REPRESENTATION SPACES OF PRETZEL KNOTS

RAPHAEL ZENTNER

Abstract. We study the representation spaces $R(K;\mathbf{i})$ as appearing in Kronheimer and Mrowka's framed instanton knot Floer homology, for a class of pretzel knots. In particular, for pretzel knots $P(p,q,r)$ with p,q,r odd, positive and pairwise coprime, these appear to be non-degenerate and comprise representations in $SU(2)$ that are not binary dihedral.

1. Introduction

Let K be a knot in the 3-sphere, and y_0 a point in its complement. Let furthermore m be a meridian to the knot. In the construction of framed instanton knot Floer homology [8–10] there appear at the chain group level representation spaces

$$R(K;\mathbf{i}) = \{\rho \in \mathrm{Hom}(\pi_1(S^3 \setminus K; y_0), SU(2)) \mid \rho(m) \sim \mathbf{i}\}$$

of knots with the meridian m (or links with each of their meridians) mapped to traceless matrices, or, equivalently, to elements that are conjugate to $\mathbf{i}$ when $SU(2)$ is viewed as the group of unit quaternions.

Our intention is to study these representation spaces for a class of pretzel knots. In particular, we describe the conjugacy classes of representations of $P(p,q,r)$ by triangles on the 2-sphere, with the binary dihedral representations appearing as degenerate triangles in Section 3. More generally, we show that for knots $K = P(p_1,\dots p_n)$ the conjugacy classes of representations are described by n-gons on the 2-sphere. In Section 6, we show that under some arithmetic conditions all the irreducible representations of $P(p,q,r)$ are non-degenerate. However, in the case $n \geq 4$ the representation spaces $R(K;\mathbf{i})$ appear to be degenerate even though they remain non-degenerate at the binary dihedral representations if certain arithmetic conditions are satisfied. For pretzel knots $P(p,q,r)$ with p,q,r odd, positive (or all of the same sign) and pairwise coprime we get a very complete picture in Section 7 below.

Kronheimer and Mrowka observed in [8, Observation 1.1] that for $(2,p)$ torus knots K there is an isomorphism

$$H_*(R(K;\mathbf{i});\mathbb{Z}) \cong Kh(K)\,. \tag{1}$$

Lewallen has proved [in the current version by use of an unpublished result of Shumakovitch [14]] that Khovanov homology of a one or two component alternating link is isomorphic to the integer homology of $R_{bd}(K;\mathbf{i})$, where $R_{bd}(K;\mathbf{i}) \subseteq R(K;\mathbf{i})$ is the subspace of binary dihedral representations [11]. This extends (1) to 2-bridge knots. Our results show that the above isomorphism does not extend to pretzel knots $P(p,q,r)$ with p,q,r positive, odd, pairwise coprime, and such that $p,q,r > 1$. These knots are still alternating and form a natural class of knots with bridge number 3. In fact, we show that these knots admit representations $\rho \in R(K;\mathbf{i})$ that are not binary dihedral, and that we have $H_*(R(K;\mathbf{i});\mathbb{Z}) \neq H_*(R_{bd}(K;\mathbf{i});\mathbb{Z})$.

We shall also outline that for $P(p,q,r)$ the associated chain complex yielding framed instanton Floer homology $FI_*(P(p,q,r))$ may be expected to have non-trivial differentials. It would be interesting to understand these differentials in detail. Possibly this could be used for investigating Question 1.2 of [8] which asks whether for all alternating knot K there is an isomorphism $FI_*(K) \cong Kh(K) \oplus Kh(K)$. This holds over $\mathbb{Q}$ for the 'reduced' versions of instanton knot Floer homology and Khovanov homology [9, Corollary 1.6]. In general, Kronheimer and Mrowka have constructed a spectral sequence starting at (reduced) Khovanov homology and converging to (reduced) framed instanton knot Floer homology [9].

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2. Presentations of pretzel knot groups

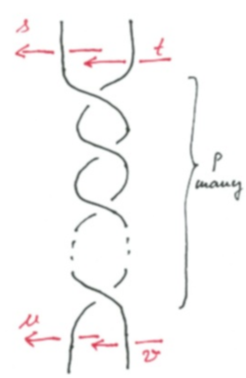
Let us consider the elementary 'p-tangle' as in the figure. This means that we have a braid with p crossings, all of them negative. Let s and t be meridians at the top as indicated (with the basepoint in front of the eye of the observer), and u and v at the bottom. Then we have

$$\begin{aligned} u &= (ts)^{-k}\ s^{-1}ts\ (ts)^k, \\ v &= (ts)^{-k}\ s\ (ts)^k \end{aligned}$$

for $p = 2k+1$ odd, and

$$\begin{aligned} u &= (ts)^{-k}\ s\ (ts)^k, \\ v &= (ts)^{-(k-1)}\ s^{-1}ts\ (ts)^{k-1} \end{aligned}$$

for $p = 2k$ even.



With these formulae at hand we get a presentation of the complement of any pretzel knot rather quickly. Indeed, consider an arbitrary pretzel knot or link, denoted $P(p_1, \ldots, p_n)$ – see for instance [12, Figure 1.7] for its 'standard diagram'. In this diagram it becomes obvious that it can be visualised as a knot or link with n bridges or a $2n$–plat. We therefore see that there is an embedded 2-sphere in the 3-sphere cutting the $P(p_1, \ldots, p_n)$ pretzel knot/link in $2n$ points such that the resulting balls each contain $2n$ unknotted arcs with boundaries on the boundary 2-sphere. We therefore see that the knot or link complement has a decomposition into two pieces which each are (or deformation retract onto) two handlebodies of genus n with common intersection a two-sphere punctured in $2n$ discs, with these discs centered at the intersection points of the knot or link with this 2-sphere. The fundamental group of each handlebody is a free group on n generators given by the meridians of the knot/link in the corresponding ball.

Now let $s_1, \ldots, s_n$ be these meridians in the upper and $u_1, \ldots, u_n$ be the merdians in the lower handlebody. We orient these meridians so that at each 'elementary p-tangle' at the position i we have s_i corresponding to s in the picture above, and s_{i+1}^{-1} corresponding to t in the picture above.

The Seifert-van-Kampen theorem now gives a presentation of the knot/link complement in a straight-forward manner. For simplicity, we give the presentation in the case where all p_i are odd and positive. This corresponds to an alternating negative 2-component link if n is even and to an alternating negative knot if n is odd.

Proposition 2.1. *The fundamental group $G(K) := \pi_1(S^3 - P(p_1, \dots, p_n), y_0)$ of the complement of the pretzel knot or link $P(p_1, \dots, p_n)$ with all $p_i = 2k_i + 1$ odd and positive is given by*

$$\begin{aligned}\langle s_1, \dots, s_n | &(s_2^{-1}s_1)^{-k_1} s_1 (s_2^{-1}s_1)^{k_1} \ (s_3^{-1}s_2)^{-k_2} s_2^{-1} s_3^{-1} s_2 (s_3^{-1}s_2)^{k_2} = 1 \ , \\ &(s_3^{-1}s_2)^{-k_2} s_2 (s_3^{-1}s_2)^{k_2} \ (s_3^{-1}s_3)^{-k_3} s_3^{-1} s_4^{-1} s_3 (s_4^{-1}s_3)^{k_3} = 1 \ , \\ &\dots \ , \\ &(s_n^{-1}s_{n-1})^{-k_{n-1}} s_{n-1} (s_n^{-1}s_{n-1})^{k_{n-1}} \ (s_1^{-1}s_n)^{-k_n} s_n^{-1} s_1^{-1} s_n (s_1^{-1}s_n)^{k_n} = 1 \ , \\ &(s_1^{-1}s_n)^{-k_n} s_n (s_1^{-1}s_n)^{k_n} \ (s_2^{-1}s_1)^{-k_1} s_1^{-1} s_2^{-1} s_1 (s_2^{-1}s_1)^{k_1} = 1\rangle.\end{aligned}$$

Each generator is a meridian, and any relation is a consequence of all others, so (any) one relation may be omitted.

Proof: The relations that are added by the Seifert-van-Kampen theorem are $v_1u_2 = 1$, $v_2u_3 = 1$, ..., $v_{n-1}u_n = 1$, $v_nu_1 = 1$. Now as $v_nu_nv_{n-1}u_{n-1} \dots v_1u_1 = 1$ it is clear that any of the n relations can be omitted. □

3. The representation space

The space $R(K; \mathbf{i})$ consists of representations $\rho \in \mathrm{Hom}(G(K), SU(2))$ with $\rho(m) \sim \mathbf{i}$, where m is a preferred element (or a set of preferred elements in the case of a link, and the condition of being conjugated to $\mathbf{i}$ is satisfied for each element of the set). Any element in $SU(2)$ with zero trace has order 4 and square -1. Therefore, any representation $\rho \in R(K, \mathbf{i})$ factors through the group

$$G(K)_{m,\mathbf{i}} := (G(K) \times \mathbb{Z}/2)/\langle m^2(-1)\rangle \ , \tag{2}$$

where $\langle m^2(-1)\rangle$ denotes the normal subgroup generated by $m^2(-1)$. This observation simplifies the description of $R(K, \mathbf{i})$ considerably. We denote by $R(G(K)_{m,\mathbf{i}})$ the representation space $\mathrm{Hom}(G(K)_{m,\mathbf{i}}, SU(2))$.

