Harmonic Maaß-Jacobi forms of degree 1 with higher rank indices

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Abstract

We define and investigate real analytic weak Jacobi forms of degree 1 and arbitrary rank. En route we calculate the Casimir operator associated to the maximal central extension of the real Jacobi group, which for rank exceeding 1 is of order 4. The notion of mixed mock modular forms is extended to Jacobi forms so as to include multivariable Appell functions in a natural way. Using the Casimir operator, we make a connection between this new notion and the notion of real analytic Jacobi forms.

1 Introduction

The theory of holomorphic Jacobi forms was developed by Eichler and Zagier in the course of their work on the Saito-Kurokawa conjecture [EZ85]. Later Berndt and Schmidt initiated a theory of real analytic Jacobi forms [BS98], which was developed further by Pitale [Pit09]. In the real analytic case, holomorphicity is replaced by the requirement that the forms be eigenfunctions of the Casimir operator, a third order operator which generates the center of the algebra of invariant operators [BCR].

Bringmann and Richter studied harmonic Maaß-Jacobi forms in the sense of Pitale [BR10], but with a weak growth condition that includes the μ-function discovered by Zwegers. Zwegers had used this function in [Zwe02] to understand the hitherto mysterious mock modular forms discovered by Ramanujan in the early 20^{th} century. His work has been the focus of intense interest, having applications to mock theta functions [Ono09], combinatorics [Bri08, BL09, BGM09, BZ10], and physics [MO10].

Zwegers has just generalized the μ-function to higher Jacobi forms [Zwe10], by demonstrating the modularity of the multivariable Appell functions arising from certain character formulas for Lie superalgebras [KW94, KW01, STT05]. It may be that these functions will have an impact comparable to that of the μ-function.

Mock modular forms are the holomorphic parts of harmonic Maaß forms. Together with real analytic Jacobi forms of degree 1 and rank 1, they have received considerable attention over the past decade; see for example [GZ98] and the references above. Recently Zagier defined a *mixed mock modular form* to be the product of a mock modular form and a holomorphic modular form [Zag09]. These developments suggest the need for a precise definition of harmonic weak Jacobi forms of higher rank, along with a "mixed mock" version of this definition capturing the essential features of mixed mock modular forms.

In the present work we generalize the notion of harmonic weak Maaß-Jacobi forms of degree 1 to arbitrary indices of higher degree, in a manner which includes the Appell functions treated in [Zwe10]. Let G^J_N be the rank N Jacobi group $\mathrm{SL}_2(\mathbb{R}) \ltimes (\mathbb{R}^N \otimes \mathbb{R}^2)$, the semidirect product action being trivial on the first factor, and let $\tilde{G}^J_N$ be its central extension by the additive group $\mathrm{M}^T_N(\mathbb{R})$ of real symmetric $N \times N$ matrices. An important ingredient of our work is the center of the universal enveloping algebra of $\tilde{G}^J_N$. Using ideas developed by Borho [Bor76], Quesne [Que88], and Campoamor-Stursburg and Low [CSL09], we prove in Section 5 that this center is the polynomial algebra generated by $\mathrm{M}^T_N(\mathbb{R})$ and one additional element $\tilde{\Omega}_N$ of degree $N+2$. We refer to $\tilde{\Omega}_N$ as the *Casimir element* of $\tilde{G}^J_N$, as it is in some sense a lift of the Casimir element of SL_2.

Given any action of $\tilde{G}^J_N$, we refer to the operator by which $\tilde{\Omega}_N$ acts as the *Casimir operator* with respect to the action. In Section 2 we give formulas for the Casimir operators with respect to the standard slash actions of $\tilde{G}^J_N$, in terms of both the usual coordinates (2.4) and the raising and lowering operators (2.6). For $N \geq 2$, these operators are of order 4.

Let Γ^J_N be the *full Jacobi group,* the integer points of G^J_N. The slash actions of $\tilde{G}^J_N$ of interest here all drop to actions of Γ^J_N. We define a *Maaß-Jacobi form* to be an eigenform of the Casimir operator with respect to such an action, invariant under Γ^J_N and satisfying a certain growth condition.

In Section 3 we build a theory of *mixed mock Jacobi forms* by imposing conditions arising from a family of Laplace operators. This theory allows for specialization to torsion points in a manner compatible with the notion of harmonic weak Maaß forms in the classical setting [BF04]. We connect mixed mock Jacobi forms with harmonic Maaß-Jacobi forms, and we show that the space of all mixed mock Jacobi forms is closed under multiplication by holomorphic Jacobi forms and, as mentioned, contains the Appell functions appearing in [Zwe10]. We also decide the question of the extent to which these functions are typical examples.

In Section 4 we investigate a distinguished subspace of the space of Maaß-Jacobi forms, the space of *semi-holomorphic forms.* We show that in the higher rank case it is connected to the space of skew-holomorphic Jacobi forms: we define a ξ-operator (4.1) which maps any harmonic Maaß-Jacobi form to the derivation of its non-holomorphic part, a skew-holomorphic Jacobi form in the sense of [Sko90, Hay06]. In Corollary 4.13 we show that all possible cuspidal non-holomorphic parts occur.

The Zagier-type dualities proved in Corollary 4.9 demonstrate the arithmetic relevance of our more general construction. As Bringmann and Richter remark in the rank 1 case [BR10], this relates holomorphic parts not only to one another, but also to non-holomorphic parts.

The paper concludes with Section 5, in which we use the algorithm developed by Helgason [Hel77] to deduce the invariant and covariant differential operators presented in Section 2.

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2 Maaß-Jacobi forms with lattice indices

We first fix some notation. All vector spaces are complex unless we indicate otherwise. Let $\mathrm{M}_{m,n}(R)$ denote the space of $m \times n$ matrices over a ring R, abbreviate $\mathrm{M}_{n,n}(R)$ as $\mathrm{M}_n(R)$, and let $\mathrm{M}_n^T(R)$ be the symmetric subspace of $\mathrm{M}_n(R)$. Write A^{T} and $\mathrm{tr}(A)$ for the transpose and (when A is square) trace of a matrix A, respectively. Regarding elements of R^m as column vectors, we will freely identify $R^m \otimes R^n$ with $\mathrm{M}_{m,n}(R)$ via $v \otimes w \mapsto vw^{\mathrm{T}}$.

Write ϵ_i for the i^{th} standard basis vector of R^m and ϵ_{ij} for the elementary matrix with $(i,j)^{\text{th}}$ entry 1 and other entries 0, the sizes of ϵ_i and ϵ_{ij} being determined by context. Let I_n be the identity matrix in $\mathrm{M}_n(R)$, and set

$$J_{2n} := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The real Jacobi group G_N^J of rank N and its subgroup Γ_N^J, the *full Jacobi group,* are

(2.1)
$$G_N^J := \mathrm{SL}_2(\mathbb{R}) \ltimes (\mathbb{R}^N \otimes \mathbb{R}^2), \qquad \Gamma_N^J := \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^N \otimes \mathbb{Z}^2).$$

The product in G_N^J arises from the natural right action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$. It can be written most simply using the above identification of $\mathbb{R}^N \otimes \mathbb{R}^2$ with $\mathrm{M}_{N,2}(\mathbb{R})$: for M, $\check{M} \in \mathrm{SL}_2(\mathbb{R})$ and X, $\check{X} \in \mathrm{M}_{N,2}(\mathbb{R})$,

$$(M, X)(\check{M}, \check{X}) = (M\check{M}, X\check{M} + \check{X}).$$

Let $\mathbb{H} := \{\tau \in \mathbb{C} : \mathfrak{Im}(\tau) > 0\}$ be the Poincaré upper half plane, and define

$$\mathbb{H}_{1,N} := \mathbb{H} \times \mathbb{C}^N.$$

We will write $\tau := x + iy$ for the $\mathbb{H}$-coordinate and $z_j := u_j + iv_j$ for the $\mathbb{C}^N$-coordinates. We will be interested in a certain family of slash actions (*i.e.,* right actions) of Γ_N^J on $C^\infty(\mathbb{H}_{1,N})$. These actions are not restrictions of actions of G_N^J, but rather quotients of restrictions of actions of a certain central extension $\tilde{G}_N^J$ of G_N^J by the additive group $\mathrm{M}_N^T(\mathbb{R})$. It will be necessary for us to work with $\tilde{G}_N^J$ in so far as we will use its Casimir element to construct for each slash action an invariant differential operator, the *Casimir operator.*

Definition 2.1. *Maintaining the* $\mathrm{M}_{N,2}(\mathbb{R})$ *identification, the centrally extended rank* N *real Jacobi group* $\tilde{G}_N^J$ *and its product are*

$$\tilde{G}_N^J := \bigl\{(M, X, \kappa) : \ (M, X) \in G_N^J,\ \kappa \in \mathrm{M}_N(\mathbb{R}),\ \kappa + \tfrac{1}{2} X J_2 X^{\mathrm{T}} \in \mathrm{M}_N^T(\mathbb{R})\bigr\},$$

$$(M, X, \kappa)(\check{M}, \check{X}, \check{\kappa}) := (M\check{M}, X\check{M} + \check{X}, \kappa + \check{\kappa} - X\check{M} J_2 \check{X}^{\mathrm{T}}).$$

Note that G_N^J is centerless, and the center of $\tilde{G}_N^J$ is $\mathrm{M}_N^T(\mathbb{R})$. As we will see in Section 5, $\tilde{G}_N^J$ is a subgroup of $\mathrm{Sp}_{2N+2}(\mathbb{R})$.

Now fix an element $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{R})$. For $\tau \in \mathbb{H}$, define

$$M\tau := (a\tau + b)(c\tau + d)^{-1}, \qquad \beta(M, \tau) := (c\tau + d)^{-1}.$$

Then $\tau \mapsto M\tau$ is the standard left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$, and β is a *scalar cocycle* with respect to it:

$$\beta(M\check{M}, \tau) = \beta(M, \check{M}\tau)\beta(\check{M}, \tau).$$

Scalar cocycles are in bijection with slash actions on scalar functions. For example, β^k is a cocycle for all $k \in \mathbb{Z}$, and the associated slash action of $\mathrm{SL}_2(\mathbb{R})$ on $C^\infty(\mathbb{H})$ is usually written

$$\phi|_k[M](\tau) := \beta^k(M, \tau)\phi(M\tau).$$

For future reference, let us mention that the algebra of differential operators on $C^\infty(\mathbb{H})$ invariant with respect to the $|_k$-action is the polynomial algebra on one variable generated by the $|_k$-Casimir operator of $\mathrm{SL}_2(\mathbb{R})$, which differs by an additive constant from the weight k hyperbolic Laplacian

(2.2)
$$\Delta_k \ := \ 4y^2 \partial_\tau \partial_{\overline{\tau}} - 2iky\partial_{\overline{\tau}}.$$

The theory of cocycles is well-known; see *e.g.* [BCR] for a brief summary. Here we will only review the method by which the scalar cocycles of a given action are classified up to cohomological equivalence. The stabilizer of $i \in \mathbb{H}$ under $\mathrm{SL}_2(\mathbb{R})$ is SO_2, and one checks that the restriction of any cocycle to $\mathrm{SO}_2 \times \{i\}$ defines a representation of SO_2 on $\mathbb{C}$. Moreover, it is a fact that two cocycles are equivalent if and only if they define equal representations of SO_2. It follows that $\{\beta^k : k \in \mathbb{Z}\}$ exhausts the cocycles of the action under consideration up to equivalence. For example, the conjugate $\overline{\beta}^{\,k}$ is also a cocycle, equivalent to β^{-k}.

Henceforth write X_1 and X_2 for the columns of any element X of $\mathrm{M}_{N,2}(\mathbb{R})$. The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ generalizes to the following well-known left action of G^J_N on $\mathbb{H}_{1,N}$:

(2.3)
$$(M, X)(\tau, z) := \big(M\tau, \beta(M, \tau)(z + X_1\tau + X_2)\big).$$

Regard this as an action of $\tilde{G}^J_N$. As such, the stabilizer of the element $(i, 0)$ of $\mathbb{H}_{1,N}$ is $\tilde{K}^J_N := \mathrm{SO}_2 \times \{0\} \times \mathrm{M}^T_N(\mathbb{R})$, and the equivalence classes of the scalar cocycles of the action are in bijection with the representations of $\tilde{K}^J_N$ on $\mathbb{C}$.

