A}_{k(s)}^{N+1}, h_s^\mu)$ for every $\mu < 1$. Using some basic properties of test ideals, we deduce that the Frobenius action on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ is bijective, where $D_s \subset \mathbf{P}_{k(s)}^N$ is the hypersurface defined by h_s. We show that this in turn implies the bijectivity of the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$, hence proves the theorem.

2. Proof of the main result

We start by recalling the definition of multiplier ideals and test ideals. Suppose first that Y is a nonsingular variety over an algebraically closed field k of characteristic zero, and $\mathfrak{a}$ is a nonzero ideal on Y. A *log resolution* of $\mathfrak{a}$ is a projective, birational morphism $\pi \colon W \to Y$, with W nonsingular, such that $\mathfrak{a} \cdot \mathcal{O}_W$ is the ideal of a divisor D on W, with $D + K_{W/Y}$ having simple normal crossings (recall that $K_{W/Y}$ denotes the relative canonical divisor of W over Y). With this notation, for every $\lambda \in \mathbf{R}_{\geq 0}$ we have

$$\mathcal{J}(Y, \mathfrak{a}^\lambda) = \pi_* \mathcal{O}_W(K_{W/Y} - \lfloor \lambda D \rfloor). \tag{2}$$

Recall that if $E = \sum_i a_i E_i$ is a divisor with $\mathbf{R}$-coefficients, then $\lfloor E \rfloor = \sum_i \lfloor a_i \rfloor E_i$, where $\lfloor t \rfloor$ is the largest integer $\leq t$. It is a well-known fact that the above definition is independent of the choice of log resolution. For this and other basic facts about multiplier ideals, see [Laz, Chapter 9].

Suppose now that $Y = \operatorname{Spec} R$ is an affine nonsingular scheme of finite type over a perfect field L of positive characteristic p (in the case of interest for us, L will be a finite field). Under these assumptions, the test ideals admit the following simpler description, that we will use, see [BMS2]. Recall that for an ideal J and for $e \geq 1$, one denotes by

$J^{[p^e]}$ the ideal $(h^{p^e} \mid h \in J)$. One can show that given an ideal $\mathfrak{b}$ in R, there is a unique smallest ideal J such that $\mathfrak{b} \subseteq J^{[p^e]}$; this ideal is denoted by $\mathfrak{b}^{[1/p^e]}$.

Suppose now that $\mathfrak{a}$ is an ideal in R and $\lambda \in \mathbf{R}_{\geq 0}$. One can show that for every $e \geq 1$ we have the inclusion

$$(\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subseteq (\mathfrak{a}^{\lceil \lambda p^{e+1} \rceil})^{[1/p^{e+1}]},$$

where $\lceil t \rceil$ denotes the smallest integer $\geq t$. Since R is Noetherian, it follows that $(\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]}$ is constant for $e \gg 0$. This is the test ideal $\tau(Y, \mathfrak{a}^{\lambda})$. For details and a discussion of basic properties of test ideals in this setting, we refer to [BMS2]. For a comparison of general properties of multiplier ideals and test ideals, see [HY] and [MY].

If $\mathfrak{a}$ is an ideal in the polynomial ring $k[x_0, \ldots, x_N]$, where k is a field of characteristic zero, a *model* of $\mathfrak{a}$ over a subring A of k, finitely generated over $\mathbf{Z}$, is an ideal $\mathfrak{a}_A$ in $A[x_0, \ldots, x_N]$ such that $\mathfrak{a}_A \cdot k[x_0, \ldots, x_N] = \mathfrak{a}$. We can obtain such a model by simply taking A to contain all the coefficients of a finite system of generators of $\mathfrak{a}$. Of course, we may always replace A by a larger ring with the same properties; in particular, we may replace A by a localization A_a at a nonzero element $a \in A$. If $s \in \operatorname{Spec} A$ and if $\mathfrak{a}_A$ is a model of $\mathfrak{a}$, then we obtain a corresponding ideal $\mathfrak{a}_s$ in $k(s)[x_0, \ldots, x_N]$. Note that if s is a closed point, then the residue field $k(s)$ is a finite field.