Proposition 3.1. *The canonical homomorphism $\pi : G(K) \to G(K)_{m,\mathbf{i}}$ induces a homeomorphism*

$$\pi^* : R(G(K)_{m,\mathbf{i}}) \to R(K; \mathbf{i}) \ ,$$

with both representation spaces seen as subspaces of $Map(G(K)_{m,\mathbf{i}}, SU(2))$, respectively $Map(G(K), SU(2))$, and with these mapping spaces topologised by the compact-open topology determined by the standard topology on $SU(2)$ and the discrete topology on the groups.

Proof: Any element in $R(K; \mathbf{i})$ is in the image of π^* by construction of $G(K)_{m,\mathbf{i}}$, and as π is surjective the map π^* is injective likewise. It is an easy matter to check continuity and openness of the map π^* directly from the definition of the compact-open topology. □

Proposition 3.2. *For the pretzel knot or link $K = P(p_1, \dots, p_n)$ with all $p_i = 2k_i + 1$ odd and positive, and with m denoting the set of (conjugacy classes of) meridians, we have a presentation of $G(K)_{m,\mathbf{i}}$ given by*

$$\langle s_1, \dots, s_n, -1 | [-1, s_i] = 1, (s_i)^2 = -1, i = 1, \dots, n, (-1)^2 = 1,$$
$$(s_1 s_2)^{p_1} = (s_2 s_3)^{p_2} = \dots = (s_{n-1} s_n)^{p_{n-1}} = (s_n s_1)^{p_n} \rangle.$$

Proof: In the presentation of Proposition 2.1 any generator is a meridian m. Therefore, the relations in the first line follow directly from the definition of $G_{m,\mathbf{i}}$. The remaining relations of the presentation 2.1 simplify because any of the elements s_i now satisfies $s_i^{-1} = -s_i$, and because any of the p_i was assumed odd. □

The elements $\rho \in R(P(p_1, \dots, p_n); \mathbf{i})$ fall into two classes, depending on whether $\rho((s_1 s_2)^{p_1}) = \pm 1$ or not. As we shall see, if this is not the case, then the representation ρ is binary dihedral, that is, it factors through a subgroup of $SU(2)$ that is conjugate to

$$Pin(2) = S^1 \coprod \mathbf{j} \cdot S^1 \ ,$$

where $SU(2)$ is seen as the unit quaternions, $S^1 \subseteq \mathbb{C} = \langle 1, \mathbf{i} \rangle \subseteq \mathbb{H}$ the unit complex numbers, and $\mathbf{j} \cdot S^1 \subseteq \langle \mathbf{j}, \mathbf{k} \rangle \subseteq \mathbb{H}$ the circle of unit complex numbers multiplied by $\mathbf{j}$, lying entirely in the space spanned by $\mathbf{j}$ and $\mathbf{k}$.

Before we proceed, we shall note a useful formula: Let $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and $\mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$ be purely imaginary quaternions. Then we have

$$\mathbf{v} \cdot \mathbf{w} = -\langle \mathbf{v}, \mathbf{w} \rangle + \mathbf{v} \times \mathbf{w} \ , \tag{3}$$

where

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2) \mathbf{i} + (v_3 w_1 - v_1 w_3) \mathbf{j} + (v_1 w_2 - v_2 w_1) \mathbf{k} \ ,$$

and where $\langle -, - \rangle$ denotes the standard scalar product. As the notation suggests, this corresponds to the usual 'cross-product' in $\mathbb{R}^3$. In particular, if $\mathbf{v}$ and $\mathbf{w}$ are linearly independent the vector $\mathbf{v} \times \mathbf{w}$ is perpendicular to the plane spanned by $\mathbf{v}$ and $\mathbf{w}$.

3.1. **A conjugacy condition.**

Lemma 3.3. *Let K be an arbitrary knot, and assume P and Q are two meridional generators of the knot group (among possibly others). Let $\rho \in R(K; \mathbf{i})$. Up to conjugation we may assume that*

$$\rho(P) = \mathbf{j}, \quad \textit{and} \quad \rho(Q) = \mathbf{j} \cdot e^{\mathbf{i}\alpha} \ ,$$

with $\alpha \in [0, 2\pi]$. Furthermore, assume that one has

$$\rho(P) = \mathbf{j}, \ \rho(Q) = \mathbf{j} \cdot e^{\mathbf{i}\alpha}, \ \textit{and}$$
$$\rho'(P) = \mathbf{j}, \ \rho'(Q) = \mathbf{j} \cdot e^{\mathbf{i}\alpha'} \ ,$$

with α, α' not both equal to 0 or both equal to π. Then ρ and ρ' are conjugated inside $SU(2)$ only if $\rho(Q) = \rho'(Q)$ or

$$\rho'(Q) = \mathbf{j} \cdot \rho(Q) \cdot \mathbf{j}^{-1} \ .$$

In the latter case $e^{\mathbf{i}\alpha'} = e^{-\mathbf{i}\alpha}$, i.e. $\rho'(Q) \in S^2 \subseteq \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ is obtained from $\rho(Q)$ by rotation around $\mathbf{j}$ by angle π.

Proof: Up to conjugation we may assume $\rho(P) = \mathbf{j}$ because the action of $SU(2)$ on the 2-sphere $S^2 \subseteq \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ of elements conjugated to $\mathbf{i}$ is transitive by definition. This assumption does not yet fix ρ up to conjugation. In fact, we have $c\,\mathbf{j}\,c^{-1} = \mathbf{j}$ if and only if $c \in SU(2)$ is of the form $c = w + y\,\mathbf{j}$ with $w, y \in \mathbb{R}$ and $w^2 + y^2 = 1$. The set of these elements is a 1-dimensional circle, and conjugating with an element c of this circle yields a rotation around the axis $\mathbf{j}$ in $\langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle \cong \mathbb{R}^3$. Therefore, we may assume that $\rho(Q)$ lies in the $\langle \mathbf{i}, \mathbf{k} \rangle$–plane, and so may be written $\rho(Q) = \mathbf{j} \cdot e^{\mathbf{i}\alpha}$ as claimed. □

3.2. **The case** $\rho((s_1 s_2)^{p_1}) = \rho((s_2 s_3)^{p_2}) = \cdots = \rho((s_n s_1)^{p_n}) = +1$**.** This class may have binary dihedral representations and may and usually does contain representations that are not binary dihedral.

Lemma 3.4. *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S^2 \subseteq \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$*, and suppose* $\mathbf{y} = \mathbf{x} \cdot e^{\mathbf{z}\alpha}$ *for some* $\alpha \in [0, 2\pi]$*. Suppose* $\mathbf{x} \neq \pm\mathbf{y}$*. Then* $\mathbf{z}$ *must be perpendicular to* $\mathbf{x}$*.*

Proof: As usually one proves that $e^{\mathbf{x}\alpha} = \cos(\alpha) + \mathbf{x}\sin(\alpha)$. Therefore we see that

$$\mathbf{x} \cdot e^{\mathbf{z}\alpha} = \mathbf{x}\,\cos(\alpha) + \mathbf{x} \cdot \mathbf{z}\,\sin(\alpha)\ . \tag{4}$$

As we were assuming $\mathbf{x} \neq \pm\mathbf{y}$ the angle α cannot be 0 or π, and so $\sin(\alpha) \neq 0$. By the above formula (3) we see that this is purely imaginary if and only if $\mathbf{x}$ and $\mathbf{z}$ are perpendicular. □

Up to conjugation we may assume that $\rho(s_1) = \mathbf{j}$, and $\rho(s_2) = \mathbf{j} \cdot e^{\mathbf{i}\alpha_{12}}$, with angle $\alpha_{12} \in [0, \pi]$ satisfying

$$p_1\,\alpha_{12} \equiv 1 \pmod{2\pi}\ .$$

We may write $\rho(s_3) = \rho(s_2) \cdot e^{\mathbf{x}\,\alpha_{23}}$, where $\mathbf{x} \in S^2 \subseteq \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ is a purely imaginary quaternion of norm 1 which is perpendicular to $\rho(s_2) \in S^2 \subseteq \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$, and with angle $\alpha_{23} \in [0, \pi]$ satisfying

$$p_2\,\alpha_{23} \equiv 1 \pmod{2\pi}\ .$$

We notice that for given angle α_{23} there is a circle of possibilities for the choice of $\rho(s_3)$ parametrised by the circle of elements $\mathbf{x}$ which are pependicular to $\rho(s_2)$, as long as α_{23} is different from 0 and π. Furthermore, α_{23} is the distance between $\rho(s_2)$ and $\rho(s_3)$ on the 2-sphere S^2 with its standard metric.