In order to describe a complete family of cocycles, define a function $a : \tilde{G}^J_N \times \mathbb{H}_{1,N} \to \mathrm{M}^T_N(\mathbb{C})$ by

$$\begin{aligned} a\big((M, X, \kappa), (\tau, z)\big) \;&:=\; \kappa + X_2X_1^{\mathrm{T}} + X_1z^{\mathrm{T}} + zX_1^{\mathrm{T}} + X_1X_1^{\mathrm{T}}\tau \\ &\quad -c\beta(M, \tau)(z + X_1\tau + X_2)(z + X_1\tau + X_2)^{\mathrm{T}} \end{aligned}$$

(recall that c is M_{21}). For $L \in \mathrm{M}^T_N(\mathbb{C})$, define $\alpha_L : \tilde{G}^J_N \times \mathbb{H}_{1,N} \to \mathbb{C}$ by

$$\alpha_L\big((M, X, \kappa), (\tau, z)\big) \;:=\; \exp\big\{2\pi i \,\mathrm{tr}\big[La\big((M, X, \kappa), (\tau, z)\big)\big]\big\}.$$

Lemma 2.2. *For all $k \in \mathbb{Z}$ and $L \in \mathrm{M}^T_N(\mathbb{C})$, $\beta^k\alpha_L$ is a scalar cocycle with respect to the action (2.3) on $\mathbb{H}_{1,N}$ of the centrally extended Jacobi group $\tilde{G}^J_N$ from Definition 2.1. Moreover, any scalar cocycle of this action is equivalent to exactly one of these cocycles.*

Proof. The proof that β^k is a cocycle of the action of $\tilde{G}^J_N$ on $\mathbb{H}_{1,N}$ is the same as the proof that it is a cocycle of the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$. The proof that α_L is a cocycle is standard in the case $N = 1$ and proceeds along the same lines in general. One must prove that $a(g\breve{g}, x) = a(g, \breve{g}x) + a(\breve{g}, x)$. First check that it suffices to prove this for both g and $\breve{g}$ in either the semisimple or the nilpotent part of $\tilde{G}^J_N$, and then check each of the resulting four cases directly. The second sentence follows immediately from the classification of representations of $\tilde{K}^J_N$.□

As a consequence of this lemma we have the following family of slash actions of $\tilde{G}^J_N$ on $C^\infty(\mathbb{H}_{1,N})$: for k, $k' \in \mathbb{Z}$ and $L \in \mathrm{M}^T_N(\mathbb{C})$,

$$\begin{aligned} \phi|_{k,k',L}[M, X, \kappa](\tau, z) \;&:=\; \phi\big((M, X, \kappa)(\tau, z)\big) \\ &\quad \times \beta^k(M, \tau)\overline{\beta}^{\,k}(M, \tau)\alpha_L\big((M, X, \kappa)(\tau, z)\big). \end{aligned}$$

Observe that since $\beta\overline{\beta}$ is positive, $|_{k,k',L}$ makes sense for all k, $k' \in \mathbb{C}$ with $k - k' \in \mathbb{Z}$. We will write $|_{k,L}$ for $|_{k,0,L}$. (Usually we will be concerned only with the case $k' = 0$, but at one point we will need the freedom to choose differently.) By Lemma 2.2, any slash action is equivalent to exactly one of the actions $|_{k,L}$; as we have mentioned, $|_{k,k',L}$ is equivalent to $|_{k-k',L}$.

Definition 2.3. *A differential operator T on $\mathbb{H}_{1,N}$ is* covariant *from $|_{k,L}$ to $|_{k',L'}$ if for all $g \in \tilde{G}^J_N$ and $f \in C^\infty(\mathbb{H}_{1,N})$, we have*

$$T\big(f|_{k,L}[g]\big) \;=\; (Tf)\big|_{k',L'}[g].$$

*Let $\mathbb{D}(k, L;\, k', L')$ be the space of covariant operators from $|_{k,L}$ to $|_{k',L'}$, and let $\mathbb{D}^r(k, L;\, k', L')$ be the space of those of order $\leq r$. When $k' = k$ and $L' = L$, we refer to such operators as $|_{k,L}$-*invariant *and write simply $\mathbb{D}_{k,L}$ and $\mathbb{D}^r_{k,L}$.*

At this point we state the main results of Section 5, Theorem 2.4 and Propositions 2.6, 2.7, and 2.8. Elements of $C^\infty(\mathbb{H}_{1,N})$ holomorphic in $\mathbb{C}^N$ will be called *semi-holomorphic.* For any $N \times N$ matrix A and any N-vector w, set

$$A[w] := w^{\mathrm{T}} A w.$$

Recall the Laplacian (2.2) and our notation $\tau := x+iy \in \mathbb{C}$ and $z := u+iv \in \mathbb{C}^N$. For brevity, write $Ł := 2\pi i L$. For L invertible, define

$$\begin{aligned} \mathcal{C}^{k,L} \;:=\; & -2\Delta_{k-N/2} + 2y^2\big(\partial_{\overline{\tau}} Ł^{-1}[\partial_z] + \partial_\tau Ł^{-1}[\partial_{\overline{z}}]\big) - 8y\partial_\tau v^{\mathrm{T}}\partial_{\overline{z}} \\ & -\tfrac{1}{2}y^2\big(Ł^{-1}[\partial_{\overline{z}}]Ł^{-1}[\partial_z] - (\partial_{\overline{z}}^{\mathrm{T}} Ł^{-1}\partial_z)^2\big) + 2y(v^{\mathrm{T}}\partial_{\overline{z}})\partial_z^{\mathrm{T}} Ł^{-1}\partial_u \\ & -\tfrac{1}{2}(2k-N+1)iy\partial_{\overline{z}}^{\mathrm{T}} Ł^{-1}\partial_u + 2v^{\mathrm{T}}(v^{\mathrm{T}}\partial_{\overline{z}})\partial_{\overline{z}} + (2k-N-1)iv^{\mathrm{T}}\partial_{\overline{z}}. \end{aligned} \tag{2.4}$$

Theorem 2.4. *For L invertible, the operator $\mathcal{C}^{k,L}$ is, up to additive and multiplicative scalars, the* Casimir operator *of $\tilde{G}^J_N$ with respect to the $|_{k,L}$-action (see Section 5). It generates the image of the $|_{k,L}$-action of the center of the universal enveloping algebra of $\tilde{G}^J_N$. In particular, it lies in the center of $\mathbb{D}_{k,L}$. Its action on semi-holomorphic functions is*

$$-2\Delta_{k-N/2} + 2y^2\partial_{\overline{\tau}} Ł^{-1}[\partial_z]. \tag{2.5}$$

Note that for $N > 1$, (2.4) is of order 4. At $N = 1$ it is of order 3 and reduces to the operator $C^{k,m}$ given in [BR10] with $L = m$. (There is a misprint in [BR10]: the term $k(z - \overline{z})\partial_{\overline{z}}$ should be $(1-k)(z-\overline{z})\partial_{\overline{z}}$. This stems in part from a similar misprint in (8) of [Pit09], where the term $(z-\overline{z})\partial_{\overline{z}}$ coming from (6) of [Pit09] is missing.)

Definition 2.5. *The* lowering operators, *X_- and Y_-, and the* raising operators, *X_+ and Y_+, are*

$$X_-^{k,L} := -2iy\big(y\partial_{\overline{\tau}} + v^{\mathrm{T}}\partial_{\overline{z}}\big), \qquad X_+^{k,L} := 2i\big(\partial_\tau + y^{-1}v^{\mathrm{T}}\partial_z + y^{-2}Ł[v]\big) + ky^{-1},$$

$$Y_-^{k,L} := -iy\partial_{\overline{z}}, \qquad Y_+^{k,L} := i\partial_z + 2iy^{-1}Łv.$$

For $N = 1$ and $L = m$, these are the operators given on page 59 of [BS98]. (There is a misprint in their formula for Y_-: the expression $\frac{1}{2}(\tau - \overline{\tau})f_{\overline{z}}$ on the far right should be multiplied by -1.) Note that $Y_\pm^{k,L}$ are actually N-vector operators. We write $Y_{\pm,j}^{k,L}$ for their entries.

Frequently we will suppress the superscript (k, L). Care must be taken with this abbreviation, as for example X_+Y_+ means $X_+^{k+1,L}Y_+^{k,L}$.

Proposition 2.6. *The spaces* $\mathbb{D}^1(k, L;\, k \pm 2, L)$ *are 1-dimensional, and the spaces* $\mathbb{D}^1(k, L;\, k \pm 1, L)$ *are* N*-dimensional. They have bases given by*

$$\mathbb{D}^1(k, L;\, k{\pm}2, L) = \mathrm{Span}\big\{X_{\pm}^{k,L}\big\}, \quad \mathbb{D}^1(k, L;\, k{\pm}1, L) = \mathrm{Span}\big\{Y_{\pm,j}^{k,L} : 1 \le j \le N\big\}.$$

The spaces $\mathbb{D}^1_{k,L}$ *are equal to* $\mathbb{D}^0_{k,L} = \mathbb{C}$*. All other* $\mathbb{D}^1(k, L;\, k', L')$ *are zero.*

The raising operators commute with one another, as do the lowering operators (but keep in mind that, for example, $X_+Y_+ = Y_+X_+$ *means* $X_+^{k+1,L}Y_+^{k,L} = Y_+^{k+2,L}X_+^{k,L}$*). The commutators between them are*

$$[X_-,\, X_+] = -k, \quad [Y_{-,j},\, Y_{+,j'}] = i\L_{jj'}, \quad [X_-,\, Y_+] = -Y_-, \quad [Y_-,\, X_+] = Y_+.$$

Proposition 2.7. *Any covariant differential operator of order* r *may be expressed as a linear combination of products of up to* r *raising and lowering operators. There is a unique such expression in which the raising operators are all to the left of the lowering operators.*

The expression of this form for the Casimir operator is

(2.6)

$$\begin{aligned} \mathcal{C}^{k,L} \;&=\; -2X_+X_- + i\big(X_+\L^{-1}[Y_-] - \L^{-1}[Y_+]X_-\big) \\ &\quad -\tfrac{1}{2}\big(\L^{-1}[Y_+]\L^{-1}[Y_-] - Y_+^{\mathrm{T}}(Y_+^{\mathrm{T}}\L^{-1}Y_-)\L^{-1}Y_-\big) \\ &\quad -\tfrac{1}{2}(2k - N - 3)iY_+^{\mathrm{T}}\L^{-1}Y_-. \end{aligned}$$

Proposition 2.8. *The algebra* $\mathbb{D}_{k,L}$ *is generated by* $\mathbb{D}^3_{k,L}$*. The spaces* $\mathbb{D}^3_{k,L}$ *and* $\mathbb{D}^2_{k,L}$ *are of dimensions* $2N^2 + N + 2$ *and* $N^2 + 2$*, respectively. Bases for them are given by the following equations:*

$$\begin{aligned} \mathbb{D}^3_{k,L} \;&=\; \mathrm{Span}\big\{X_+Y_{-,i}Y_{-,j},\ Y_{+,i}Y_{+,j}X_- : 1 \le i \le j \le N\big\} \oplus \mathbb{D}^2_{k,L}, \\ \mathbb{D}^2_{k,L} \;&=\; \mathrm{Span}\big\{1,\ X_+X_-,\ Y_{+,i}Y_{-,j} : 1 \le i, j \le N\big\}. \end{aligned}$$

The focus of this paper is the space of harmonic Maaß-Jacobi forms of index L and weight k. In order to define it, fix $k \in \mathbb{Z}$ and a positive definite integral even lattice L of rank N. We will identify L with its Gram matrix with respect to a fixed basis, a positive definite symmetric matrix with entries in $\frac{1}{2}\mathbb{Z}$ and diagonal entries in $\mathbb{Z}$. Write $|L|$ for the covolume of the lattice, the determinant of the Gram matrix.

The full Jacobi group Γ_N^J defined in (2.1) clearly has a central extension by $\mathrm{M}_N^T(\mathbb{Z})$ which is a subgroup of $\tilde{G}_N^J$. It is easy to check that when L is a Gram matrix, the cocycle α_L is trivial on $\mathrm{M}_N^T(\mathbb{Z})$. Therefore the $|_{k,L}$-action factors through to an action of Γ_N^J, which we will also denote by $|_{k,L}$.

Definition 2.9 (Maaß-Jacobi forms). *A Maaß-Jacobi form of weight* k *and index* L *is a function* $\phi \in C^\infty(\mathbb{H}_{1,N})$ *satisfying the following conditions:*

(i) For all $A \in \Gamma_N^J$*, we have* $\phi|_{k,L}[A] = \phi$*.*

(ii) ϕ *is an eigenfunction of* $\mathcal{C}^{k,L}$*.*

(iii) For some $a > 0$, $\phi(\tau, z) = O\left(e^{ay} e^{2\pi L[v]/y}\right)$ as $y \to \infty$.