Suppose now that $X \subseteq \mathbf{P}^N_k$ is a projective subscheme defined by the homogeneous ideal $\mathfrak{a} \subseteq k[x_0, \ldots, x_N]$. If $\mathfrak{a}_A \subseteq A[x_0, \ldots, x_N]$ is a model of $\mathfrak{a}$ over A, which we may assume homogeneous, then the subscheme X_A of $\mathbf{P}^N_A$ defined by $\mathfrak{a}_A$ is a model of X over A. If $s \in \operatorname{Spec} A$, then the subscheme $X_s \subseteq \mathbf{P}^N_{k(s)}$ is defined by the ideal $\mathfrak{a}_s$. We refer to [MS, §2.2] for some of the standard facts about reduction to positive characteristic. We note that given $\mathfrak{a}$ as above, we may consider simultaneously all the reductions $\mathcal{J}(\mathbf{A}^{N+1}_k, \mathfrak{a}^{\lambda})_s$ for all $\lambda \in \mathbf{R}_{\geq 0}$. This is due to the fact that for bounded λ we only have to deal with finitely many ideals, while for $\lambda \gg 0$, the multiplier ideals are determined by the lower ones via a Skoda-type theorem (see [MS, §3.2] for details).

We can now give the proof of our main result stated in Introduction.

Proof of Theorem 1.3. Let X be a smooth, irreducible n-dimensional projective variety over an algebraically closed field k of characteristic zero, with $n \geq 1$. It is clear that the assertion we need is independent on the model X_A that we choose. Consider a closed embedding $X \hookrightarrow \mathbf{P}^N_k$. After replacing this by a composition with a d-uple Veronese embedding, for $d \gg 0$, we may assume that the saturated ideal $\mathfrak{a} \subset R = k[x_0, \ldots, x_N]$ defining X is generated by homogeneous polynomials of degree two (see [ERT, Proposition 5]). Furthermore, we may clearly assume that $r := N - n \geq n + 1$. Under these assumptions, it is easy to determine the multiplier ideals of $\mathfrak{a}$ of exponent $< r$.

Lemma 2.1. *With the above notation, if* $\mathfrak{m} = (x_0, \ldots, x_N)$, *then*

$$\mathcal{J}(\mathbf{A}^{N+1}_k, \mathfrak{a}^{\lambda}) = \begin{cases} R, & \textit{if } 0 \leq \lambda < \frac{N+1}{2}; \\ \mathfrak{m}^{\lfloor 2\lambda \rfloor - N}, & \textit{if } \frac{N+1}{2} \leq \lambda < r. \end{cases}$$

Proof. Let us fix $\lambda \in \mathbf{R}_{\geq 0}$, with $\lambda < r$. We denote by Z the subscheme of $\mathbf{A}_k^{N+1}$ defined by $\mathfrak{a}$. Let $\varphi\colon W \to \mathbf{A}_k^{N+1}$ be the blow-up of the origin, with E the exceptional divisor. Since $\mathfrak{a}$ is generated by homogeneous polynomials of degree two, it follows that $\mathfrak{a} \cdot \mathcal{O}_W = \mathcal{O}_W(-2E) \cdot \widetilde{\mathfrak{a}}$, where $\widetilde{\mathfrak{a}}$ is the ideal defining the strict transform $\widetilde{Z}$ of Z on W. We have $K_{W/\mathbf{A}_k^{N+1}} = NE$, hence the change of variable formula for multiplier ideals (see [Laz, Theorem 9.2.33]) implies

$$\mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda) = \varphi_*\left(\mathcal{J}(W, (\mathfrak{a} \cdot \mathcal{O}_W)^\lambda) \otimes \mathcal{O}_W(NE)\right). \tag{3}$$