This process continues, and the last congruence to satisfy is

$$p_n\,\alpha_{n1} \equiv 1 \pmod{2\pi}\ ,$$

with $\alpha_{n1} \in [0, \pi]$ now such that $\rho(s_1) = \rho(s_n) \cdot e^{\mathbf{y}\,\alpha_{n1}}$. In particular, having $\rho(s_n)$ fixed there is only one possibility of choosing $\mathbf{y} \in S^2$ - instead of a whole circle - as we have to 'come back' to $\rho(s_1)$ that was already fixed. Using the above Lemma 3.3 for fixing the conjugacy classes, we can now summarise our discussion in the following

Proposition 3.5. *The set of conjugacy classes of representations* $\rho \in R(K; \mathbf{i})$ *such that* $\rho((s_1 s_2)^{p_1}) = \rho((s_2 s_3)^{p_2}) = \cdots = \rho((s_n s_1)^{p_n}) = +1$ *is bijective to the ordered subsets* $(\rho_1, \ldots, \rho_n)$ *of points* $\rho_1, \ldots, \rho_n \in S^2 \subseteq \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ *such that*

(1) *the distance between* ρ_i *and* ρ_{i+1} *is given by* $\alpha_{i,i+1} \in [0, \pi]$ *satisfying the congruence*

$$p_i \, \alpha_{i,i+1} \equiv 1 \; (mod\ 2\pi) \ ,$$

for $i = 1, \dots, n$ *with* $n + 1 = 1$ *understood, and*

(2) *these points satisfy the following 'conjugacy class fixing condition':*

$$\rho_1 = \mathbf{j}\,, \quad \rho_{j+1} = \rho_j \cdot e^{\mathbf{i}\,\alpha_{j,j+1}},$$

for $j = 1, \dots, l$ *where* l *is the smallest integer such that* $\alpha_{l,l+1} \neq \pi$*, if it exists, or* $l = n$ *if not.*

Furthermore, a representation ρ *determined by an* n*-tuple* $(\alpha_{12}, \alpha_{23}, \dots, \alpha_{n1})$ *is non-abelian unless all angles* $\alpha_{i,i+1}$ *are equal to* π *(the case* $\alpha_{i,i+1} = 0$ *is excluded by the congruences to satisfy). It is binary dihedral if and only if all points* ρ_i *lie on the great circle lying in the* $\langle \mathbf{j}, \mathbf{k} \rangle$*–plane. Reflection on this plane induces an involution on the space of conjugacy classes of representations with fixed point set precisely the binary dihedral representations. In particular, conjugacy classes of non-abelian representations that are not binary dihedral come in pairs.*

□

3.3. **The case** $\rho((s_1 s_2)^{p_1}) = \rho((s_2 s_3)^{p_2}) = \cdots = \rho((s_n s_1)^{p_n}) = -1$**.** This case is entirely analogous to the preceeding one, and the corresponding statement is given by

Proposition 3.6. *The set of conjugacy classes of representations* $\rho \in R(K; \mathbf{i})$ *such that* $\rho((s_1 s_2)^{p_1}) = \rho((s_2 s_3)^{p_2}) = \cdots = \rho((s_n s_1)^{p_n}) = -1$ *is bijective to the ordered subsets* $(\rho_1, \dots, \rho_n)$ *of points* $\rho_1, \dots, \rho_n \in S^2 \subseteq \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ *such that*

(1) *the distance between* ρ_i *and* ρ_{i+1} *is given by* $\alpha_{i,i+1} \in [0, \pi]$ *satisfying the congruence*

$$p_i \, \alpha_{i,i+1} \equiv 0 \; (mod\ 2\pi) \ ,$$

for $i = 1, \dots, n$ *with* $n + 1 = 1$ *understood, and*

(2) *these points satisfy the following 'conjugacy class fixing condition':*

$$\rho_1 = \mathbf{j}\,, \quad \rho_{j+1} = \rho_j \cdot e^{\mathbf{i}\,\alpha_{j,j+1}},$$

for $j = 1, \dots, l$ *where* l *is the smallest integer such that* $\alpha_{l,l+1} \neq 0$*, if it exists, or* $l = n$ *if not.*

Furthermore, a representation ρ *determined by an* n*-tuple* $(\alpha_{12}, \alpha_{23}, \dots, \alpha_{n1})$ *is non-abelian unless all angles* $\alpha_{i,i+1}$ *are equal to* 0 *(the case* $\alpha_{i,i+1} = \pi$ *is excluded by the congruences to satisfy). It is binary dihedral if and only if all points* ρ_i *lie on the great circle lying in the* $\langle \mathbf{j}, \mathbf{k} \rangle$*–plane. Reflection on this plane induces an involution on the space of conjugacy classes of representations with fixed point set precisely the binary dihedral representations. In particular, conjugacy classes of non-abelian representations that are not binary dihedral come in pairs.*

□

3.4. **The case $\rho((s_1s_2)^{p_1}) = \rho((s_2s_3)^{p_2}) = \cdots = \rho((s_ns_1)^{p_n}) \neq \pm 1$.** Under this assumption we have $\rho(s_i) \neq \pm\rho(s_{i+1})$ modulo n. Up to conjugation we may assume $\rho(s_1) = \mathbf{j}$ and $\rho(s_2) = \mathbf{j}\, e^{\mathbf{i}\alpha}$, with $\alpha \neq 0 \pmod{\pi}$.

We therefore have $\rho(s_1s_2) = (-1)\, e^{\mathbf{i}\alpha}$. The image of this element under $SU(2) \to SO(3)$ is given by rotation by 2α around the $\mathbf{i}$ axis, where we consider $\mathbb{R}^3$ as the span of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ inside $\mathbb{H}$. By assumption, $e^{\mathbf{i}\beta} := \rho((s_1s_2)^{p_1}) \neq \pm 1$, so $\rho((s_2s_3))^{p_2}$ must be a non-trivial rotation around the same axis, the one spanned by $\mathbf{i}$. By the formula (3) we therefore see that $\rho(s_3)$ must lie in the plane $\langle \mathbf{j}, \mathbf{k} \rangle$ as well, and so may be written as $\rho(s_3) = \rho(s_2)\, e^{\mathbf{i}\alpha'}$. Inductively, we see that all elements $\rho(s_i)$ are of the form $\mathbf{j}\, e^{\mathbf{i}\alpha}$ for some angle $\alpha \in [0, 2\pi]$. This means that the presentation ρ is binary dihedral, so we have shown:

Proposition 3.7. *The set of conjugacy classes of representations $\rho \in R(K; \mathbf{i})$ such that $\rho((s_1s_2)^{p_1}) = \rho((s_2s_3)^{p_2}) = \cdots = \rho((s_ns_1)^{p_n})$ is conjugate to $e^{\mathbf{i}\beta}$ with $\beta \notin \{0, \pi\}$ is in one-to-two correspondance with the ordered subsets $(\rho_1, \ldots, \rho_n)$ of points $\rho_1, \ldots, \rho_n \in S^2 \subseteq \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ such that*

$$\rho_1 = \mathbf{j}\,, \quad \rho_{i+1} = \rho_i \cdot e^{\mathbf{i}\, \alpha_{i,i+1}},$$

with the angle $\alpha_{i,i+1} \in [0, 2\pi]$ satisfying the congruence

$$p_i\, \alpha_{i,i+1} \equiv \beta \ (mod\ 2\pi)\ ,$$

for $i = 1, \ldots, n$ with $n + 1 = 1$ understood.

All these representations are binary dihedral and non-abelian.

□

By the methods of Section 5 below we can see explicitely that there are only finitely many possible values for β in the formula $e^{\mathbf{i}\beta} = \rho((s_1s_2)^{p_1}) = \cdots = \rho((s_ns_1)^{p_n})$. However, this also follows from [7, Theorem 10].