If ϕ is annihilated by the Casimir operator $\mathcal{C}^{k,L}$, it is said to be a harmonic Maaß-Jacobi form. We denote the space of all harmonic Maaß-Jacobi forms of fixed weight k and index L by $\mathbb{J}_{k,L}$.

Remark 1. *Adapting the proof in [BS98, Section 2.6], which is based on [LV80, Section 1.3] and [MVW87, Section 2.I.2], we see that any automorphic representation of $\tilde{G}^J_N$ is a tensor product $\tilde{\pi} \otimes \pi^L_{\mathrm{SW}}$. Here $\tilde{\pi}$ is a genuine representation of the metaplectic cover of SL_2, and π^L_{SW} is the Schrödinger-Weil representation of central character L. The latter is the extension to the metaplectic cover of the Jacobi group of the Schrödinger representation of the Heisenberg group, which is induced from the character $e^{2\pi i \mathrm{tr}(L\kappa)}$ of its center. Thus, as in [Pit09], semi-holomorphic forms play an important role in the representation-theoretic treatment of harmonic Maaß-Jacobi forms.*

For later use we set

$$e(r) := e^{2\pi i r},\ q := e(\tau),\ \zeta^r := \prod_{i=1}^{N} e(z_i r_i).$$

3 Mixed mock Jacobi forms

The Maaß-Jacobi forms introduced in the last section completely capture the spectral aspects of the Jacobi group. However, for arithmetic applications the conditions in Definition 2.9 are too weak. Indeed, even harmonic Maaß-Jacobi forms yield a partial differential equation of order 4 that is imposed on $a_{n,r}(y, v)$ in the Fourier addend $a_{n,r}(y, v) e(nx + ru)$.

To get an arithmetically significant subspace it is necessary to impose further conditions. It is highly desirable that this leads to finite dimensional spaces of solutions $a_{n,r}(y, v)$ for each n and r. Starting with the Laplace operator, we impose conditions ensuring that specialization to torsion points yields the Fourier expansions of harmonic Maaß forms over GL_2. Later we will attach certain polynomials to each of the resulting space of solutions. After fixing these polynomials, these spaces of solutions are indeed finite dimensional.

There is a family of $\tilde{G}^J_N$-invariant metrics on $\mathbb{H}_{1,N}$. To make their expression more readable we use the *S-coordinates* (p, q) on $\mathbb{C}^n$ defined by $z = p\tau + q$ with p and q real (see [BS98]).

Proposition 3.1. *For any positive definite symmetric matrix $C \in \mathrm{M}^T_N(\mathbb{R})$,*

$$ds^2 = y^{-2} d_\tau d_{\overline{\tau}} + y^{-1}(\tau\overline{\tau}\, C[\partial_p] + 2x\, \partial_q^{\mathrm{T}} C \partial_p + C[\partial_q])$$

is an invariant metric. The associated Laplace operator is $X_+ X_- + Y_+^{\mathrm{T}} C Y_-$.

Proof. The invariance with respect to $\mathrm{SL}_2(\mathbb{R}) \subseteq G^J_N$ follows as for $N = 1$. The invariance with respect to the Heisenberg group we can see by choosing an appropriate basis of $\mathbb{C}^N$ and again following the calculation for $N = 1$. The Laplace operator can be seen to be attached to ds^2 by choosing a basis of $\mathbb{C}^N$, such that C is becomes a diagonal matrix. $\square$

A function $\phi \in C^\infty(\mathbb{H}_{1,N})$ is harmonic with respect to all Laplace operators in Proposition 3.1 if and only if it vanishes under the operators $X_+ X_-$ and $Y_{+,i} Y_{-,j}$ for all $i, j \in \{1, \ldots, N\}$. Note that this is equivalent to vanishing under all elements of $\mathbb{D}^2_{k,L}$ which annihilate constants.

The following definition is not standard; we use it to determine a particular subspace of modular forms.

Definition 3.2. *A function $\phi \in C^\infty(\mathbb{H}_{1,N})$ is* polynomially torsion harmonic *if and only if there is an absolutely convergent series representation $\phi = \sum_{h \in \mathbb{N}} \phi_h$ such that for each $h \in \mathbb{N}$ there are nonzero polynomials p_X and $p_{Y,i}$ in N variables, $1 \le i \le N$, satisfying*

$$Y_{-,i}\, \phi_h \in p_{Y,i}(v/y) \ker(Y_{+,i}), \quad X_-\, \phi_h \in p_X(v/y) \ker(X_+).$$

The next lemma justifies this definition by connecting it with the order 1 covariant operators. We will see below that the μ-function and the Appell functions are typical examples of polynomially torsion harmonic functions.

Lemma 3.3. *For $l, h \in \mathbb{Q}$, any function $a(y, v) e(lx + hv)$ in the intersection $\ker(X_+) \cap \bigcap_{i=1}^N \ker(Y_{+,i})$ of all kernels of the order 1 raising operators is a scalar multiple of*

$$y^{-k} e(l\overline{\tau} + h\overline{z} + 4L[v] \,/\, y)$$

In what follows we need the space $\mathrm{TH}_{L'}$ of images of polynomially torsion harmonic functions under X_-.

Definition 3.4. *A* completed mixed mock Jacobi form of weight k and index L and harmonic index L' *is a function $\phi \in C^\infty(\mathbb{H}_{1,N})$ satisfying the following conditions:*

(i) For all $A \in \Gamma_N^J$, we have $\phi|_{k,L}[A] = \phi$.

(ii) For all $i \in \{1, \ldots, N\}$ and some $L' \in \mathrm{M}_N^T(\mathbb{Z})$, we have $Y_{-,i}\phi \in \mathbb{J}_{k,L} \otimes \mathrm{TH}_{L'}$ and $X_-\phi \in \mathbb{J}_{k,L} \otimes \mathrm{TH}_{L'}$.

(iii) For some $a > 0$, $\phi(\tau, z) = O\left(e^{ay} e^{2\pi L[v]/y}\right)$ as $y \to \infty$.

Remark 2. *This notion of a mixed mock modular form is based on a definition introduced by Zagier in a seminar [Zag09]. It encompasses products of mock and holomorphic modular forms, which have applications in physics.*

Remark 3. *Because X_- and $Y_{-,i}$ are anti-holomorphic differential operators, the space of mixed mock Jacobi forms is preserved under multiplication by $\mathbb{J}_{k,L}$ for all k, L.*

Remark 4. *Example 3.6 will show that we need the freedom to choose $L' \neq L$ in (ii). Since the Laplace operators are not in the center of the universal enveloping algebra, this is permissible, but it is an interesting phenomenon which might inspire the construction of further examples of harmonic Maaß-Jacobi forms.*

We are mainly interested in mixed mock Jacobi forms whose completion vanishes under the Casimir operator.

Proposition 3.5. *The Fourier expansion of a completed mixed mock Jacobi form with constant polynomials $p_{Y,i}$ and p_X of degree 1 is of the form*

$$\sum_{\substack{n\in\mathbb{Z}\\ r\in\mathbb{Z}^N}} c^+(D)\, q^n\zeta^r \;+\; \sum_{\substack{n,\nu\in\mathbb{Z}\\ r\in\mathbb{Z}^N,\, h\in\mathbb{R}^N}} c^-(D)\, E((\nu + hv\,/\,y)\sqrt{2y})\, q^n\zeta^r,$$

where $E(z) = 2\int_0^z e^{-\pi u^2}\, du$. The index h runs over all values yielding a fixed value of $h^{\mathrm{T}}h$, the harmonicity index *of the mixed mock Jacobi form. These forms are eigenfunctions of the Casimir operator if and only if $k = 1$.*

Proof. It is easy to see that the Fourier expansion maps to the kernels of X_+ and $Y_{+,i}$ under X_- and $Y_{-,i}$, and the module of solutions has rank 2 over the holomorphic functions. For the second statement, apply the decomposition in (2.6) and use the assumption on p_X and the $p_{Y,i}$. □

The first sum in the preceding lemma only involves holomorphic functions, and we will call this part of a completed mixed mock Jacobi form a *mixed mock Jacobi form.*

Example 3.6. In [Zwe10] Zwegers investigated the higher Appell sums $A_{Q,\xi}$, which Kac and Wakimoto had previously related to affine Lie superalgebras [KW01]. These sums are examples of mixed mock Jacobi forms (with Definition 3.4 (i) holding for a congruence subgroup). With Zwegers's considerations in mind, it is not hard to see that all Fourier coefficients of the meromorphic Jacobi forms

$$\frac{\eta(\tau)^l f(u+z) g(v-\xi z)}{f(u)\theta(z)}$$

with a sufficiently large $l \in \mathbb{N}$ are mixed mock Jacobi forms. Here f is an arbitrary holomorphic Jacobi form for rank 1, and g is a holomorphic Jacobi form of arbitrary rank. □

Theorem 3.7. *Given a mixed mock Jacobi form ϕ for any $\lambda, \mu \in \mathbb{Q}^N$, the function $\tau \mapsto \phi(\tau, \lambda\tau + \mu)$ is a mixed mock modular form. For any $N' < N$ and any matrix $M \in \mathbb{Q}^{(N,N')}$, the function $\phi : \mathbb{H}_{1,N'} \to \mathbb{C}, (\tau, z') \mapsto \phi(\tau, Mz')$ is a mixed mock Jacobi form for an appropriate congruence subgroup.*

Proof. We need only check the images under the elliptic ξ-operators or under X_- and $Y_{-,i}$, respectively. In the case that we specialize to torsion points $z = \mu + \lambda\tau$, the result holds, as $\partial_{\overline{z}}$ corresponds to $\sum \lambda_i \partial_{\overline{\tau}}$ in the specialization. The second case reduces to linearity of differential operators. □

4 Semi-holomorphic forms

Recall that a function on $\mathbb{H}_{1,N}$ holomorphic in $z \in \mathbb{C}^N \subseteq \mathbb{H}_{1,N}$ is called semi-holomorphic. We will denote the space of semi-holomorphic harmonic Maaß-Jacobi forms by $\mathbb{J}^{\mathrm{semi}}_{k,L}$. Semi-holomorphic forms vanish under Y_-, and X_- acts on them by $\partial_{\overline{\tau}}$. In particular, semi-holomorphic forms do not fall under Definition 3.4 unless they are holomorphic.

The theory of semi-holomorphic forms essentially mimics that of harmonic weak Maaß forms. Indeed, in Theorem 4.5 we will see that the θ-decomposition gives a well-behaved bijection between vector-valued weak harmonic Maaß forms and harmonic semi-holomorphic Maaß-Jacobi forms.

We first discuss semi-holomorphic Fourier expansions of Maaß-Jacobi forms. The negative discriminant of a Fourier index (n, r) is denoted by

$$D := D_L(n, r) := |L|(4n - L^{-1}[r])$$

By analogy with [BF04, page 9], define a function

$$H(y) := e^{-y} \int_{-2y}^{\infty} e^{-t} t^{-k-N/2}\, dt.$$

Proposition 4.1. *Any semi-holomorphic harmonic Maaß-Jacobi form has a Fourier expansion of the form*

$$y^{N/2-k} \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^N \\ s.t.\ D=0}} c^0(n, r)\, q^n \zeta^r + \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^N \\ s.t.\ D\gg -\infty}} c^+(n, r)\, q^n \zeta^r + \sum_{\substack{n\in\mathbb{Z}, r\in\mathbb{Z}^N \\ s.t.\ D\ll \infty}} c^-(n, r)\, H(\pi D y \,/\, 2|L|)\, e\big(-iDy \,/\, 4|L|\big)\, q^n \zeta^r.$$

Proof. This can be proved as in the case of rank 1 lattices, by solving the differential equation for the coefficients coming from the Casimir operator and then imposing the growth condition. □

Our investigation will concentrate on semi-holomorphic harmonic Maaß-Jacobi forms, and in particular their relation to skew-holomorphic forms. To state this relation we must define a ξ-operator. Proceeding as in [BR10, Section 4], we first define the lowering operator

$$D_-^{(L)} := -2iy\big(y\, \partial_{\overline{\tau}} + v^{\mathrm{T}} \partial_{\overline{z}} - \tfrac{1}{4} y \L^{-1}[\partial_{\overline{z}}]\big) \;=\; X_- - \tfrac{i}{2} \L^{-1}[Y_-].$$

Using this operator, we define the ξ-operator by

$$\xi_{k,L} := y^{k-5/2} D_-^{(L)}. \tag{4.1}$$

This is an analog of the ξ-operator in [Maa49]. The latter sends Maaß forms to their shadows, which are holomorphic if they have harmonic preimages. In our setting skew-holomorphic forms take the place of holomorphic ones.