It is clear that $\widetilde{Z}$ is nonsingular over $\mathbf{A}_k^{N+1} \smallsetminus \{0\}$. Since $\widetilde{Z} \cap E \subsetneq E \simeq \mathbf{P}^N$ is isomorphic to the scheme X, hence it is nonsingular, it follows that $\widetilde{Z}$ is nonsingular, and $\widetilde{Z}$ and E have simple normal crossings. Let $\psi\colon \widetilde{W} \to W$ be the blow-up of W along $\widetilde{Z}$, with exceptional divisor F, and let $\widetilde{E}$ be the strict transform of E. Note that $\widetilde{W}$ is nonsingular, and $\widetilde{E} + F$ has simple normal crossings. We have $K_{\widetilde{W}/W} = (r-1)F$ and $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{W}} = \mathcal{O}_{\widetilde{W}}(-2\widetilde{E} - F)$. Therefore ψ is a log resolution of $\mathfrak{a} \cdot \mathcal{O}_W$, and by definition we have

$$\mathcal{J}(W, (\mathfrak{a} \cdot \mathcal{O}_W)^\lambda) = \psi_*(\mathcal{O}_{\widetilde{W}}(-(\lfloor \lambda \rfloor - r + 1)F - \lfloor 2\lambda \rfloor \widetilde{E}) = \mathcal{O}_W(-\lfloor 2\lambda \rfloor E) \tag{4}$$

(recall that $\lambda < r$). The formula in the lemma follows from (3), (4), and the fact that $\varphi_*(\mathcal{O}_W(-iE)) = \mathfrak{m}^i$ for every $i \in \mathbf{Z}_{\geq 0}$. □

Let $f_1, \ldots, f_m$ be a system of generators of $\mathfrak{a}$, with each f_i homogeneous of degree two. If $g_1, \ldots, g_r$ are linear combinations of the f_i with coefficients in k, and if $h = g_1 \cdots g_r$, then

$$\mathcal{J}(\mathbf{A}_k^{N+1}, \mathfrak{a}^\lambda) = \mathcal{J}(\mathbf{A}_k^{N+1}, h^{\lambda/r}) \tag{5}$$

for every $\lambda < r$ (see [Laz, Proposition 9.2.28]).

Suppose now that $\mathfrak{a}_A$ and h_A are homogeneous models of $\mathfrak{a}$, and respectively h, over A. Let $X_A, D_A \subset \mathbf{P}_A^N$ be the projective schemes defined by $\mathfrak{a}_A$ and h_A, respectively. Note that $g_1, \ldots, g_r$ being general linear combinations of the f_i, the subscheme $V(g_1, \ldots, g_r) \subset \mathbf{P}_k^N$ has pure codimension r. Therefore we may assume that for every $s \in \operatorname{Spec} A$, the scheme $V((g_1)_s, \ldots, (g_r)_s)$ has pure codimension r in $\mathbf{P}_{k(s)}^N$. We need to show that given models as above, there is a dense set of closed points $S \subset \operatorname{Spec} A$ such that the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective for every $s \in S$. The next lemma shows that in fact, it is enough to find S as above such that the Frobenius action on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ is bijective for all $s \in S$.

Lemma 2.2. *Let L be a finite field, and $D_1, \ldots, D_r$ hypersurfaces in $\mathbf{P}^N = \mathbf{P}_L^N$ such that the intersection scheme $Y = D_1 \cap \ldots \cap D_r$ has pure codimension r in $\mathbf{P}^N$. If the Frobenius acts bijectively on $H^{N-1}(D, \mathcal{O}_D)$, where $D = \sum_{i=1}^r D_i$, then for every closed subscheme X of Y, the Frobenius action on $H^{N-r}(X, \mathcal{O}_X)$ is bijective.*

Proof. For every subset $J \subseteq \{1, \ldots, r\}$, let $D_J = \bigcap_{j \in J} D_j$. By assumption, Y is a complete intersection, hence there is an exact complex

$$\mathcal{C}^\bullet: \quad 0 \to \mathcal{C}^0 \overset{d^0}{\to} \mathcal{C}^1 \overset{d^1}{\to} \ldots \overset{d^{r-1}}{\to} \mathcal{C}^r \to 0,$$

where $\mathcal{C}^0 = \mathcal{O}_D$, and $\mathcal{C}^m = \bigoplus_{|J|=m} \mathcal{O}_{D_J}$ for $m \geq 1$. Note that we have a morphism of complexes $\mathcal{C}^\bullet \to F_*(\mathcal{C}^\bullet)$, where F is the absolute Frobenius morphism on X. It follows that if we break-up $\mathcal{C}^\bullet$ into short exact sequences, the maps in the corresponding long exact sequences for cohomology are compatible with the Frobenius action.