Observation 3.8. *A corollary from the above explicit description is the following: If we denote by $\mathscr{R}(K; \mathbf{i})$ the space $R(K; \mathbf{i})/SU(2)$, the quotient by the action given by conjugation, then for $n = 3$ the conjugacy classes of representations ρ appear to be isolated in $\mathscr{R}(K; \mathbf{i})$ – essentially because a triangle on the sphere is 'rigid' given the lengths of its sides and keeping two edge points fixed – whereas for $n \geq 4$ the conjugacy classes in $\mathscr{R}(K; \mathbf{i})$ will in general come in positive dimensional families – because one may 'move' the n-gon even if two consecutive points of it are fixed. This intuitive conclusion will be settled rigorously in Section 6 below.*

3.5. **Orbits of the conjugacy action.** A representation $\rho \in \mathrm{Hom}(G, SU(2))$ is called *irreducible* if there is no proper subspace V of $\mathbb{C}^2$ that is invariant under ρ, in the sense that $\rho(g)V = V$ for all $g \in G$. Otherwise it is called reducible. It is not hard to see that a representation into $SU(2)$ is irreducible if and only if it is non-abelian.

Let us consider the action of $SU(2)$ on $\mathrm{Hom}(G, SU(2))$ given by conjugation. A representation ρ is irreducible if and only if its stabiliser $\Gamma_\rho \subseteq SU(2)$ is equal to the centre $\mathbb{Z}/2$. The trivial representation and representations with image inside the centre of $SU(2)$ have stabiliser $SU(2)$, and a reducible representation that acts non-trivially on a proper subspace has stabiliser isomorphic to $U(1)$. Therefore, the orbit $[\rho]$ of an irreducible representation is isomorphic to $SU(2)/\mathbb{Z}/2 \cong \mathbb{RP}^3$, and the orbit of a reducible representation that acts non-trivially on a proper subspace is homeomorphic to $SU(2)/U(1) \cong S^2$. In the situation of the representation spaces $R(K;\mathbf{i})$ that we consider the reducible representations with stabiliser $SU(2)$ do not appear.

4. Abelian representations

The reducible/abelian representations inside $R(K;\mathbf{i})$ are quite easily described for the pretzel knots that we consider.

Proposition 4.1. *Let $K = P(p_1,\dots,p_n)$ be a pretzel knot with all p_i odd and positive. Then there are, up to conjugation, precisely two abelian representations in $R(K;\mathbf{i})$ for n even (the situation of a 2-component link), and there is precisely one abelian representation when n is odd.*

Proof: Let $\rho \in R(K;\mathbf{i})$ be an abelian representation. We must then have $\rho(s_{i+1}) = \pm\rho(s_i)$ for all i because otherwise the representation would be non-abelian.

Suppose $\rho(s_2) = -\rho(s_1)$. Then $\rho((s_1s_2))^{p_1}) = 1 = \rho((s_2s_3))^{p_2}) = \cdots = \rho((s_ns_1)^{p_n})$. In particular, we must have $\rho(s_{i+1}) = -\rho(s_i)$ for $i = 1,\dots,n$ modulo n. For n odd we obtain a contradiction. For n even there is precisely one abelian representation, up to conjugation, for which we have $\rho(s_2) = -\rho(s_1)$.

Suppose now that we have $\rho(s_2) = \rho(s_1)$. Then $\rho((s_1s_2))^{p_1}) = -1 = \rho((s_2s_3))^{p_2}) = \cdots = \rho((s_ns_1)^{p_n})$, and this defines precisely one abelian representation, up to conjugation, regardless on the parity of n. Clearly, for n even, this representation is not conjugated to the preceeding one. □

5. Arithmetic properties

We show that under a simple arithmetic condition on the numbers $p_1,\dots,p_n$ irreducible representations with certain 'degenerate properties' may be avoided, with the best possible situation for $n = 3$.

Proposition 5.1. *Suppose the positive odd numbers $p_1,\dots,p_n$ are pairwise coprime. Then there is no representation ρ determined by angles*

$$(\alpha_{12},\dots,\alpha_{n,1})$$

with only two of the angles $\alpha_{i,i+1}$ not in the set $\{0,\pi\}$.

In particular, if $n = 3$, then any ρ that is non-abelian must have angles $\alpha_{12},\alpha_{23},\alpha_{31}$ none of which lies in $\{0,\pi\}$.

Proof: Notice that if ρ is not such that $\rho((s_1s_2)^{p_1}) = \cdots = \rho((s_ns_1)^{p_n}) = \pm 1$ then none of the angles $\alpha_{i,i+1}$ may be in the set $\{0,\pi\}$. So we may assume ρ to be such that this equation holds, and therefore in the situation of Proposition 3.5 or 3.6 above.

Suppose $\alpha_{i,i+1},\alpha_{j,j+1}$ with $i \neq j$ are such that they both do not lie in $\{0,\pi\}$, and all others do. These two angles must satisfy

$$p_i\,\frac{\alpha_{i,i+1}}{\pi} \equiv p_j\frac{\alpha_{j,j+1}}{\pi} \equiv 0 \pmod{\mathbb{Z}}.$$

Therefore we may assume $\alpha_{i,i+1}/\pi = k_i/p_i$, and $\alpha_{j,j+1}/\pi = k_j/p_j$ with numbers $k_i,k_j \in \mathbb{Z}$. On the other hand it is easy to see that the points $\rho_i = \rho(s_i)$ map to precisely two points under $S^2 \to \mathbb{RP}^2$, and in particular lie on a geodesic circle joining two non-antipodal points on S^2, of length 2π. Therefore we must have the congruence

$$\frac{\alpha_{i,i+1}}{\pi} + \frac{\alpha_{j,j+1}}{\pi} \equiv 0 \ (\mathrm{mod}\ \mathbb{Z}\,).$$

This implies that there is an integer $n \in \mathbb{Z}$ such that

$$k_i p_j + k_j p_i = n\, p_i p_j\ ,$$

and so p_j divides $k_j p_i$. As p_i and p_j are assumed to be coprime p_j must divide k_j. But this contradicts that $\alpha_j = \pi \frac{k_j}{p_j}$ does not lie in $\{0, \pi\}$. $\square$

Our next arithmetic result concerns binary dihedral representations.

Proposition 5.2. *Suppose the positive odd numbers $p_1, \ldots, p_n$ are pairwise coprime, and suppose ρ is a binary dihedral representation. Then either ρ is abelian or we must have (up to conjugation)*

$$\rho((s_1 s_2)^{p_1}) = \cdots = \rho((s_n s_1)^{p_n}) = e^{\mathbf{i}\,\beta}$$

with $\beta \notin \{0, \pi\}$. In other words, the situation in Section 3.4 above is the one that must occur for all non-abelian binary dihedral representations under this assumption.

Proof: Suppose ρ is binary dihedral and $\frac{\beta}{\pi}$ is an integer. We may suppose that $\rho(s_1) = \mathbf{j}$ and

$$\rho(s_{i+1}) = \rho(s_i)\, e^{\mathbf{i}\,\alpha_{i,i+1}}\ ,$$

with angle $\alpha_{i,i+1} \in [0, 2\pi]$, for $i = 1, \ldots, n$ and $n+1 = 1$ understood. This implies that the sum of the angles must be a multiple of 2π,

$$\frac{\alpha_{1,2}}{\pi} + \cdots + \frac{\alpha_{n-1,n}}{\pi} + \frac{\alpha_{n,1}}{\pi} \equiv 0 \ (\mathrm{mod}\ 2\mathbb{Z}\,). \tag{5}$$

In addition, we must have the congruences

$$p_i \frac{\alpha_{i,i+1}}{\pi} \equiv 0 \ (\mathrm{mod}\ \mathbb{Z}\,)$$

for $i = 1, \ldots, n$. Putting $\alpha_{i,i+1}/\pi = k_i/p_i$ with $k_i \in \mathbb{Z}$, $i = 1, \ldots, n$, inserting this in equation (5), and multiplying this equation by $p_1 \cdot \cdots \cdot p_n$ we see that p_j divides $k_j\, p_1 \cdot \cdots \cdot \widehat{p}_j \cdot \cdots \cdot p_n$, with the hat on $\widehat{p}_j$ indicating that this factor is omitted. By the condition on pairwise coprimeness we see that p_j must in fact divide k_j, and this for $j = 1, \ldots, n$. As a consequence, each angle $\alpha_{i,i+1}$ must be 0 or π, and so the representation ρ is abelian. $\square$

6. Non-degeneracy conditions

The local structure of the representation variety $\mathrm{Hom}(G, SU(2))$ of a discrete group G was first studied by Weil, see [15, 16]. Given a group G the representation variety $\mathrm{Hom}(G, SU(2))$ has the structure of an algebraic variety. Given a presentation $\langle g_1, \ldots, g_n | r_1, \ldots, r_m \rangle$ of G, the space $\mathrm{Hom}(G, SU(2))$ is homeomorphic to $F^{-1}(1, \ldots, 1)$, where $F : SU(2)^n \to SU(2)^m$ is given by the polynomials determined by the relations that the generators have to satisfy. The (Zariski) tangent space at an element $\rho \in \mathrm{Hom}(G, SU(2))$ is the kernel of dF at ρ. Of course it depends a priori on the explicit presentation of G.