Definition 4.2 (Skew-holomorphic Jacobi forms). *A skew-holomorphic Jacobi form of weight k and index L is a semi-holomorphic function $\phi \in C^\infty(\mathbb{H}_{1,N})$ satisfying the following conditions. First, for all $A \in \Gamma_N^J$ the equation $\phi|_{1/2,k-1/2,L}\, A = \phi$ holds. Second, the Fourier expansion of ϕ has the form*

$$\phi(\tau, z) = \sum_{\substack{n \in Z, r \in \mathbb{Z}^N \\ s.t.\ D \gg -\infty}} c(n, r)\, e(-iDy \,/\, 2|L|)\, q^n \zeta^r.$$

We write $\mathbb{J}^{\mathrm{sk}}_{k,L}$ for the space of all such forms.

Remark 5. *Skew-holomorphic Jacobi forms were first introduced by Skoruppa in [Sko90]. There are several articles treating a slightly more general notion than that we have given. See in particular [Hay06].*

Remark 6. *The Fourier expansion condition can be stated in terms of annihilation by the heat operator $2\partial_\tau - \L^{-1}[\partial_z] \,/\, 2$.*

Proposition 4.3. *If $\phi \in \mathbb{J}^{\mathrm{semi}}_{k,L}$, then $\xi_{k,L}\phi$ is an element of $\mathbb{J}^{\mathrm{sk}}_{3-k,L}$.*

Proof. By Proposition 2.6, $D_-^{(L)}$ is a covariant operator from $|_{k,L}$ to $|_{k-2,L}$. Applying $\xi_{k,L}$ to the Fourier expansion of a Maaß-Jacobi form as in Proposition 4.1 shows that the Fourier expansion of $\xi_{k,L}\phi$ has the correct form. □

The ξ-operator is compatible with the θ-decomposition. To state this precisely, let Γ be the elliptic metaplectic group with the same level as L. Denote the spaces of vector-valued harmonic Maaß forms for the Weil representation ρ_L by $[\Gamma, k - 1\,/\,2, \rho_L]^{\mathrm{Maaß}}$. For weakly holomorphic vector-valued Maaß forms change the superscript to $*^!$. The ξ-operator $\xi_{k-1/2} = y^{k-1/2}\, \overline{\partial_{\overline{\tau}} \,\cdot}$ maps this space of harmonic Maaß forms to the space of weakly holomorphic forms.

To revise the θ-decomposition we need the following θ-series for $\mu \in \mathbb{Z}^N$:

$$\theta_{L,\mu}(\tau, z) := \sum_{r \in \mathbb{Z}^N,\, r \equiv \mu (L\mathbb{Z}^N)} q^{L^{-1}[r]/4} \zeta^r. \tag{4.2}$$

Definition 4.4 ($\boldsymbol{\theta}$-decomposition). *The Maaß-Jacobi θ-decomposition is the map $\theta_L^{\mathrm{semi}} : \mathbb{J}^{\mathrm{semi}}_{k,L} \to [\Gamma, k - 1\,/\,2, \rho_L]$ defined by*

$$f(\tau, z) = \sum_{\mu (\mathbb{Z}^N / L\mathbb{Z}^N)} \theta_L^{\mathrm{semi}}(f)_\mu(\tau)\; \theta_{L,\mu}(\tau, z).$$

Similarly, the skew-holomorphic θ-decomposition map $\theta_L^{\mathrm{sk}} : \mathbb{J}^{\mathrm{sk}}_{k,L} \to [\Gamma, k - 1\,/\,2, \rho_L]$ is defined by

$$f(\tau, z) = \sum_{\mu (\mathbb{Z}^N / L\mathbb{Z}^N)} \overline{\theta_L^{\mathrm{sk}}(f)_\mu(\tau)}\; \theta_{L,\mu}(\tau, z).$$

Remark 7. *The existence of a θ-decomposition for a harmonic Maaß form is equivalent to its semi-holomorphicity.*

Theorem 4.5. *If k is even, the θ-decomposition of forms in $\mathbb{J}^{\mathrm{semi}}_{k,L}$ and $\mathbb{J}^{\mathrm{sk}}_{3-k,L}$ commutes with the ξ-operators $\xi_{k,L}$ and $\xi_{k-1\,/\,2}$. More precisely, the following diagram is commutative:*

$$\begin{array}{ccc}
\mathbb{J}^{\mathrm{semi}}_{k,L} & \xrightarrow{\quad \xi_{k,L} \quad} & \mathbb{J}^{\mathrm{sk}}_{3-k,L} \\
\Big\downarrow{\scriptstyle \theta^{\mathrm{semi}}_L} & & \Big\downarrow{\scriptstyle \theta^{\mathrm{sk}}_L} \\
[\Gamma, k-1\,/\,2, \rho_L]^{\mathrm{Maaß}} & \xrightarrow{\quad \xi_{k-1\,/\,2} \quad} & [\Gamma, 5\,/\,2-k, \rho_L]^{!}
\end{array}.$$

Proof. This is a calculation analogous to that in [BR10, Section 6]. □

Before we consider the Poincaré series we define a special part of the space of semi-holomorphic harmonic Maaß-Jacobi forms. We will show that it maps surjectively to the space of skew-holomorphic Jacobi forms with cuspidal shadow.

Definition 4.6 (Maaß-Jacobi forms with cuspidal shadow). *The inverse image under $\xi_{k,L}$ of $\mathbb{J}^{\mathrm{sk,cusp}}_{k,L}$, the cuspidal subspace of $\mathbb{J}^{\mathrm{sk}}_{k,L}$, is denoted by $\mathbb{J}^{\mathrm{semi,cusp}}_{k,L}$. It is the space of semi-holomorphic harmonic Maaß-Jacobi forms with cuspidal shadow.*

4.1 Poincaré series

In [BR10, Section 5] the authors define Maaß-Poincaré series for the Jacobi group. They restrict to Jacobi indices of rank one. In this section we generalize their considerations to arbitrary lattice indices.

We use the notation of Section 2; in particular, L is an integral lattice and k is in $\mathbb{Z}$. Throughout this section n will be an integer and r will be in $\mathbb{Z}^N$. Maintain D as above and set h as follows:

$$D := D_L(n,r) := |L|(4n - L^{-1}[r]), \qquad h := h_L(r) := |L|\, L^{-1}[r].$$

The standard scalar product of two N-vectors λ and z will be written as λz.

Using the M-Whittaker function $M_{\nu,\mu}$ (see [WW96]), we define

$$\mathcal{M}_{s,\kappa}(t) := |t|^{-\kappa/2} M_{\mathrm{sgn}(t)\kappa/2,\, s-1/2}(|t|), \tag{4.3}$$

$$\phi^{(n,r)}_{k,L,s}(\tau, z) := \mathcal{M}_{s,k-N/2}(\pi D y\,/\,|L|)\; e(rz + iL^{-1}[r]y\,/\,4 + nx). \tag{4.4}$$

Lemma 4.7. *The function $\phi^{(n,r)}_{k,L,s}$ defined in (4.4) is an eigenfunction of the Casimir operator $\mathcal{C}^{k,L}$ in Theorem 2.4, with eigenvalue*

$$-2s(1-s) - \tfrac{1}{2}\Big(k^2 - k(N+2) + \tfrac{1}{4}N(N+4)\Big). \tag{4.5}$$

Proof. Factor ϕ as follows:

$$\phi_{k,L,s} = e(rz + \tau L^{-1}[r] / 4) \cdot e(-Dx / 4|L|) \, \mathcal{M}_{s,k-N/2}(-\pi Dy / |L|).$$

The first factor is holomorphic in τ and the second is constant in z. Hence in applying $\mathcal{C}^{k,L}$ the contribution of the first factor cancels. We need only consider $-2\Delta_{k-N/2}$, yielding (4.5). □

We will study the Poincaré series

$$P^{(n,r)}_{k,L,s} := \sum_{A \in \Gamma^J_{N,\infty} \backslash \Gamma^J_N} \phi^{(n,r)}_{k,L,s}\Big|_{k,L} A, \tag{4.6}$$

which is semi-holomorphic. The usual estimate

$$\mathcal{M}_{s,k-N/2}(y) \ll y^{\mathfrak{Re}(s)-(2k-N)/4} \quad \text{as} \quad y \to 0$$

ensures absolute and uniform convergence for $\mathfrak{Re}(s) > 1 + N / 2$. Of particular interest will be the case $s \in \{k/2 - N/4,\, 1 + N/4 - k/2\}$, where the Poincaré series is annihilated by the Casimir operator.

We need to compute the Fourier expansions of the Poincaré series, which involve the I-Bessel function as well as the J-Bessel function. The following W-Whittaker function, θ-series, and higher Kloosterman sum will also arise. To make the notation more natural we renormalize the Whittaker function:

$$\mathcal{W}_{s,\kappa}(t) := |t|^{-\kappa/2} \, W_{\mathrm{sgn}(t)\kappa/2,\, s-1/2}(|t|), \tag{4.7}$$

$$\theta^{(r)}_{k,L} := \sum_{\lambda \in \mathbb{Z}^N} q^{L[\lambda]} \zeta^{2L\lambda} \, (q^{r\lambda} \zeta^r + (-1)^k q^{-r\lambda} \zeta^r), \tag{4.8}$$

$$K_{c,L}(n,r,n',r') := e(-rL^{-1}r' / 2c) \sum_{d(c)^\times,\, \lambda \in \mathbb{Z}^N / c\mathbb{Z}^N} e(\overline{d}L[\lambda] / c + n'd - r'\lambda + \overline{d}n + \overline{d}r\lambda), \tag{4.9}$$

where $\overline{d}$ is an integer inverse of d modulo c.

Theorem 4.8. *The Poincaré series (4.6) has the Fourier expansion*

$$P^{(n,r)}_{k,L,s}(\tau,z) = \mathcal{M}_{s,k-N/2}(\pi Dy / |L|) \, e(-iDy / 4|L|) \, \theta^{(r)}_{k,L}(\tau,t) q^n + \sum_{n' \in \mathbb{Z}, r' \in \mathbb{Z}^N} c_{y,s}(n',r') \, q^{n'} \zeta^{r'}. \tag{4.10}$$

Here the θ-series $\theta^{(r)}_{k,L}$ is defined in (4.8), and the coefficients $c_{y,s}$ are

$$c_{y,s}(n',r') := b_{y,s}(n',r') + (-1)^k b_{y,s}(n',-r'),$$

with $b_{y,s}$ depending on D and $D' = |L|(4n' + L^{-1}[r'])$.

For $D' = 0$, there is a constant $a_s(n', r')$ such that

$$b_{y,s}(n', r') = \frac{y^{1+N/4-k/2-s}}{\Gamma(s + k/2 - N/4)\,\Gamma(s - k/2 + N/4)}\; a_s(n', r').$$

For $D' \neq 0$,

$$\begin{aligned} b_{y,s}(n', r') = {} & 2^{1-N/2}\pi i^{-k}\, |L|^{-1/2}\, \Big(\Gamma(2s) \,/\, \Gamma\big(s - \mathrm{sgn}(D')(k/2 - N/4)\big)\Big) \\ & \cdot (D'/D)^{k/2-(N+2)/4}\;\; e(-iD'y \,/\, 4|L|)\, \mathcal{W}_{s,k-N/2}(\pi D'y \,/\, |L|) \\ & \cdot \sum_{c \in \mathbb{Z}_{>0}} c^{-(N+2)/2} K_{c,L}(n, r, n', r') \\ & \qquad\qquad \cdot \begin{cases} J_{2s-1}(\pi\sqrt{D'D}/(c|L|)) & \text{if } DD' > 0, \\ I_{2s-1}(\pi\sqrt{D'D}/(c|L|)) & \text{if } DD' < 0. \end{cases} \end{aligned}$$

The Kloosterman sum $K_{c,L}$ is defined in (4.9), and the W-Whittaker function $\mathcal{W}_{s,k-N/2}$ is given in (4.7).

Before proving this expansion, we use it to prove the Zagier-type duality announced in the introduction.