Let $\mathcal{M}^i = \mathrm{Im}(d^i)$, hence $\mathcal{M}^0 \simeq \mathcal{C}^0 = \mathcal{O}_D$ and $\mathcal{M}^{r-1} = \mathcal{C}^r = \mathcal{O}_Y$. Since each D_J is a complete intersection in $\mathbf{P}^N$, it follows that $H^i(D_J, \mathcal{O}_{D_J}) = 0$ for every i with $1 \leq i < \dim(D_J) = N - |J|$. We deduce that for every i with $0 \leq i \leq r-2$, the short exact sequence

$$0 \to \mathcal{M}^i \to \mathcal{C}^{i+1} \to \mathcal{M}^{i+1} \to 0$$

gives an exact sequence

$$0 = H^{N-i-2}(\mathbf{P}^N, \mathcal{C}^{i+1}) \to H^{N-i-2}(\mathbf{P}^N, \mathcal{M}^{i+1}) \to H^{N-i-1}(\mathbf{P}^N, \mathcal{M}^i).$$

Therefore we have a sequence of injective maps

$$H^{N-r}(Y, \mathcal{O}_Y) \hookrightarrow H^{N-r+1}(\mathbf{P}^N, \mathcal{M}^{r-2}) \hookrightarrow \ldots \hookrightarrow H^{N-2}(\mathbf{P}^N, \mathcal{M}^1) \hookrightarrow H^{N-1}(D, \mathcal{O}_D),$$

compatible with the Frobenius action. Since this action is bijective on $H^{N-1}(D, \mathcal{O}_D)$ by hypothesis, it follows that it is bijective also on $H^{N-r}(Y, \mathcal{O}_Y)$ (see, for example, [MS, Lemma 2.4]).

On the other hand, since $\dim(Y) = N - r$, the surjection $\mathcal{O}_Y \to \mathcal{O}_X$ induces a surjection $H^{N-r}(Y, \mathcal{O}_Y) \to H^{N-r}(X, \mathcal{O}_X)$, compatible with the Frobenius action. As we have seen, the Frobenius action is bijective on $H^{N-r}(Y, \mathcal{O}_Y)$, hence on every quotient (see [MS, Lemma 2.4]). This completes the proof of the lemms. □

Returning to the proof of Theorem 1.3, we see that it is enough to show that there is a dense set of closed points $S \subset \operatorname{Spec} A$ such that Frobenius acts bijectively on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ for $s \in S$. We assume that Conjecture 1.1 holds, hence there is a dense set of closed points $S \subset \operatorname{Spec} A$ such that $\tau(\mathbf{A}_{k(s)}^{N+1}, h_s^\lambda) = \mathcal{J}(\mathbf{A}_k^{N+1}, h^\lambda)_s$ for every $\lambda \in \mathbf{R}_{\geq 0}$ and every $s \in S$. In particular, it follows from Lemma 2.1 and (5) that $(x_0, \ldots, x_N)^{2r-N-1} \subseteq \tau(\mathbf{A}_{k(s)}^{N+1}, h_s^\lambda)$ for every $\lambda < 1$. Since $\deg(h_s) = 2r \geq (N+1)$, Proposition 2.3 below implies that the Frobenius action on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ is bijective for all $s \in S$. As we have seen, this completes the proof of Theorem 1.3. □