Adapted to our situation is the following notion: We call a representation *non-degenerate* if its only possible deformations by 1-parameter families inside $\mathrm{Hom}(G, SU(2))$ are induced from the action of $SU(2)$ on itself by conjugation.

Definition 6.1. *A map $\xi : G \to \mathfrak{su}(2)$ is called a cocycle at ρ if one has*

$$\xi(gh) = \xi(g) + Ad_{\rho(g)}\xi(h) \tag{6}$$

for any $g, h \in G$. An element $\zeta \in \mathfrak{su}(2)$ defines a coboundary $\zeta^{\#} : G \to \mathfrak{su}(2)$ at a representation ρ by the formula

$$\zeta^{\#}(g) = \zeta - Ad_{\rho(g)}\zeta \tag{7}$$

for $g \in G$. Coboundaries are cocycles. Coboundaries are inifinitessimal deformations of ρ that are induced by conjugating ρ by elements of $SU(2)$.

To motivate this definition, let ρ be a representation. A deformation into nearby representations ρ_t may be written as $\rho_t(g) = \rho(g) + t\,\xi(g)\rho(g) + o(t)$. It is then easy to check that the requirement of ρ_t to be a group homomorphism implies for ξ to be a cocycle. Likewise, the derivative of $t \mapsto e^{t\zeta}\rho(g)e^{-t\zeta}$ at 0 yields the coboundary $\zeta^{\#}$ associated to $\zeta \in \mathfrak{su}(2)$. Therefore, a representation ρ is non-degenerate if and only if the space of cocycles at ρ is equal to the space of coboundaries. Orbits $[\rho]$ of non-degenerate representations ρ are isolated in $\mathscr{R}(K; \mathbf{i})$.

Remark 6.2. *The quotient of the space of cocycles $Z_\rho(G; \mathfrak{su}(2))$ by the space of coboundaries $B_\rho(G; \mathfrak{su}(2))$ at ρ turns out to be isomorphic to the first cohomology group $H^1(G; \mathfrak{su}(2)_\rho)$ with twisted coefficients defined by the adjoint action of G on $\mathfrak{su}(2)$ determined by ρ, see* [16].

In this section we shall study the local structure of $R(K; \mathbf{i})$, seen as $R(G(K)_{m,\mathbf{i}}) = \mathrm{Hom}(G(K)_{m,\mathbf{i}}, SU(2))$ by Proposition 3.1 above. In particular, a non-degenerate representation $\rho \in R(K; \mathbf{i})$ may well have deformations not coming from the action by conjugation when seen as an element of the bigger representation space $\mathrm{Hom}(G(K), SU(2))$ (without the assumption that meridians are mapped onto trace-free matrices).

Observation 6.3. *If ρ is a non-abelian representation then a coboundary ξ at ρ necessarily satisfies $\xi((-1)) = 0$.*

In fact, by the cocycle condition we must have

$$\xi(g\,(-1)) = \xi(g) + \mathrm{Ad}_{\rho(g)}\xi((-1)) = \xi((-1)) + \mathrm{Ad}_{\rho((-1))}\xi(g) = \xi((-1)\,g)$$

for all $g \in G(K)_{m,\mathbf{i}}$ as (-1) commutes with all elements in $G(K)_{m,\mathbf{i}}$. Clearly the endomorphism $\mathrm{Ad}_{\rho((-1))}$ is the identity. Therefore $\mathrm{Ad}_{\rho(g)}\xi((-1)) = \xi((-1))$ for all $g \in G(K)_{m,\mathbf{i}}$. If ρ is non-abelian this implies $\xi((-1)) = 0$.

Observation 6.4. *If ξ is a cocycle at ρ then for any element h that is (conjugated to) a meridional element m the element $\xi(h) \in \mathfrak{su}(2)$ must be perpendicular to $\rho(h) \in S^2 \subseteq \langle\, \mathbf{i}, \mathbf{j}, \mathbf{k}\,\rangle = \mathfrak{su}(2) = \mathbb{R}^3$.*

Indeed, as we have the relation $h^2 = -1$ in $G(K)_{m,\mathbf{i}}$ the cocycle condition together with the preceeding Observation implies

$$0 = \xi(h^2) = (1 + \mathrm{Ad}_{\rho(h)})\,\xi(h)\ .$$

By assumption $\mathrm{Ad}_{\rho(h)}$ is a rotation by π around the axis $\rho(h)$. Consequently $\xi(h)$ must lie in the plane annihilated by $(1 + \mathrm{Ad}_{\rho(h)})$ which is precisely the plane of elements perpendicular to $\rho(h)$.

Proposition 6.5. *Let $K = P(p_1, \dots, p_n)$ be a pretzel knot or link with p_i odd and positive, $i = 1, \dots, n$ (without any arithmetic assumption). Let $\rho \in R(K; \mathbf{i})$ be a non-abelian representation which is binary dihedral, and which satisfies $\rho((s_1 s_2)^{p_1}) = \cdots = \rho((s_n s_1)^{p_n}) \neq \pm 1$. Then the tangent space of $R(K; \mathbf{i})$ at ρ is equal to the space of coboundaries determined by ρ. Equivalently, the tangent space to $\mathscr{R}(K; \mathbf{i})$ at the conjugacy class $[\rho]$ of ρ is zero-dimensional.*

Proof: Up to conjugation we may assume that $\rho(s_1) = \mathbf{j}$ and $\rho(s_{i+1}) = \rho(s_i)\, e^{\mathbf{i}\,\alpha_{i,i+1}}$ with angles $\alpha_{i,i+1} \in [0, 2\pi]$ for $i = 1, \dots, n$ and $n+1 = 1$ understood. The assumption implies that there is an angle $\beta \in [0, 2\pi]$, different from 0 and π, such that $p_i\, \alpha_{i,i+1} \equiv \beta \pmod{2\pi\mathbb{Z}}$ and $\rho((s_1 s_2)^{p_1}) = \cdots = \rho((s_n s_1)^{p_n}) = (-1)\, e^{\mathbf{i}\beta}$. Suppose ξ is a cocycle at ρ. Notice that we have

$$\xi((s_i\, s_{i+1})^{p_i}) = \underbrace{(1 + \mathrm{Ad}_{\rho(s_i s_{i+1})} + \cdots + \mathrm{Ad}^{p_i - 1}_{\rho(s_i s_{i+1})})}_{=: B_{i,i+1}}\, \xi(s_i\, s_{i+1}) \ , \tag{8}$$

with Ad denoting the adjoint action of $SU(2)$ on the Lie algebra $\mathfrak{su}(2)$.

Lemma 6.6. *The endomorphism $B_{i,i+1}$ of $\mathfrak{su}(2)$ is an automorphism.*

Proof of the Lemma: Notice that we have the equation

$$1 - \mathrm{Ad}_{e^{\mathbf{i}\beta}} = B_{i,i+1}(1 - \mathrm{Ad}_{\rho(s_i s_{i+1})})$$

for any $i = 1, \dots, n$. Now $\mathrm{Ad}_{e^{\mathbf{i}\beta}}$ is rotation by angle $2\beta \notin \{0, 2\pi\}$ around the $\mathbf{i}$–axis. Therefore $(1 - \mathrm{Ad}_{e^{\mathbf{i}\beta}})$ has the subspace spanned by $\mathbf{i} \in \mathfrak{su}(2) \cong \mathbb{R}^3$ in its kernel and maps the whole space onto the plane perpendicular to $\mathbf{i}$. Therefore $B_{i,i+1}$ must have rank at least 2. Likewise, $\mathrm{Ad}_{\rho(s_i s_{i+1})}$ is a non-trivial rotation around the $\mathbf{i}$–axis, and so $(1 - \mathrm{Ad}_{\rho(s_i s_{i+1})})$ maps the plane perpendicular to $\mathbf{i}$ onto itself. Therefore $B_{i,i+1}$ must map the plane perpendicular to $\mathbf{i}$ onto itself. On the other hand, $B_{i,i+1}$ restricted to the subspace spanned by $\mathbf{i}$ is just multiplication by the number p_i, and so $B_{i,i+1}$ is an automorphism. $\square$

For an element $\lambda \in \mathfrak{su}(2) = \langle\, \mathbf{i}, \mathbf{j}, \mathbf{k}\,\rangle$ we denote by $\lambda\|_{\mathbf{i}}$ its projection onto the subspace generated by $\mathbf{i}$.