Corollary 4.9. *The Fourier coefficients $c^{(n,r)}_{k,L,s}(n', r')$ of the Poincaré series $P^{(n,r)}_{k,L,s}$ of Proposition 4.1 satisfy a Zagier-type duality with dual weights k and $N + 2 - k$. More precisely, suppose that $\mathfrak{Re}(s) > 1 + N\,/\,2$, in which case the Poincaré series converges. Then if $D, D' < 0$, there is a constant $h_{k,s}$ depending only on (k, s) such that*

$$c^{(n,r)}_{k,L,s}(n', r') = h_{k,s}\, c^{(n',r')}_{N+2-k,L,s}(n, r),$$

while if $D < 0$, $D' > 0$, there is a constant $h'_{k,s}$ depending only on (k, s) such that

$$c^{(n,r)}_{k,L,s}(n', r') = h'_{k,s}\, c^{(n',r')}_{N+2-k,L,s}(n, r).$$

Proof. The corollary follows immediately from Theorem 4.8 if we show that $K_{c,L}(n, r, n', r') = K_{c,L}(n', r', n, r)$. This can be seen by changing λ to $-d\lambda$ in (4.9). □

Corollary 4.10. *If $s = k/2 - N/4$ (respectively, $1 + N/4 - k/2$), then the Poincaré series $P^{(n,r)}_{k,L,s}$ converges for $k > 2 + N$ (respectively, $k < 0$). In both* *cases it is a Maaß-Jacobi form in $\mathbb{J}^{\mathrm{semi}}_{k,L}$ with Fourier expansion*

(4.11)

$$P^{(n,r)}_{k,L,s}(\tau,z) = \mathcal{M}_{s,k-N/2}(\pi Dy\,/\,|L|)\; e(-iDy\,/\,4|L|)\; \theta^{(r)}_{k,L}(\tau,t)\; q^n$$

$$+ \sum_{n'\in\mathbb{Z}, r'\in\mathbb{Z}^N, D'=0} c^k(n',r')\; q^{n'}\zeta^{r'}$$

$$+ \sum_{n'\in\mathbb{Z}, r'\in\mathbb{Z}^N, D'\neq 0} c^k(n',r')\; e(-iD'y\,/\,4|L|)\, \mathcal{W}_{s,k-N/2}(\pi D'y/|L|)\; q^{n'}\zeta^{r'}.$$

Proof (of Theorem 4.8). As usual we can choose as a system of representatives of $\Gamma^J_{N,\infty}\backslash\Gamma^J_N$ the elements

$$\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (a\lambda, b\lambda)\Big),$$

where (c,d) runs through coprime integers, λ varies in $\mathbb{Z}^N$, and $a,b\in\mathbb{Z}$ are fixed for each pair (c,d) such that $ad-bc=1$.

The θ-series in (4.10) arises naturally as the contribution of the representatives with $c=0$. Hence we must compute the contribution of representatives with $c\neq 0$. Since the calculation is similar for both signs of c, we focus on $c>0$. We proceed by calculating the integral occurring in the Fourier transform. First we separate as many terms as possible. Toward this aim, note

$$\frac{a\tau+b}{c\tau+d} = \frac{a}{c} - \frac{1}{c(c\tau+d)}, \qquad \frac{z}{c\tau+d} + \lambda\frac{a\tau+b}{c\tau+d} = \frac{z-\lambda\,/\,c}{c\tau+d} + \frac{a\lambda}{c},$$

$$\frac{a\tau+b}{c\tau+d}\,\frac{L[\lambda]}{2} + \frac{\lambda L z + zL\lambda}{c\tau+d} - \frac{cL[z]\,/\,2}{c\tau+d} = \frac{-cL[z-\lambda\,/\,c]\,/\,2}{c\tau+d} + \frac{aL[\lambda]}{2c}.$$

Splitting fractions into their fractional and integer parts yields

$$\sum_{\substack{c\in\mathbb{Z}_{>0} \\ d(c)^\times \\ \lambda\in\mathbb{Z}^N \bmod c\mathbb{Z}^N \\ \alpha\in\mathbb{Z},\, \beta\in\mathbb{Z}^N}} c^{-k}(\tau+c\,/\,d+\alpha)^{-k}\; e\Big(\frac{-L[z-\lambda\,/\,c+\beta]}{\tau+d\,/\,c+\alpha} + aL[\lambda]\,/\,c\Big)$$

$$\cdot\, \phi^{(n,r)}_{k,L,s}\Big(a\,/\,c - \frac{1}{c^2(\tau+d\,/\,c+\alpha)},\; \frac{z-\lambda\,/\,c-\beta}{c(\tau+d\,/\,c+\alpha)} + a\lambda\,/\,c\Big).$$

Fixing the fractional parts and subsuming them under τ, we need only compute the Fourier transform of

$$\sum_{a\in\mathbb{Z},\, \beta\in\mathbb{Z}^N} (\tau+\alpha)^{-k}\; e\Big(\frac{-L[z-\beta]}{\tau+\alpha}\Big)\; \phi^{(n,r)}_{k,L,s}\Big(a\,/\,c - \frac{1}{c^2(\tau+\alpha)},\; \frac{z-\beta}{c(\tau+\alpha)} + a\lambda\,/\,c\Big).$$

By the Poisson summation formula for L', the Fourier coefficients are

$$a_y(n',r') = \int\limits_{\mathbb{R}\times\mathbb{R}^N} t^{-k}\, e(L[w]\,/\,t)\;\; \phi^{(n,r)}_{k,L,s}(a\,/\,c - 1\,/\,c^2 t,\; w\,/\,ct + a\lambda\,/\,c)$$
$$\cdot\, e(-n'x' + r'w)\; dx' du',$$

where we use the variables $w = u' + iv' \in \mathbb{C}^N$ and $t = x' + iy' \in \mathbb{C}$.

Separating the real and imaginary part of

$$a\,/\,c - 1\,/\,c^2 t = a\,/\,c - x'\,/\,c^2|t|^2 + i(y\,/\,c^2|t|^2),$$

we find that the integral equals

$$e(na\,/\,c + ar\lambda\,/\,c) \int\limits_{\mathbb{R}} t^{-k}\; \mathcal{M}_{s,k-N\,/\,2}(\pi D y\,/\,|L| c^2 |t|^2)$$
$$\cdot\, e(-n'x' + iL^{-1}[r] y\, c^2\, |t|^2 - nx'\,/\,c^2|t|^2)$$
$$\cdot \int\limits_{\mathbb{R}^N} e(-r'w - L[w]\,/\,t + rw\,/\,ct)\; du' dx'.$$

Evaluating the inner integral leaves us with

$$\int\limits_{\mathbb{R}} t^{-(k-N/2)}\; \mathcal{M}_{s,k-N\,/\,2}(\pi D y\,/\,|L|\, c^2\, |t|^2)$$
$$\cdot\, e(D'x'\,/\,4|L| - Dx'\,/\,4|L|\, c^2\, |t|^2)\; dx',$$

which can be evaluated using [Fay77, page 176]. □

As a tool for the next proposition we will need skew-holomorphic Poincaré series. For $k \geq 3$, set

$$P^{(n,r),\mathrm{sk}}_{k,L} := \sum_{A \in \Gamma^J_{N,\infty} \backslash \Gamma^J_N} e_{n,r,L}\, |_{1/2,k-1/2,L}\, A, \tag{4.12}$$

where

$$e_{n,r,L}(\tau,z) := e(n\tau + rz)\; e(-iDy\,/\,2|L|).$$

Theorem 4.11. *The Poincaré series $P^{(n,r),\mathrm{sk}}_{k,L}$ defined in (4.12) is an element of $\mathbb{J}^{\mathrm{sk,cusp}}_{k,L}$ and has Fourier expansion*

$$P^{(n,r),\mathrm{sk}}_{k,L}(\tau,z) = e(-iDy\,/\,2|L|)\; \theta^{(r)}_{k-1,L}(\tau,z)\; q^n$$
$$+ \sum_{\substack{n' \in \mathbb{Z}, r' \in \mathbb{Z}^N \\ D > 0}} c(n',r')\; e(-iD'y\,/\,2|L|)\; q^{n'} \zeta^{r'}.$$

Here the θ-series is defined in (4.8), and the coefficients $c(n',r')$ are

$$c(n',r') = b(n',r') + (-1)^k b(n',-r'),$$

where b depends on D and D'. We have

$$b(n',r') = 2^{1-N/2}\pi i^{-k+1}\,|L|^{-1\,/\,2}\,(D'\,/\,D)^{k/2-(N+2)/4} \sum_{c\in\mathbb{Z}_{>0}} c^{-(N+2)/2}\,K_{c,L}(n,r,n',-r')\,J_{k-(N+2)/2}(\pi\sqrt{DD'}\,/\,|L|c),$$

the Kloosterman sum $K_{c,L}$ being defined in (4.9).

Proof. The proof is analogous to that of Theorem 4.8. □

Remark 8. *The Poincaré series $P^{(n,r),\mathrm{sk}}_{k,L}$ span $\mathbb{J}^{\mathrm{sk,cusp}}_{k,L}$. This can be seen as in [Sko90]: evaluation of the Petersson scalar product of an arbitrary form f with $P^{(n,r),\mathrm{sk}}_{k,L}$ yields the $(n,r)^{\mathrm{th}}$ Fourier coefficient of f. Hence any cusp form orthogonal with respect to the Petersson scalar product to all Poincaré series vanishes.*

Proposition 4.12. *The Maaß-Jacobi Poincaré series $P^{(n,r)}_{k,L,k\,/\,2-N\,/\,4}$ with $k > 2+N$ is meromorphic.*

For $k<0$, ξ_L maps the Maaß-Jacobi Poincaré series $P^{(n,r)}_{k,L,1+N\,/\,4-k\,/\,2}$ to the skew-holomorphic Poincaré series $P^{(n,r),\mathrm{sk}}_{3-k,L}$ up to a constant factor.

Proof. This follows immediately if we recall that

$$\mathcal{W}_{1+N\,/\,4-k\,/\,2,k-N\,/\,2}(y) = \begin{cases} e^{-y\,/\,2} & \text{if } y>0,\\ e^{-y\,/\,2}\Gamma((N+2)\,/\,2-k,-y) & \text{otherwise.}\end{cases}$$ □

We obtain the following simple but important corollary.

Corollary 4.13. *The restriction of the ξ-operator to $\mathbb{J}^{\mathrm{semi,cusp}}_{3-k,L}$ is surjective onto $\mathbb{J}^{\mathrm{sk,cusp}}_{k,L}$.*

Proof. There are no skew-holomorphic forms of negative weight, so we can restrict to $k>0$. Applying the ξ-operator and exploiting the fact that skew-holomorphic Poincaré series span $\mathbb{J}^{\mathrm{sk,cusp}}_{k,L}$, the result follows from Proposition 4.12. □

5 The Casimir operator

The purpose of this section is to prove Theorem 2.4 and Propositions 2.6, 2.7, and 2.8. Recall the real Jacobi group $\tilde{G}^J_N$ defined in Section 2, and let $\tilde{\mathfrak{g}}^J_N$ be its complexified Lie algebra. Making identifications as in Definition 2.1, we find

$$\tilde{\mathfrak{g}}^J_N := \big\{(M,X,\kappa) : M\in\mathfrak{sl}_2(\mathbb{C}),\, X\in\mathrm{M}_{N,2}(\mathbb{C}),\, \kappa\in\mathrm{M}^T_N(\mathbb{C})\big\},$$

$$\big[(M,X,\kappa),\,(\check{M},\check{X},\check{\kappa})\big] = \big([M,\check{M}],\, X\check{M}-\check{X}M,\, \check{X}J_2X^{\mathrm{T}} - XJ_2\check{X}^{\mathrm{T}}\big).$$

The exponential map $\exp : \tilde{\mathfrak{g}}^J_N \to \tilde{G}^J_N$ is

(5.1)
$$\exp(M, X, \kappa) \,=\, (e^M,\, Xg(M),\, \kappa - Xh(M)J_2X^{\mathrm{T}}),$$

where $g(z) := (e^z - 1)/z$ and $h(z) := (e^z - z - 1)/z^2$. This formula can be proven efficiently by checking that $\tilde{\mathfrak{g}}^J_N$ embeds in $\mathfrak{gl}_{2N+2}(\mathbb{R})$ as follows. For reference, we also give the embedding of $\tilde{G}^J_N$ into $\mathrm{GL}_{2N+2}(\mathbb{R})$:

$$(M, X, \kappa)_{\tilde{\mathfrak{g}}^J_N} \mapsto \begin{pmatrix} 0 & X & \kappa \\ & M & -J_2X^{\mathrm{T}} \\ & & 0 \end{pmatrix}, \quad (M, X, \kappa)_{\tilde{G}^J_N} \mapsto \begin{pmatrix} I_N & X & \kappa \\ & M & -MJ_2X^{\mathrm{T}} \\ & & I_N \end{pmatrix}.$$

Observe that conjugation by the matrix of the cyclic permutation $(1, 2, \ldots, N+1)$ carries $\tilde{G}^J_N$ into a minimal parabolic subgroup of the standard realization of $\mathrm{Sp}_{2N+2}(\mathbb{R})$, the image being everything but the center.