Proposition 2.3. *Let L be a perfect field of characteristic $p > 0$, and $h \in R = L[x_0, \ldots, x_N]$ a homogeneous polynomial of degree $d \geq N+1$, with $N \geq 2$. If $(x_0, \ldots, x_N)^{d-N-1} \subseteq \tau(\mathbf{A}_L^{N+1}, h^{1-\frac{1}{p}})$, then the Frobenius action on $H^{N-1}(D, \mathcal{O}_D)$ is bijective, where $D \subset \mathbf{P}_L^N$ is the hypersurface defined by h.*

Proof. In the case $d = N+1$, this is a reformulation of a well-known fact due to Fedder [Fe]. We follow the argument from [MTW, Proposition 2.16], that extends to our more general setting. It is enough to show that the Frobenius action on $H^{N-1}(D, \mathcal{O}_D)$ is injective (see [MS, §2.1]).

Note first that $\tau(\mathbf{A}_L^{N+1}, h^{1-\frac{1}{p}}) = (h^{p-1})^{[1/p]}$ (see [BMS1, Lemma 2.1]), hence by assumption $\mathfrak{m}^{d-N-1} \subseteq (h^{p-1})^{[1/p]}$, where $\mathfrak{m} = (x_0, \ldots, x_N)$. It is convenient to use the interpretation of the ideal $(g^{p-1})^{[1/p]}$ in terms of local cohomology. Let $E = H_{\mathfrak{m}}^{N+1}(R)$.

Recall that this is a graded R-module, carrying a natural action of the Frobenius, that we denote by F_E. There is an isomorphism

$$E \simeq R_{x_0\cdots x_N} / \sum_{i=0}^{N} R_{x_0\cdots\widehat{x_i}\cdots x_N}.$$

Via this isomorphism, F_E is induced by the Frobenius morphism on $R_{x_0\cdots x_N}$.

The annihilator of $(h^{p-1})^{[1/p]}$ in E is equal to $\mathrm{Ker}(h^{p-1}F_E)$ (see, for example, [BMS2, §2.3]). Therefore we have

$$\mathrm{Ker}(h^{p-1}F_E) \subseteq \mathrm{Ann}_E(\mathfrak{m}^{d-N-1}) = \bigoplus_{i\geq -d+1} E_i. \tag{6}$$

On the other hand, the exact sequence

$$0 \to R(-d) \xrightarrow{h} R \to R/(h) \to 0$$

induces an isomorphism

$$H^N_{\mathfrak{m}}(R/(h)) \simeq \{u \in E \mid hu = 0\}(-d),$$

such that the Frobenius action on $H^N_{\mathfrak{m}}(R/(h))$ is given by $h^{p-1}F_E$. Since $H^{N-1}(D, \mathcal{O}_D) \simeq H^N_{\mathfrak{m}}(R/(h))_0 \hookrightarrow E_{-d}$, (6) implies that the Frobenius action is injective on $H^{N-1}(D, \mathcal{O}_D)$. This completes the proof of the proposition. □

Remark 2.4. In the proof of Theorem 1.3 we only used the inclusion "$\subseteq$" in Conjecture 1.1. However, this is the interesting inclusion: the reverse one is known, see [HY] or [MS, Proposition 4.2]. It is more interesting that we only used Conjecture 1.1 when $Y = \mathbf{A}_k^{N+1}$, $\mathfrak{a}$ is principal and homogeneous, and $\lambda = 1 - \frac{1}{p}$. By combining Theorem 1.3 with the main result in [MS], we see that in order to prove Conjecture 1.1 in general, it is enough to consider the case when $Y = \mathbf{A}_k^n$, $\mathfrak{a} = (f)$ is principal and homogeneous, and show the following: if $\mathfrak{b} = \mathcal{J}(Y, \mathfrak{a}^{1-\varepsilon})$ for $0 < \varepsilon \ll 1$, and if $f_A \in A[x_1, \ldots, x_n]$ is a model for f, then there is a dense set of closed points $S \subset \mathrm{Spec}\, A$ such that

$$\mathfrak{b}_s \subseteq (f_s^{p-1})^{[1/p]}$$

for every $s \in S$, where $p = \mathrm{char}(k(s))$.

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