Lemma 6.7. *For a cocycle ξ at ρ the following equation holds for the $\mathbf{i}$ projections:*

$$(p_n + p_1)\xi(s_1)_{\|\mathbf{i}} = (p_1 + p_2)\xi(s_2)_{\|\mathbf{i}} = \cdots = (p_{n-1} + p_n)\xi(s_n)_{\|\mathbf{i}} \ .$$

Proof of the Lemma: Recall that the automorphisms $B_{i,i+1}$ introduced above respect the splitting of $\mathfrak{su}(2)$ into the span of $\mathbf{i}$ and its orthogonal complement, and that $B_{i,i+1}$ restricted to the span of $\mathbf{i}$ is just multiplication by p_i. Now expressing the cocyle ξ at the element $(s_1 s_2)^{p_1} = \cdots = (s_n s_1)^{p_n}$ in the $\mathbf{i}$–direction and using this fact we just obtain

$$p_1(\xi(s_1)_{\|\mathbf{i}} - \xi(s_2)_{\|\mathbf{i}}) = p_2(\xi(s_2)_{\|\mathbf{i}} - \xi(s_3)_{\|\mathbf{i}}) = \cdots = p_n(\xi(s_n)_{\|\mathbf{i}} - \xi(s_1)_{\|\mathbf{i}}) \ ,$$

which is equivalent to the formula in the statement. $\square$

Lemma 6.8. *We may choose an element $\zeta \in \mathfrak{su}(2)$ such that the coboundary $\zeta^{\#}$ satisfies $\zeta^{\#}(s_1) = \xi(s_1)$ and $\zeta^{\#}(s_2) = \xi(s_2)$.*

Proof of Lemma: By Observation 6.4 above we know that $\xi(s_1)$ must be perpendicular to $\rho(s_1) = \mathbf{j}$. Now for any $\zeta \in \mathfrak{su}(2)$ we have $\zeta^\#(s_1) = (1 - \mathrm{Ad}_{\rho(s_1)})\,\zeta$, and $(1 - \mathrm{Ad}_{\rho(s_1)})$ has kernel given by the span of $\mathbf{j}$ and is an automorphism when restricted to the plane perpendicular to $\mathbf{j}$. We decompose ζ as

$$\zeta = \zeta_{\|_\mathbf{j}} + \zeta_{\perp_\mathbf{j}} \ ,$$

where $\zeta_{\|_\mathbf{j}}$ is parallel to $\mathbf{j}$ and $\zeta_{\perp_\mathbf{j}}$ lies in the plane orthogonal to $\mathbf{j}$. Therefore we may choose $\zeta_{\perp_\mathbf{j}}$ such that $\zeta^\#(s_1) = \xi(s_1)$, thereby leaving us the possibility to fix $\zeta_{\|_\mathbf{j}}$ later.

By Lemma 6.7 above we see that the $\mathbf{i}$ component of $\xi(s_2)$ is determined by that of $\xi(s_1)$. The element $\xi(s_2)$, being perpendicular to $\rho(s_2) \in S^2 \subseteq \langle\, \mathbf{i}, \mathbf{j}, \mathbf{k}\,\rangle = \mathfrak{su}(2)$ therefore only has its 1-dimensional component $\xi(s_2)_{\perp \mathbf{i}, \rho(s_2)}$, perpendicular to $\mathbf{i}$ and $\rho(s_2)$, as remaining degree of liberty. However, as $\rho(s_2)$ is not parallel to $\rho(s_1)$ by assumption, it is easy to see that it is indeed possible to choose $\zeta_\|$ so that $\zeta^\#(s_2) = \xi(s_2)$.

□

We may now proceed with the proof of Proposition 6.5. We suppose now that ζ is chosen according to Lemma 6.8 for a given cocycle ξ. Therefore the cocycles $\zeta^\#$ and ξ satisfy $\xi((s_1 s_2)^{p_1}) = \zeta^\#((s_1 s_2)^{p_1})$. As $(s_1 s_2)^{p_1} = (s_2 s_3)^{p_2}$, they also satisfy $\xi((s_2 s_3)^{p_2}) = \zeta^\#((s_2 s_3)^{p_2})$, or equivalently

$$B_{2,3}\, \xi(s_2 s_3) = B_{2,3}\, \zeta^\#(s_2 s_3) \ .$$

By Lemma 6.6 above $B_{2,3}$ is an automorphism, and so we must have

$$\xi(s_2 s_3) = \xi(s_2) + \mathrm{Ad}_{\rho(s_2)} \xi(s_3) = \zeta^\#(s_2) + \mathrm{Ad}_{\rho(s_2)} \zeta^\#(s_3) = \zeta^\#(s_2 s_3) \ .$$

As we already have $\xi(s_2) = \zeta^\#(s_2)$ it follows that we also must have $\xi(s_3) = \zeta^\#(s_3)$. Using the same argument, we see inductively that we also have $\xi(s_4) = \zeta^\#(s_4)$, ..., $\xi(s_n) = \zeta^\#(s_n)$. □

Proposition 6.9. *Let $K = P(p, q, r)$ be a pretzel knot with each p, q, r positive, odd and pairwise coprime. Then the tangent space of $R(K; \mathbf{i})$ at any non-abelian representation ρ is of dimension 3. The tangent space to $\mathscr{R}(K; \mathbf{i})$ at any conjugacy class $[\rho]$ of a non-abelian representation ρ is zero-dimensional.*

Proof: By Proposition 6.5 we only have to check the claim at non-abelian representations ρ with $\rho((s_1 s_2)^p) = \rho((s_2 s_3)^q) = \rho((s_3 s_1)^r) = \pm 1$ so we shall assume that this holds.

Up to conjugation we may assume that $\rho(s_1) = \mathbf{j}$ and $\rho(s_2) = \mathbf{j}\, e^{\mathbf{i}\, \alpha_{12}}$ with angle $\alpha_{12} \in [0, 2\pi]$. Suppose ξ is a cocycle at ρ. The conclusions of Observation 6.3 and 6.4 remain valid in this situation. Therefore, the dimension of the space of cocylces at ρ is at most 6. However, instead of Lemma 6.6 we now have:

Lemma 6.10. *Suppose that we have $\rho((s_i s_{i+1})^{p_i}) = \pm 1$ and that $\rho(s_i) \neq \pm \rho(s_{i+1})$. Then the endomorphism $B_{i,i+1}$ of $\mathfrak{su}(2)$ defined in equation (8) above has rank 1. More precisely, we have $B_{i,i+1} = p_i\, \Pi_{\rho(s_i) \times \rho(s_{i+1})}$, where $\Pi_{\rho(s_i) \times \rho(s_{i+1})}$ is the projection onto the space spanned by $\rho(s_i) \times \rho(s_{i+1}) \in \mathfrak{su}(2)$.*

Proof: As we now have $\mathrm{Ad}_{\rho((s_i s_{i+1})^{p_i}} = \pm 1$ we can conclude that

$$0 = B_{i,i+1}(1 - \mathrm{Ad}_{\rho((s_i s_{i+1}))}) \ .$$

By the arithmetic assumption we made we know that $\mathrm{Ad}_{\rho((s_i s_{i+1}))})$ is a non-trivial element in $SO(\mathfrak{su}(2))$ by Proposition 5.1. Therefore the image of $\mathrm{Ad}_{\rho((s_i s_{i+1}))}$ must have rank 2, and so the kernel of $B_{i,i+1}$ must at least contain the 2-dimensional subspace of $\mathfrak{su}(2)$ that is perpendicular to the rotation axis $\langle\, \rho(s_i) \times \rho(s_{i+1})\,\rangle$ of $\mathrm{Ad}_{\rho(s_i s_{i+1})}$. On the other hand, it is immediate from the definition of $B_{i,i+1}$ that it is given by multiplication by p_i when restricted to the 1-dimensional subspace $\langle\, \rho(s_i) \times \rho(s_{i+1})\,\rangle$. □

Because of $(s_1 s_2)^p = (s_2 s_3)^q = (s_3 s_1)^r$ the cocycle must satisfy

$$\begin{aligned} B_{12}\,\xi(s_1 s_2) - B_{23}\,\xi(s_2 s_3) &= 0 \\ B_{23}\,\xi(s_2 s_3) - B_{31}\,\xi(s_3 s_1) &= 0 \ . \end{aligned}$$

From our arithmetic assumptions it follows from Proposition 5.2 that the three axes $\rho(s_1) \times \rho(s_2)$, $\rho(s_2) \times \rho(s_3)$, $\rho(s_3) \times \rho(s_1)$ are pairwise linearly independent. Therefore, the last equations are equivalent to

$$\begin{aligned} B_{12}\,\xi(s_1 s_2) &= 0 \\ B_{23}\,\xi(s_2 s_3) &= 0 \\ B_{31}\,\xi(s_3 s_1) &= 0 \ . \end{aligned}$$

Equivalently, the element $(\xi(s_1), \xi(s_2), \xi(s_3)$ lies in the kernel of the linear map $L : \mathfrak{su}(2)^3 \to \mathfrak{su}(2)^2$ given by

$$(\xi_1, \xi_2, \xi_3) \mapsto \underbrace{\begin{pmatrix} B_{12} & 0 & 0 \\ 0 & B_{23} & 0 \\ 0 & 0 & B_{31} \end{pmatrix}}_{:=B} \underbrace{\begin{pmatrix} 1 & \mathrm{Ad}_{\rho(s_1)} & 0 \\ 0 & 1 & \mathrm{Ad}_{\rho(s_2)} \\ \mathrm{Ad}_{\rho(s_3)} & 0 & 1 \end{pmatrix}}_{:=C} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \ .$$

The linear map defined by B has rank 3. Likewise, the map C is an automorphism. Therefore, the homomorphism L has rank 3. Let $H_i \subseteq \mathfrak{su}(2)$ be the subspace in which the cocycle $\xi(s_i)$ must lie according to Observation 6.4. It is not hard to show that the restriction of L to $H_1 \oplus H_2 \oplus H_3$ still has rank 3. So the fact that $(\xi(s_1), \xi(s_2), \xi(s_3))$ must lie in the kernel of L then implies that the space of cocycles at ρ is 3-dimensional. □

Remark 6.11. *It is interesting to notice that instead of computing the Zariski tangent space explicitely one may also get non-degeneracy results by studying the topology of the 3-manifold one gets from taking the double branched cover of S^3, branched along the knot, see the corresponding results of* [4].