We note that the formula for the exponential function given in Section 5.2 of [BCR07] and Section 5.1 of [BCR] is incorrect. It should be identical to (5.1), but the $h(M)$ term was missed. The embedding π_4 on p. 148 of [BCR07] is also wrong: the $(1, 4)$ and $(3, 4)$ entries should make up $-MJ_2X^{\mathrm{T}}$ rather than $-J_2X^{\mathrm{T}}$.

In order to fix a basis for $\tilde{\mathfrak{g}}^J_N$, recall that we write ϵ_{ij} for the elementary matrix with $(i, j)^{\mathrm{th}}$ entry 1 and other entries 0, the size of the matrix being determined by the context. The standard basis of $\mathfrak{sl}_2$ is of course $E := \epsilon_{12}$, $F := \epsilon_{21}$, and $H := \epsilon_{11} - \epsilon_{22}$. For a basis of $\mathrm{M}_{N,2}$ we take $e_i := \epsilon_{i2}$ and $f_i := \epsilon_{i1}$, and for $\mathrm{M}^{\mathrm{T}}_N$ we take $Z_{ij} := \frac{1}{2}(\epsilon_{ij} + \epsilon_{ji})$. Then $\tilde{\mathfrak{g}}^J_N$ has basis

$$\big\{E, F, H;\ e_i, f_i,\, 1 \leq i \leq N;\ Z_{ij},\, 1 \leq i \leq j \leq N\big\}.$$

The brackets of this basis are as follows. Those on $\mathfrak{sl}_2$ are standard, and $\mathrm{M}^{\mathrm{T}}_N$ is the center $\mathfrak{z}(\tilde{\mathfrak{g}}^J_N)$. Under $\mathrm{ad}(H)$, the e_i are of weight 1 and the f_i are of weight -1. The $\mathrm{ad}(E)$ and $\mathrm{ad}(F)$ actions are given by

$$\mathrm{ad}(E):\ e_i \mapsto 0,\ f_i \mapsto -e_i; \quad \mathrm{ad}(F):\ e_i \mapsto -f_i,\ f_i \mapsto 0.$$

Finally, $[e_i, f_j] = -2Z_{ij}$.

The first step in proving Theorem 2.4 is to compute the center $\mathfrak{Z}(\tilde{\mathfrak{g}}^J_N)$ of the universal enveloping algebra $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$. Towards this end, let us write e and f for the column vectors with entries e_i and f_i, respectively, and Z for the symmetric matrix with entries Z_{ij}. This permits us to write conveniently such elements of $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$ as $e^{\mathrm{T}}Zf$ and $\det(Z)$, the determinant of Z.

Observe that $\det(Z)e^{\mathrm{T}}Z^{-1}f$ is a well-defined element of $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$. We will need in addition the following more subtle fact:

$$P \,:=\, \det(Z)\big(e^{\mathrm{T}}(e^{\mathrm{T}}Z^{-1}f)Z^{-1}f - (e^{\mathrm{T}}Z^{-1}e)(f^{\mathrm{T}}Z^{-1}f)\big)$$

is an element of $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$ of degree $N + 2$. To prove this, note that $\det(Z)P$ is clearly in $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$. Check that if we specialize Z to a diagonal matrix of scalars, then $\det(Z) = 0$ implies $\det(Z)P = 0$. Since the symmetric determinant is an irreducible polynomial, the result follows.

We now define the *Casimir element* of $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$, which has degree $N+2$:

$$\begin{aligned}\Omega_N \;:=\; \det(Z)\Big(& H^2 - (N+2)H + 4EF \\ & - \big(H - \tfrac{1}{2}(N+3)\big)e^{\mathrm{T}}Z^{-1}f + Ef^{\mathrm{T}}Z^{-1}f - e^{\mathrm{T}}Z^{-1}eF \\ & + \tfrac{1}{4}e^{\mathrm{T}}(e^{\mathrm{T}}Z^{-1}f)Z^{-1}f - \tfrac{1}{4}(e^{\mathrm{T}}Z^{-1}e)(f^{\mathrm{T}}Z^{-1}f)\Big).\end{aligned}$$

Theorem 5.1. *The center $\mathfrak{Z}(\tilde{\mathfrak{g}}^J_N)$ is polynomial on $1 + \binom{N+1}{2}$ generators:*

$$\mathfrak{Z}(\tilde{\mathfrak{g}}^J_N) \;=\; \mathbb{C}[\Omega_N, Z_{ij} : 1 \le i \le j \le N].$$

In order to prove this theorem we will describe *virtual copies of semisimple subalgebras,* which provide a simple but very clever way to compute the centers of the universal enveloping algebras of a certain class of semidirect sum Lie algebras. This idea was discovered by Borho [Bor76] for $\tilde{\mathfrak{g}}^J_1$, and independently by Quesne [Que88] for various Lie algebras of 1-dimensional center. It was generalized by Campoamor-Stursburg and Low [CSL09] and applied to further examples.

Before we give the proof of Theorem 5.1 arising from the virtual copy of $\mathfrak{sl}_2$, let us outline a straightforward but considerably less efficient proof. For any Lie algebra $\mathfrak{g}$, the *symmetrizer map* Sym from the symmetric algebra $\mathcal{S}(\mathfrak{g})$ to $\mathfrak{U}(\mathfrak{g})$ restricts to a vector space isomorphism from the invariants $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ to $\mathfrak{Z}(\mathfrak{g})$. The strategy is to compute $\mathcal{S}(\tilde{\mathfrak{g}}^J_N)^{\mathfrak{sl}_2}$ and then locate $\mathcal{S}(\tilde{\mathfrak{g}}^J_N)^{\tilde{\mathfrak{g}}^J_N}$ inside it.

Define the following elements of $\mathcal{S}(\tilde{\mathfrak{g}}^J_N)$: $Q_0 := H^2 + 4FE$, and

$$Q_{ij} \;:=\; \tfrac{1}{2}(f_i e_j - f_j e_i), \quad C_{ij} \;:=\; Ef_if_j - \tfrac{1}{2}H(f_ie_j + f_je_i) - Fe_ie_j$$

for $1 \le i, j \le N$. It can be shown that together with the Z_{ij}, these elements generate $\mathcal{S}(\tilde{\mathfrak{g}}^J_N)^{\mathfrak{sl}_2}$. For reference, we note that they satisfy the relations

$$Q_{ij}Q_{kl} + Q_{il}Q_{jk} + Q_{ik}Q_{lj} \;=\; 0, \quad C_{ij}C_{kl} - C_{ik}C_{jl} + Q_0Q_{il}Q_{jk} \;=\; 0$$

for $1 \le i, j, k, l \le N$. Thus $\mathcal{S}(\tilde{\mathfrak{g}}^J_N)^{\tilde{\mathfrak{g}}^J_N}$ is the kernel of the actions of the $\mathrm{ad}(e_i)$ and $\mathrm{ad}(f_i)$ on the algebra generated by $\big\{Q_0, Q_{ij}, C_{ij}, Z_{ij}\big\}_{ij}$.

Suppose that P is an element of this kernel. Regard it as a polynomial in the Z_{ij} with coefficients generated by $\big\{Q_0, Q_{ij}, C_{ij}\big\}_{ij}$. One can use the $\mathrm{ad}(e_i)$- and $\mathrm{ad}(f_i)$-invariance of P to prove that the coefficients of the monomials of top Z-degree lie in $\mathbb{C}[Q_0]$.

Now let Q and C be the matrices with entries Q_{ij} and C_{ij}, and define

$$P_N \;:=\; \det(Z)\big(Q_0 + \mathrm{tr}(Z^{-1}C) + \tfrac{1}{2}\mathrm{tr}(Z^{-1}Q)^2\big).$$

This is polynomial in Z because Q is skew-symmetric, and direct verification shows that it is $\tilde{\mathfrak{g}}^J_N$-invariant.

Collecting these results, one finds easily that P_N generates $\mathcal{S}(\tilde{\mathfrak{g}}_N^J)^{\tilde{\mathfrak{g}}_N^J}$ over the extension of $\mathbb{C}[Z_{ij}]$ by $\det(Z)^{-1}$. With a little more work one can drop the necessity to adjoin $\det(Z)^{-1}$.

The last step is to symmetrize P_N. A long computation leads to $\mathrm{Sym}(P_N) = \Omega_N + \frac{1}{4}N(N+3)\det(Z)$, proving Theorem 5.1.

5.1 Virtual semisimple subalgebras

This section is an exposition of the results of [Bor76, Que88, CSL09]. Let $\mathfrak{g}$ be any complex finite dimensional Lie algebra. Write $\mathfrak{s} \oplus_s \mathfrak{r}$ for its semidirect sum Levi decomposition, in which its semisimple part $\mathfrak{s}$ acts on its solvable part $\mathfrak{r}$. Let $\mathfrak{z}$ be the center $\mathfrak{z}(\mathfrak{g})$, which is of course contained in $\mathfrak{r}$. For any subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{z}$, define

$$\mathfrak{U}_{\mathfrak{z}}(\mathfrak{h}) := \mathfrak{U}(\mathfrak{h}) \otimes_{\mathfrak{U}(\mathfrak{z})} \mathrm{Frac}\big(\mathfrak{U}(\mathfrak{z})\big), \quad \mathfrak{Z}_{\mathfrak{z}}(\mathfrak{h}) := \mathfrak{Z}(\mathfrak{h}) \otimes_{\mathfrak{U}(\mathfrak{z})} \mathrm{Frac}\big(\mathfrak{U}(\mathfrak{z})\big).$$

Throughout this section we make the following assumption:

(5.2)

There is a Lie algebra homomorphism $\eta : \mathfrak{g} \to \mathfrak{U}_{\mathfrak{z}}(\mathfrak{r})$ with $\eta|_{\mathfrak{r}} = 1$.

We will see that under this assumption, there is a *virtual copy* $\mathfrak{s}_\nu$ of $\mathfrak{s}$ in $\mathfrak{U}_{\mathfrak{z}}(\mathfrak{g})$ such that $\mathfrak{Z}(\mathfrak{s}_\nu) \otimes \mathfrak{Z}_{\mathfrak{z}}(\mathfrak{r})$ and $\mathfrak{Z}_{\mathfrak{z}}(\mathfrak{g})$ are equal *as algebras*. This greatly simplifies the deduction of $\mathfrak{Z}(\mathfrak{g})$.

To our knowledge, general conditions on $\mathfrak{g}$ under which η exists are not known. Such conditions would be interesting. The case that $\mathfrak{g}$ is perfect and maximally centrally extended, *i.e.,* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $H^2(\mathfrak{g}, \mathbb{C}) = 0$, seems to be of particular significance. However, although $\tilde{\mathfrak{g}}_N^J$ has these properties, η does not exist for all such Lie algebras. The semidirect product of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ with its 4-dimensional irreducible representation is a counter example.

Lemma 5.2. *Assuming (5.2), η commutes with the* ad*-action of $\mathfrak{g}$.*

Proof. First prove that η is an $\mathfrak{r}$-map, and note that $\eta|_{\mathfrak{r}} = 1$ is an $\mathfrak{s}$-map. Then prove that $[S, \Theta] = [\eta(S), \Theta]$ for all $S \in \mathfrak{s}$ and $\Theta \in \mathfrak{U}_{\mathfrak{z}}(\mathfrak{r})$. Finally, prove that $\eta|_{\mathfrak{s}}$ is an $\mathfrak{s}$-map. □

The proofs of the next two results are left to the reader.

Corollary 5.3. *Assuming (5.2), the map $\nu := 1 - \eta : \mathfrak{g} \to \mathfrak{U}_{\mathfrak{z}}(\mathfrak{g})$ is a $\mathfrak{g}$-map and a Lie algebra homomorphism. It annihilates $\mathfrak{r}$ and is injective on $\mathfrak{s}$. Its image $\mathfrak{s}_\nu := \nu(\mathfrak{s})$, the* virtual copy *of $\mathfrak{s}$, commutes with $\mathfrak{r}$. The natural multiplication map defines algebra isomorphisms*

$$\mathfrak{U}(\mathfrak{s}_\nu) \otimes \mathfrak{U}_{\mathfrak{z}}(\mathfrak{r}) \cong \mathfrak{U}_{\mathfrak{z}}(\mathfrak{g}), \quad \mathfrak{Z}(\mathfrak{s}_\nu) \otimes \mathfrak{Z}_{\mathfrak{z}}(\mathfrak{r}) \cong \mathfrak{Z}_{\mathfrak{z}}(\mathfrak{g}).$$

Lemma 5.4. *Assumption (5.2) holds for $\tilde{\mathfrak{g}}_N^J$: define η by*

$$\eta(E) := \tfrac{1}{4} e^{\mathrm{T}} Z^{-1} e, \qquad \eta(F) := -\tfrac{1}{4} f^{\mathrm{T}} Z^{-1} f,$$

$$\eta(H) := \tfrac{1}{4}(e^{\mathrm{T}} Z^{-1} f + f^{\mathrm{T}} Z^{-1} e) = \tfrac{1}{2}(N + e^{\mathrm{T}} Z^{-1} f).$$

Proof (of Theorem 5.1). Take $\mathfrak{g} = \tilde{\mathfrak{g}}^J_N$, where $\mathfrak{s} = \mathfrak{sl}_2$. As is well known, the Casimir operator $\Omega_{\mathfrak{sl}_2} := H^2 - 2H + 4EF$ generates $\mathfrak{Z}(\mathfrak{s})$, and so $\nu(\Omega_{\mathfrak{sl}_2}) = \nu(H)^2 - 2\nu(H) + 4\nu(E)\nu(F)$ generates $\mathfrak{Z}(\mathfrak{s}_\nu)$. A short calculation gives $\nu(\Omega_{\mathfrak{sl}_2}) = \det(Z)^{-1}\Omega_N + \frac{1}{4}N(N+4)$. It is not hard to verify that $\mathfrak{Z}(\mathfrak{r})$ is $\mathfrak{U}(\mathfrak{z})$. Clearing denominators and noting that we have polynomial independence, the result follows. □

5.2 The slash actions of $\tilde{\mathfrak{g}}^J_N$

Recall from Section 2 that associated to each scalar cocycle $\gamma : \tilde{G}^J_N \to C^\infty(\mathbb{H}_{1,N})$ there is a right action $|_\gamma$ of $\tilde{G}^J_N$ on $C^\infty(\mathbb{H}_{1,N})$, defined for $g \in \tilde{G}^J_N$ and $x \in \mathbb{H}_{1,N}$ by $\phi|_\gamma[g](x) := \gamma(g,x)\phi(gx)$.

We will denote the differential of γ at the identity, a linear map from $\tilde{\mathfrak{g}}^J_N$ to $C^\infty(\mathbb{H}_{1,N})$, by the same symbol γ. Thus $\gamma(Y,x) := \partial_t|_{t=0}\gamma(e^{tY},x)$ for $Y \in \tilde{\mathfrak{g}}^J_N$. The differential right action $|_\gamma$ of $\tilde{\mathfrak{g}}^J_N$ on $C^\infty(\mathbb{H}_{1,N})$ is

$$\phi|_\gamma[Y](x) \;:=\; \partial_t|_{t=0}\big(\gamma(e^{tY},x)\,\phi(e^{tY}x)\big) \;=\; \gamma(Y,x)\,\phi(x) + \partial_t|_{t=0}\,\phi(e^{tY}x).$$

This action extends to $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$ as usual, and elements of $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$ of order r act by differential operators of order $\leq r$.

Following Definition 2.3, let $\mathbb{D}_\gamma$ denote the algebra of differential operators on $\mathbb{H}_{1,N}$ invariant with respect to the action $|_\gamma$ of $\tilde{G}^J_N$. Since $\tilde{G}^J_N$ is connected, $\mathbb{D}_\gamma$ is precisely those operators commuting with the $|_\gamma$-action of $\tilde{\mathfrak{g}}^J_N$. In particular, $|_\gamma$ maps the center $\mathfrak{Z}(\tilde{\mathfrak{g}}^J_N)$ of $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$ into the center $Z(\mathbb{D}_\gamma)$ of $\mathbb{D}_\gamma$. (In [BCR] it is proven that $\mathfrak{Z}(\tilde{\mathfrak{g}}^J_1)$ covers $Z(\mathbb{D}_\gamma)$ for all scalar cocycles of $\tilde{G}^J_1$. It would be interesting to decide this question for $N > 1$.)

Proof (of Theorem 2.4). Recall the cocycles $\beta^k\alpha_L$ from Lemma 2.2 defining the slash actions $|_{k,L}$. Easy calculations using (5.1) give the actions of our basis of $\tilde{\mathfrak{g}}^J_N$: writing $\L$ for $2\pi i L$ and $A[w]$ for $w^{\mathrm{T}}Aw$ as earlier,
(5.3)

$$\begin{aligned}
|_{k,L}[E] &= 2\mathfrak{Re}(\partial_\tau),\\
|_{k,L}[F] &= -2\mathfrak{Re}\,\tau(\tau\partial_\tau + z^{\mathrm{T}}\partial_z) - k\tau - \L[z],\\
|_{k,L}[H] &= 2\mathfrak{Re}(2\tau\partial_\tau + z^{\mathrm{T}}\partial_z) + k,\\
|_{k,L}[0,X,\kappa] &= 2\mathfrak{Re}(X_1\tau + X_2)^{\mathrm{T}}\partial_z + 2X_1^{\mathrm{T}}\L z + \mathrm{tr}(\kappa\L).
\end{aligned}$$

In using these formulas to compute the $|_{k,L}$-actions of elements of $\mathfrak{U}(\tilde{\mathfrak{g}}^J_N)$, care must be taken to reverse the order of multiplication, because $|_{k,L}$ is a right action but the formulas (5.3) are given in left-acting notation. Thus for example $\phi|[Y_1Y_2]$ is $\big(\phi|[Y_1]\big)|[Y_2]$.

Since $|_{k,L}[Z_{ij}] = \L_{ij}$, Theorem 5.1 shows that $|_{k,L}[\Omega_N]$ generates the $|_{k,L}$-action of $\mathfrak{Z}(\tilde{\mathfrak{g}}^J_N)$. Using (5.3), a straightforward but long computation gives □

$$|_{k,L}[\Omega_N] \;=\; \det(\L)\big(k(k-N-2) - 2\mathcal{C}^{k,L}\big).$$

5.3 Covariant differential operators

In order to prove Propositions 2.6, 2.7, and 2.8, we must recall the algebraic side of the general theory of invariant differential operators (IDOs) developed by Helgason in the 1950's (see, *e.g.,* Section 2 of [Hel77]). Here we will adapt the framework of Section 4 of [BCR] from IDOs to *covariant differential operators* (CDOs), by regarding them as nilpotent IDOs on the direct sum of the range and domain spaces. Thus to treat scalar-valued CDOs we must consider vector-valued IDOs.

Let G be a real Lie group, K a closed subgroup, and V a complex vector space. Given $x \in G$, denote the coset xK by $\overline{x}$. A *V-valued 1-cocycle of G on G/K* is a smooth function

$$\gamma : G \times G/K \to \mathrm{GL}(V) \text{ satisfying } \gamma(gh, \overline{x}) = \gamma(h, \overline{x})\gamma(g, h\overline{x})$$

for all g, h, $x \in G$. The associated right action of G on $C^\infty(G/K) \otimes V$ is

$$f|_\gamma[g](\overline{x}) \;:=\; \gamma(g, \overline{x}) f(g\overline{x}),$$

and the associated representation of K on V is $\pi_\gamma(k) := \gamma(k, \overline{e})^{-1}$.

Suppose that V' is a vector space of the same dimension as V, and γ' is a V'-valued 1-cocycle of G on G/K. Then γ' is said to be *cohomologous* to γ if there is a smooth map b from G/K to the set of invertible linear maps from V to V' such that $\gamma'(g, \overline{x}) b(g\overline{x}) = b(\overline{x})\gamma(g, \overline{x})$. In this case $f \mapsto bf$ is an equivalence from $|_\gamma$ to $|_{\gamma'}$, and $b(\overline{e})$ is an equivalence from π_γ to $\pi_{\gamma'}$. Conversely, if π_γ and $\pi_{\gamma'}$ are equivalent, then γ and γ' are cohomologous. If G/K is simply connected, then given any complex finite dimensional representation π of K there exists a cocycle γ such that $\pi_\gamma = \pi$, and so one has a natural bijection between slash actions of G on G/K and representations of K.

We now define CDOs in the general setting. Let V and V' be any two vector spaces, not necessarily related. Fix 1-cocycles γ and γ' of G on G/K taking values in V and V', respectively.

Definition 5.5. *A differential operator $T : C^\infty(G/K) \otimes V \to C^\infty(G/K) \otimes V'$ is* covariant from $|_\gamma$ to $|_{\gamma'}$ *if for all $g \in G$ and $f \in C^\infty(G/K) \otimes V$, we have*

$$T\big(f|_\gamma[g]\big) \;=\; (Tf)\big|_{\gamma'}[g].$$

Let $\mathbb{D}_{\gamma,\gamma'}(G/K)$ be the space of CDOs from $|_\gamma$ to $|_{\gamma'}$, and let $\mathbb{D}^r_{\gamma,\gamma'}(G/K)$ be the space of those of order $\le r$. When $\gamma = \gamma'$, we refer to such operators as $|_\gamma$-invariant *and write simply $\mathbb{D}_\gamma(G/K)$ and $\mathbb{D}^r_\gamma(G/K)$.*

The following proposition adapts Section 4.3 of [BCR] to CDOs. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the complexified Lie algebras of G and K, and assume that the pair $K \subseteq G$ is *reductive, i.e.,* there exists a K-splitting $\mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g}$. Recall that $\mathcal{S}$ denotes the symmetric algebra and superscripts indicate invariants.

Proposition 5.6. *There exists a filtration-preserving linear bijection*

$$\mathrm{CDO}_{\gamma,\gamma'} : \big(\mathcal{S}(\mathfrak{m}) \otimes \mathrm{Hom}(V, V')\big)^K \to \mathbb{D}_{\gamma,\gamma'}(G/K).$$

It is compatible with multiplication at the symbol level: if γ'' is a third cocycle taking values in a space V'' and Θ and Θ' are K-invariant elements of $\mathcal{S}(\mathfrak{m}) \otimes \mathrm{Hom}(V, V')$ and $\mathcal{S}(\mathfrak{m}) \otimes \mathrm{Hom}(V', V'')$, respectively, then $\mathrm{CDO}_{\gamma,\gamma''}(\Theta'\Theta)$ and $\mathrm{CDO}_{\gamma',\gamma''}(\Theta') \circ \mathrm{CDO}_{\gamma,\gamma'}(\Theta)$ have the same symbol.

Proof. Construct a $V \oplus V'$-valued cocycle $\gamma \oplus \gamma'$ in the obvious way. Equation (38) of [BCR] defines a filtration-preserving linear bijection

$$\mathrm{IDO}_{\gamma\oplus\gamma'} : \big(\mathcal{S}(\mathfrak{m}) \otimes \mathrm{End}(V \oplus V')\big)^K \to \mathbb{D}_{\gamma\oplus\gamma'}(G/K)$$

which is an algebra isomorphism at the symbol level. Tracing the definitions leading to (38) shows that $\mathrm{IDO}_{\gamma\oplus\gamma'}$ restricts to the desired map $\mathrm{CDO}_{\gamma,\gamma'}$. □

We now give a general result (probably already known) of independent interest: in the reductive case, the CDOs of order 1 generate all CDOs. We do not know if it holds in the absence of reductivity.

Proposition 5.7. *Let $K \subseteq G$ be reductive. Then all CDOs of order r are linear combinations of compositions of up to r CDOs of order 1.*

Proof. By Proposition 5.6, it suffices to show that $\big(\mathcal{S}^r(\mathfrak{m}) \otimes \mathrm{Hom}(V, V')\big)^K$ is contained in the product

$$\mathrm{Hom}(V_r, V')^K \big(\mathfrak{m}\otimes\mathrm{Hom}(V_{r-1}, V_r)\big)^K \cdots \big(\mathfrak{m}\otimes\mathrm{Hom}(V_1, V_2)\big)^K \big(\mathfrak{m}\otimes\mathrm{Hom}(V, V_1)\big)^K$$

for some representations $V_1, \ldots, V_r$ of K (the first factor contains order 0 operators and can be merged with the second factor).