7. The pretzel knots $P(p, q, r)$

We get a complete picture of the representation space $R(K; \mathbf{i})$ for the pretzel knots $K = P(p, q, r)$ for p, q, r odd, pairwise coprime, and of the same sign.

Proposition 7.1. *Let K be the pretzel knot $P(p,q,r)$ for p,q,r odd, pairwise coprime, and of the same sign. Then the representation space $R(K;\mathbf{i})$ is isomorphic to the disjoint union*

$$S^2 \coprod \left(\coprod_I \mathbb{RP}^3 \right) ,$$

where the finite index set I parametrises the conjugacy classes of all non-abelian representations. Among these there are $(|\Delta_{P(p,q,r)}(-1)|-1)/2 = 1/2(pq+qr+rp-1)$ many binary dihedral ones, as well as the non-abelian non-binary dihedral representations that are described in Proposition 3.5 and 3.6 above.

If one of p,q,r is equal to 1, then there are no representations that are not binary dihedral. If all of p,q,r are strictly bigger than 1, then there always are representations that are not binary dihedral, and so fall in the situation of Proposition 3.5 or 3.6 above.

Proof: The fact that there is a single orbit homeomorphic to S^2 follows from Lemma 4.1 above. That there are only finitely many isolated orbits homeomorphic to $\mathbb{RP}^3$ follows from the results of the preceeding sections. The number of non-abelian binary dihedral conjugacy classes was determined by Klassen [7].

If p,q or r is equal to 1, then Proposition 5.1 comes to bear. In fact, suppose without loss of generality that $p=1$. Then any representation ρ occuring in Proposition 3.5 or 3.6 above has to satisfy $\alpha_{12} \in \{0,1\}$. By Proposition 5.1 the representation ρ would then be abelian.

Suppose now that p,q,r all are strictly bigger than 1. We may without loss of generality assume that $r \geq q \geq p$. We show that there is a non-abelian representation ρ as in Proposition 3.5. We fix the first two angles α_{pq} and α_{qr} which have to satisfy the congruence $p\,\alpha_{pq} \equiv 1 \pmod{2\pi}$ and $q\,\alpha_{qr} \equiv 1 \pmod{2\pi}$. We just pick $\alpha_{pq} = \pi/p$ and $\alpha_{qr} = \pi/q$. What remains to show is that we can find a distance $\alpha_{qr} \in (0,\pi)$ which satisfies the congruence $q\,\alpha_{qr} \equiv 1 \pmod{2\pi}$, and such that the triangle inequality

$$|\alpha_{pq} - \alpha_{qr}| \leq \alpha_{rp} \leq \alpha_{pq} + \alpha_{qr}$$

holds, or equivalently, with the already chosen angles, such that

$$\frac{q-p}{pq} \leq \frac{\alpha_{rp}}{\pi} \leq \frac{q+p}{pq} .$$

The interval $[\frac{q-p}{pq}, \frac{q+p}{pq}]$ has length $\frac{2}{q}$. But as we have assumed $r \geq q$, there certainly are two integer multiples of $\frac{\pi}{r}$ inside this interval, one of which will satisfy the congruence $r\,\alpha_{rp} \equiv 1 \pmod{2\pi}$. □

It is interesting that we get the following Corollary from the method of the preceeding proof. The result is not new, see [2].

Corollary 7.2. *Let K be the pretzel knot $P(p,q,r)$ with p,q,r odd and of the same sign. Then $P(p,q,r)$ has bridge number 3 if and only if all of p,q,r are strictly bigger than 1.*

Proof: Assume that p,q,r are all odd and strictly bigger than 1. By the preceeding proof we see that there are representations that are not binary dihedral. However, a 2-bridge knot K only has representations in $R(K;\mathbf{i})$ that are binary dihedral, which is easy to see. On the other hand, if one of the numbers p,q,r is 1, then it is easy to see that the knot actually is 2-bridge. □

As an example, we shall now compute the non-abelian representations of $P(3,5,7)$ that are not binary dihedral. As a matter of notation, we shall write $\overline{\alpha}_{i,i+1} := \alpha_{i,i+1}/\pi$ for the angles occuring in Proposition 3.5 and 3.6 above. These have to satisfy the congruences in these Propositions, and as distances between the points ρ_1, ρ_2, and ρ_3, they have to satisfy the triangle inequality.

The representations ρ with $\rho((s_1s_2)^3) = \rho((s_2s_3)^5 = \cdots = \rho((s_3s_1)^7) = +1$ are determined by the following table, that first lists all possible combinations of angles satisfying the congruencies, and then checks the triangle-inequality on each.

$\overline{\alpha}_{12}$	$\overline{\alpha}_{23}$	$\overline{\alpha}_{31}$	$\lvert\overline{\alpha}_{23} - \overline{\alpha}_{31}\rvert$	$\overline{\alpha}_{23} + \overline{\alpha}_{31}$	Δ–inequality
1/3	1/5	1/7	2/35	12/35	no
		3/7	8/35	22/35	yes
		5/7	18/35	32/35	no
	3/5	1/7	16/35	26/35	no
		3/7	6/35	36/35	yes
		5/7	4/35	46/35	yes.

Likewise, the representations ρ with $\rho((s_1s_2)^3) = \rho((s_2s_3)^5 = \cdots = \rho((s_3s_1)^7) = -1$ are determined by the following table.

$\overline{\alpha}_{12}$	$\overline{\alpha}_{23}$	$\overline{\alpha}_{31}$	$\lvert\overline{\alpha}_{23} - \overline{\alpha}_{31}\rvert$	$\overline{\alpha}_{23} + \overline{\alpha}_{31}$	Δ–inequality
2/3	2/5	2/7	4/35	24/35	yes
		4/7	6/35	34/35	yes
		6/7	16/35	44/35	yes
	4/5	2/7	18/35	38/35	yes
		4/7	8/35	48/35	yes
		6/7	2/35	58/35	yes.

Each combination of angles that gives rise to a non-abelian representation that is not binary dihedral yields precisely two different conjugacy classes of representations. Therefore, there are in total 18 such conjugacy classes for the knot $P(3,5,7)$.

8. Perspectives and connections with related results

8.1. **Relation to Lin's knot invariant.** Lin has defined a knot invariant, that he denotes $h(K)$, from the representation space $\mathscr{R}(K;\mathbf{i}) = R(K;\mathbf{i})/SU(2)$ considered here. In the case that all irreducible representations are non-degenerate, and so these are isolated points in $\mathscr{R}(K;\mathbf{i})$, the number $h(K)$ is just a signed count of these conjugacy classes of irreducible representations. Surprisingly, this is related to the signature of K in the following way [13]:

$$h(K) = \frac{1}{2}\,\mathrm{sign}(K)\ .$$

The signature of a knot is just the signature of the symmetric bilinear form given by the matrix $V + V^t$, with V denoting a Seifert matrix of the knot.

For the pretzel knots $P(p,q,r)$ with p,q,r odd the signature is easily computed. Indeed, a Seifert matrix is given by

$$V = \frac{1}{2}\begin{pmatrix} p+q & q+1 \\ q-1 & q+r \end{pmatrix}\ ,$$

see for instance [12, Example 6.9]. If p, q, r are all odd and positive then the signature is just 2, and so Lin's invariant $h(K)$ equals 1, and is in particular odd. We may therefore draw the conclusion that the index set I appearing in Proposition 7.1 as the index set for the conjugacy classes of irreducible represention has odd cardinality. This is not such a surprise. In fact, as indicated earlier, the non-abelian non-binary dihedral representations come in pairs, and so modulo 2 the cardinality of the index set I is just equal to the absolute value of the determinant $|\Delta_K(-1)|$ which, for a knot, is always an odd integer.