Set $V_s := \mathcal{S}^s(\mathfrak{m}^*) \otimes V$. Fix any basis $\{X_j\}$ of $\mathfrak{m}$, and let $\{X_j^*\}$ be the dual basis of $\mathfrak{m}^*$. Let I_s be $\sum_j X_j X_j^*$, regarded as an element of $\mathfrak{m} \otimes \mathrm{Hom}(V_{s-1}, V_s)$ in the natural way. Verify that as such, it is K-invariant. We will in fact prove that $\big(\mathcal{S}^r(\mathfrak{m}) \otimes \mathrm{Hom}(V, V')\big)^K$ is equal to $\mathrm{Hom}(V_r, V')^K I_r \cdots I_1$.

For this, it is enough to show that right composition with $I_r \cdots I_1$ is an injection from $\mathrm{Hom}(V_r, V')$ to $\mathcal{S}^r(\mathfrak{m}) \otimes \mathrm{Hom}(V, V')$, as these two representations of K are equivalent. Using (subscript) monomial notation X_J and X_J^*, observe that $I_r \cdots I_1$ may be written as $\sum_J \binom{r}{J} X_J X_J^*$. Given H in $\mathrm{Hom}(V_r, V')$, regard it as a map from $\mathcal{S}^r(\mathfrak{m}^*)$ to $\mathrm{Hom}(V, V')$. Then $H I_r \cdots I_1$ is $\sum_J \binom{r}{J} X_J \otimes H(X_J^*)$, proving the injectivity. □

The construction in the preceding proof is inefficient in practice. Let us describe a more useful approach in the case that K is abelian, G/K is simply connected, and we restrict to cocycles γ such that π_γ is a completely reducible representation of K. Here the irreducible representations of K are 1-dimensional, so it suffices to prove the result for CDOs between scalar slash actions. Given a scalar 1-cocycle γ, write $\mathbb{C}_\gamma$ for $\mathbb{C}$ endowed with the K-action π_γ, and 1_γ for $1 \in \mathbb{C}_\gamma$. If γ' is a second cocycle, then γ'/γ is again a cocycle and $\mathrm{Hom}(\mathbb{C}_\gamma, \mathbb{C}_{\gamma'})$ is K-equivalent to $\mathbb{C}_{\gamma'/\gamma}$. Therefore by Proposition 5.6, there is an order-preserving bijection

$$\mathbb{D}_{\gamma,\gamma'}(G/K) \cong \big(\mathcal{S}(\mathfrak{m}) \otimes \mathbb{C}_{\gamma'/\gamma}\big)^K.$$

The space on the right is essentially the $\pi_{\gamma/\gamma'}$-isotype of $\mathcal{S}(\mathfrak{m})$. Let $\{X_j\}$ be a K-eigenbasis of $\mathfrak{m}$, and for each j let χ_j be a scalar cocycle such that $\mathbb{C}X_j$ is a copy of π_{1/χ_j} under K (such χ_j exist because G/K is simply connected). Then for each scalar cocycle γ we have the order 1 CDO

(5.4)
$$X_j^\gamma \;:=\; \mathrm{CDO}_{\gamma,\gamma\chi_j}(X_j \otimes 1_{\chi_j}) \;\in\; \mathbb{D}^1_{\gamma,\gamma\chi_j}(G/K).$$

Note that by Lemma 4.6 of [BCR], the symbol of X_j^γ is independent of γ. For clarity we will sometimes use the notation X_j^{CDO} for the diagonal action of $\oplus_\gamma X_j^\gamma$ on the algebraic direct sum of all the scalar slash actions. The various X_j^{CDO} do not necessarily commute, but since we are in the scalar setting their commutators are of order ≤ 1. More precisely,

(5.5)
$$X_{j'}^{\gamma\chi_j} \circ X_j^\gamma - X_j^{\gamma\chi_{j'}} \circ X_{j'}^\gamma \;\in\; \mathrm{Span}\big\{X_{j''}^\gamma : \chi_{j''} = \chi_{j'}\chi_j\big\} \;\oplus\; \delta_{1,\chi_{j'}\chi_j}\mathbb{C}1.$$

(The Kronecker delta coefficient of the final summand $\mathbb{C}1$ indicates that it should be omitted unless $\chi_{j'}\chi_j = 1$.) This discussion gives the following corollary of Proposition 5.7 (the last statement of the corollary follows from an examination of the definition of $\mathrm{CDO}_{\gamma,\gamma'}$; see Proposition 5.6 and [BCR]).

Corollary 5.8. *Assume that $K \subseteq G$ is reductive, K is abelian, and G/K is simply connected. Let γ and γ' be scalar cocycles. Then*

$$\mathbb{D}^r_{\gamma,\gamma'}(G/K) \;\text{ has basis }\; \big\{\textstyle\prod_j (X_j^{\mathrm{CDO}})^{J_j} : \prod_j \chi_j^{J_j} = \gamma'/\gamma,\ \sum_j J_j \leq r\big\}.$$

The symbol of X_j^{CDO} at the base point $\overline{e}$ coincides with that of the left action of X_j, which in turn coincides with that of $-|_\gamma[X_j]$ for all γ.

We now specialize to the setting of Section 2. Recall that the subgroup of $\tilde{G}^J_N$ stabilizing the point $(i,0)$ in $\mathbb{H}_{1,N}$ is $\tilde{K}^J_N = \mathrm{SO}_2 \times \{0\} \times \mathrm{M}^T_N(\mathbb{R})$. This group has complex Lie algebra $\tilde{\mathfrak{k}}^J_N := \mathfrak{o}_2 \times \{0\} \times \mathrm{M}^T_N(\mathbb{C})$.

Define a linear map $\tau : \tilde{\mathfrak{g}}^J_N \to \tilde{\mathfrak{g}}^J_N$ by $\tau(X) = \tilde{X}$, where

$$\begin{aligned} \tilde{H} &:= i(F-E), & \tilde{E} &:= \tfrac{1}{2}\big(H + i(F+E)\big), & \tilde{F} &:= \tfrac{1}{2}\big(H - i(F+E)\big), \\ \tilde{Z}_{jk} &:= \tfrac{1}{2} i Z_{jk}, & \tilde{e}_j &:= \tfrac{1}{2}(f_j + ie_j), & \tilde{f}_j &:= \tfrac{1}{2}(f_j - ie_j). \end{aligned}$$

One checks that there is a unique $\tilde{K}^J_N$-splitting $\tilde{\mathfrak{k}}^J_N \oplus \tilde{\mathfrak{m}}^J_N$ of $\tilde{\mathfrak{g}}^J_N$, given by

$$\mathfrak{k}^J_N = \mathrm{Span}\big\{\tilde{H},\, \tilde{Z}_{jk} : 1 \leq j \leq k \leq N\big\}, \quad \mathfrak{m}^J_N := \mathrm{Span}\big\{\tilde{E},\, \tilde{F},\, \tilde{e}_j,\, \tilde{f}_j : 1 \leq j \leq N\big\}.$$

It is simple to verify that τ is an automorphism, and so the given basis of $\tilde{\mathfrak{m}}_N^J$ is a $\tilde{K}_N^J$-eigenbasis: the $\tilde{H}$-weights of $\tilde{E}$, $\tilde{F}$, $\tilde{e}_j$, and $\tilde{f}_j$ are 2, -2, 1, and -1, respectively.

Write $\pi_{k,L}$ for the $\tilde{K}_N^J$-character $\pi_{\beta^k\alpha_L}$ associated to the scalar cocycle $\beta^k\alpha_L$ defining the slash action $|_{k,L}$, and $\mathbb{C}_{k,L}$ for its space. Check that $\pi_{k,L}$ is determined by $\tilde{H} \mapsto -k$ and $\tilde{Z}_{ij} \mapsto \pi L_{ij}$. Hence $\tilde{E}$, $\tilde{F}$, $\tilde{e}_j$, and $\tilde{f}_j$ span copies of $\mathbb{C}_{-2,0}$, $\mathbb{C}_{2,0}$, $\mathbb{C}_{-1,0}$, and $\mathbb{C}_{1,0}$, respectively, and so $\tilde{E}^{\mathrm{CDO}}$, $\tilde{F}^{\mathrm{CDO}}$, $\tilde{e}_j^{\mathrm{CDO}}$, and $\tilde{f}_j^{\mathrm{CDO}}$ are CDOs from $|_{k,L}$ to $|_{k+2,L}$, $|_{k-2,L}$, $|_{k+1,L}$, and $|_{k-1,L}$, respectively.

We remark that τ maps the Casimir element Ω_N to $\left(\frac{i}{2}\right)^N \Omega_N$. The second order operators $|_{k,L}[\nu(\tilde{H})]$, $|_{k,L}[\nu(\tilde{E})]$, and $|_{k,L}[\nu(\tilde{F})]$ are the analogs for $N > 1$ of the operators Δ_1 and $D_\pm$ given on p. 38 of [BS98].

Proof (of Proposition 2.6). By the first paragraph of Corollary 5.8, $\{\tilde{E}^{\mathrm{CDO}}\}$, $\{\tilde{F}^{\mathrm{CDO}}\}$, $\{\tilde{e}_j^{\mathrm{CDO}} : 1 \le j \le N\}$, and $\{\tilde{f}_j^{\mathrm{CDO}} : 1 \le j \le N\}$ are bases of $\mathbb{D}^1(k, L;\, k+2, L)$, $\mathbb{D}^1(k, L;\, k-2, L)$, $\mathbb{D}^1(k, L;\, k+1, L)$, and $\mathbb{D}^1(k, L;\, k-1, L)$, respectively, and there are no other CDOs of order 1.

By (5.3) and Corollary 5.8, the symbols of the order 1 CDOs at $(i, 0)$ are (using vector notation for e and f)

$$\tilde{E}^{\mathrm{CDO}} \equiv -2i\partial_\tau, \quad \tilde{F}^{\mathrm{CDO}} \equiv 2i\partial_{\overline{\tau}}, \quad \tilde{e}^{\mathrm{CDO}} \equiv -i\partial_z, \quad \tilde{f}^{\mathrm{CDO}} \equiv i\partial_{\overline{z}}.$$

Once we prove that $X_\pm$ and $Y_\pm$ really are CDOs, it will follow that

$$\tilde{E}^{\mathrm{CDO}} = -X_+, \quad \tilde{F}^{\mathrm{CDO}} = -X_-, \quad \tilde{e}^{\mathrm{CDO}} = -Y_+, \quad \tilde{f}^{\mathrm{CDO}} = -Y_-,$$

and the first paragraph of the proposition will be proven. One could carry this out by applying the map $\mathrm{CDO}_{\gamma,\gamma'}$. Our method was to guess the CDOs from the $N = 1$ case given in [BS98] and check them with a computer. One can also proceed as follows: note that any order 1 CDO from $|_{k,L}$ to $|_{k',L'}$ lies in the $C^\infty(\mathbb{H}_{1,N})$-span of $\{1, \partial_\tau, \partial_{\overline{\tau}}, \partial_{z_j}, \partial_{\overline{z}_j} : 1 \le j \le N\}$, and use covariance to solve for the coefficients. For example, covariance with respect to E and e implies that the coefficients depend only on y and v. Covariance with respect to Z and H implies that $L' = L$ and the entire operator is of weight $k - k'$ under the Euler operator $2\tau\partial_\tau + 2\overline{\tau}\partial_{\overline{\tau}} + z^{\mathrm{T}}\partial_z + \overline{z}^{\mathrm{T}}\partial_{\overline{z}}$. The condition for covariance with respect to F and f is harder but can be deduced by hand, leading to the formulas for $X_\pm$ and $Y_\pm$.

To prove the second paragraph of the proposition, use (5.5). It implies that X_+ and $Y_{+,j}$ commute among themselves, X_- and $Y_{-,j}$ commute among themselves, $[X_+, X_-]$ and $[Y_{+,j}, Y_{-,k}]$ are constants, and $[X_\pm, Y_{\mp,j}]$ is in the span of the $Y_{\pm,k}$. To deduce the constants and the coefficients, apply the commutators to 1, or if that fails, to $\overline{z}$. □

Proof (of Proposition 2.7). The first paragraph holds by Corollary 5.8 and the fact that both the raising operators and the lowering operators commute among themselves. For the second paragraph, use (2.4) and match the τ- and z-symbols separately to see (easily) that $\mathcal{C}^{k,L}$ minus the first two lines on the right of (2.6) is in the span of the operators $Y_{+,j}Y_{-,k}$. The coefficients can be deduced by applying both sides to $\overline{z}$. $\square$

Proof (of Proposition 2.8). This follows easily from Proposition 2.6 and Corollary 5.8. $\square$

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