8.2. **Relation to Khovanov-homology.** As we have mentioned in the introduction, there is an isomorphism of abelian groups

$$Kh(K) \cong H_*(R(K; \mathbf{i}); \mathbb{Z}) \ , \tag{9}$$

where $Kh(K)$ denotes the Khovanov homology [6] of K, for certain knots. For torus knots of type $(2, p)$ this was observed by Kronheimer and Mrowka [8, Observation 1.1]. For an arbitrary 2-bridge knot or 2-component link this was proved by Lewallen [11](in the current version by use of an unpublished result of Shumakovitch [14]). More precisely, he shows that Khovanov homology of a one or two component alternating link is isomorphic to the integer homology of $R_{bd}(K; \mathbf{i})$, where $R_{bd}(K; \mathbf{i}) \subseteq R(K; \mathbf{i})$ is the subspace of binary dihedral representations. Our explicit description in Proposition 7.1 allows us to draw the following conclusion:

Proposition 8.1. *Let K be the alternating pretzel knot $P(p, q, r)$ for p, q, r odd, pairwise coprime, positive, and such that $p, q, r > 1$. Then*

$$\mathrm{Kh}(K) \ncong H_*(R(K; \mathbf{i}); \mathbb{Z}) \ , \tag{10}$$

i.e. these two abelian groups are not isomorphic.

Proof: In fact we have $R(K; \mathbf{i}) \cong R_{bd}(K; \mathbf{i}) \coprod (\coprod_{I'} \mathbb{RP}^3)$, where I' parametrises the non-empty set of conjugacy classes of non-binary dihedral representations. □

8.3. **Relation to Kronheimer and Mrowka's framed instanton Floer homology.** In [8] Kronheimer and Mrowka construct an abelian group called the framed instanton Floer homology $FI_*(Y, K)$ of knots K in a 3-manifold Y. This is the Morse homology of the Chern-Simons functional CS defined on a space of connections on Y with certain singularities along K. However, there are conditions on the 3-manifold Y in this construction. In particular one must have $b_1(Y) > 0$. For classical knots in S^3 the construction is not directly applicable, but it is applied to a connected sum of (S^3, K) with the pair $(T^3, \emptyset)$. If one applies their construction to $SU(2)$ connections, the space of critical points $\mathfrak{C}_{CS}$ for this connected sum is given by

$$R(K; \mathbf{i}) \coprod R(K; \mathbf{i})$$

in our notation. In particular it is not non-degenerate, and so the Chern-Simons functional CS has to be perturbed. If the critical space is non-degenerate in the Morse-Bott sense [1] one may proceed as outlined in the following, see for instance [3, Section 4] and [1, Section 3.4]: One chooses a Morse-function g on the critical space $\mathfrak{C}_{CS}$ and extends it to a regular neighbourhood of $\mathfrak{C}_{CS}$ inside the space of connections considered. One then studies the perturbed Chern-Simons functional

$CS_g = CS + \varepsilon\psi\, g$, where $\varepsilon > 0$ and ψ is a bump function on the regular neighbourhood above. For small enough ε it will only have non-degenerate critical points, and these generate the instanton Floer chain complex of $FI_*(K)$, which by definition is FI_* applied to $(S^3 \# T^3, K)$. Of course, there are compactness and transversality issues to be studied if one wants to settle this further.

Presumably our results of Section 6 imply that for the pretzel knots $P(p,q,r)$ with p,q,r positive, odd, and pairwise coprime, the critical space

$$R(P(p,q,r);\mathbf{i}) \coprod R(P(p,q,r);\mathbf{i})$$

is non-degenerate in the sense of [8] in the normal directions, and so non-degenerate in the Morse-Bott sense for the setting of [8]. If this is the case then our results of Proposition 7.1 indicate that the instanton Floer chain complex of $FI_*(P(p,q,r))$ has non-trivial differentials. In fact, for quasi-alternating knots K the rank of $FI_*(K)$ is known [8–10] to be equal to $2|\Delta_K(-1)| + 2$, whereas the rank of

$$H_*(R(P(p,q,r);\mathbf{i});\mathbb{Z}) \oplus H_*(R(P(p,q,r);\mathbf{i});\mathbb{Z})$$

is strictly bigger if p,q and r are all strictly bigger than 1. As the number of critical points of a Morse-function on $R(P(p,q,r);\mathbf{i})$ is at least as high as the rank of its homology, the claim about the non-triviality of the differentials follows.

Thinking further, it would be interesting to compute the differentials yielding to $FI_*(K)$ explicitly and to study Question 1.2 of [8] explicitly on the class of pretzel knots considered here.

8.4. **Parallels with representation spaces of Brieskorn homology spheres.** There appear to be parallels between the representation spaces $R(K;\mathbf{i})$ for $K = P(p_1,\dots,p_n)$ and the representation spaces $R(Y) = \mathrm{Hom}(\pi_1(Y); SU(2))$ for $Y = \Sigma(p_1,\dots,p_n)$ a Seifert fibred homology sphere [3]. In both cases the representation space is non-degenerate for $n = 3$ and degenerate for $n \geq 4$, with a similar growth in the dimensions. Possibly this fact could be used to compute various versions of the instanton Floer homology of the knots $P(p,q,r)$, and we hope to be able to investigate this further. However, the analogies between the two cases also have limitations: In the case of the Brieskorn homology spheres the Floer gradings of the critical points all have the same parity, so there are no non-zero differentials in the instanton Floer chain complex, and the Floer homology is just isomorphic to the chain complex. As indicated above, this cannot be expected from the instanton Floer chain complex of the pretzel knots $P(p,q,r)$ with p,q,r all odd, pairwise coprime, and of the same sign.

8.5. **Relation to results of Heusener and Kroll.** In [4] Heusener and Kroll extend Lin's result to the situation of studying spaces of representations ρ modulo conjugation, such that $\rho(m) \sim e^{\mathbf{i}\alpha} \in SU(2)$. They define an invariant $h^\alpha(K)$ and establish $h^\alpha(K) = \frac{1}{2}\mathrm{sign}_K(e^{\mathbf{i}2\alpha})$. Here $\mathrm{sign}_K : S^1 \setminus \{1\} \to \mathbb{Z}$ is the Levine-Tristram signature function, i.e. $\mathrm{sign}_K(\omega)$ is the signature of the Hermitian form $(1-\omega)V + (1-\overline{\omega})V^t$, where $\omega \in S^1 \setminus \{1\} \subseteq \mathbb{C}$, and V is a Seifert matrix of K.

Our notion of non-degeneracy implied that a non-degenerate representation $\rho \in R(K;\mathbf{i})$ has its conjugacy class $[\rho] \in \mathscr{R}(K;\mathbf{i})$ isolated. Below Remark 6.2 we were pointing out that this does not imply that it is isolated when seen as element of the representation space $\mathscr{R}(K)$ of all representations of the knot group in $SU(2)$, up to conjugation. In fact, the following result of Heusener and Kroll has interesting conclusion for our situation.

Proposition 8.2. [4, Corollary 3.9] *Let K be a knot, Δ_K its Alexander polynomial and $sign_K$ its Levine-Tristram signature function. If $\Delta_K(e^{2\mathbf{i}\alpha}) \neq 0$ and $sign_K(e^{2\mathbf{i}\alpha}) \neq 0$ for $\alpha \in (0, \pi)$ then there is an irreducible representation $\rho_0 \in Hom(G_K, SU(2))$ with $tr(\rho_0(m)) = 2\cos(\alpha)$. Furthermore this deforms to an arc inside $\mathscr{R}(K)$: There is some $\varepsilon > 0$ and a continous arc $\rho_t \in Hom(G_K, SU(2))$ for $t \in [-\varepsilon, \varepsilon]$, extending ρ, and such that $tr(\rho_{-\varepsilon}(m)) < 2\cos(\alpha) < tr(\rho_\varepsilon(m))$.*

This applies to our situation for $\alpha = \pi/2$. For this value $\Delta_K(e^{2\mathbf{i}\alpha}) = \Delta_K(-1)$ is just the determinant of the knot K and $\mathrm{sign}_K(e^{2\mathbf{i}\alpha}) = \mathrm{sign}(K)$ is just the ordinary signature. For $P(p, q, r)$ with p, q, r all odd and of the same sign both values are non-zero. Therefore we may conclude that there must be non-abelian representations $\rho \in R(P(p, q, r); \mathbf{i})$ that have 1–parameter deformations inside $\mathscr{R}(P(p, q, r))$.

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Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany

E-mail address: `raphael.zentner@math.uni-muenster.de`