Properties of the $1/\cosh(t)$ Laser Pulse, Supersymmetry, and the sine-Gordon Equation

Andrew Koller[1] and Maxim Olshanii[1]

[1]*Department of Physics, University of Massachusetts Boston, Boston MA 02125, USA*

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In this Letter, we explain a well-known but little understood effect in Laser Physics: when a two-level atom, initially in the ground state, is subjected to a $1/\cosh(t)$ laser pulse, it remains in the ground state after the pulse has been applied, for *any* choice of the laser detuning. To this end, we reinterpret the effect as the absence of reflection in a corresponding stationary wave scattering problem and then—inspired by the numerous examples of the reflectionless scattering in quantum mechanics, wave optics, and linearized mean-field many-body dynamics—identify a two-step quantum-mechanical supersymmetric (QM SUSY) chain that links the Hamiltonian of the system to a potential-free problem. At the same time, we observe that our problem trivially maps to the first member of the Lax pair for the kink soliton solution of the sine-Gordon equation. This allows us to conjecture that, by analogy with the Korteweg-deVries equation, the multi-soliton solutions of the sine-Gordon equation can also be systematically generated via supersymmetric chains.

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Introduction.– Historically, reflectionless problems were first introduced and partially classified in the context of light propagation in a spatially inhomogeneous dielectric media [1]. Mathematically, this is equivalent to finding reflectionless potentials for the one-dimensional non-relativistic Schrödinger equation, as addressed in Refs. [2, 3]. In a parallel development, an algorithm producing multi-soliton solutions of the Korteweg-deVries equation was proposed [4, 5] which was intimately connected to the reflectionless potentials for the Schrödinger equation. Another procedure for producing reflectionless problems can be found in Ref. [6]. The relativistic Dirac equation in scalar and pseudoscalar external potentials is also shown to exhibit cases of reflectionless scattering [7, 8].

The fact that the phenomenon of reflectionless scattering is so peculiar and rare hints to a hidden nontrivial algebraic structure. Indeed, in the aforementioned cases, the explanation comes from the algebra of quantum-mechanical supersymmetry (QM SUSY) [9], which links—via a finite number of intermediate steps—the reflectionless Hamiltonians to their respective potential-free supersymmetric partners [10–14]. Potential-free Hamiltonians are, in turn, inherently reflectionless.

At least three examples of reflectionless problems outside of the standard stationary quantum mechanics (non-relativistic or relativistic) are present in the literature. No SUSY interpretation is known for any of them. The reflectionless time-dependent perturbations to the time-dependent Schrödinger equation were used to generate multi-soliton solutions of the Kadomtsev-Petviashvili-I equation [15]. Note that in this case, unlike in all other known cases, the reflectionless problem is set in two spatial dimensions. Next, the Bogoliubov-de Gennes (BdG) Liouvillian, representing excitations above the ground state of a one-dimensional attractive Bose condensate, is reflectionless at all energies [16, 17].

The third example comes from Laser Physics. Consider a two-level atom, initially in the ground state, and apply a laser pulse of a finite duration. Generally, the probability of finding the atom in the excited state after the pulse is applied is an intricate function of the pulse intensity, its duration, and the laser detuning. However, it is known [18] that for pulses of a $1/\cosh(t)$ shape, the excitation probability factorizes into a product of a function of the pulse area and a function of the detuning. Furthermore, for a discrete series of values of the pulse area the former function vanishes, and as a result, no excitation is observed for any choice of detuning.

In this Letter, we reinterpret the no-excitation property of the $1/\cosh(t)$ pulse as absence of reflection in a stationary scattering problem with an effective non-Hermitian Hamiltonian acting on two-component wave functions. We show that our Hamiltonian is linked, via a two-step SUSY chain, to a potential-free supersymmetric partner; this provides an explanation for the reflectionless property of the original Hamiltonian in question. Furthermore, a trivial transformation links our scattering problem to the first member of the Lax pair for the sine-Gordon equation [19], with the sine-Gordon field in the state of a stationary kink soliton. This observation indicates that, similarly to the Korteweg-deVries case, the solitonic solutions of the sine-Gordon equation could be systematically generated using supersymmetric chains.

Two-Level System with $1/\cosh(t)$ Pulse.– Consider a two-level atom subjected to a time-dependent pulse of the form $V_{eg}(t) = V/\cosh(t/\tau)$ and detuning detuning Δ. Here V is the amplitude of the pulse, τ is its duration, and $|e\rangle$ and $|g\rangle$ are the excited and ground states respectively. The time-dependence of this system can be solved exactly in terms of hypergeometric functions, and it is known that for specific values of the pulse amplitude, the transition probability is zero regardless of the detuning choice Δ [18]. We consider the first of these amplitudes given by $V = \hbar/\tau$. If we represent the probability amplitudes of the ground and excited states by ψ_g and ψ_e, respectively, the dynamics of the system will obey

$$\begin{aligned} i\frac{d}{dt}\psi_g &= +\frac{\Delta}{2}\psi_g + \frac{1}{\tau\cosh(t/\tau)}\psi_e \\ i\frac{d}{dt}\psi_e &= +\frac{1}{\tau\cosh(t/\tau)}\psi_g - \frac{\Delta}{2}\psi_e \end{aligned} \quad (1)$$

The remarkable property of this pulse is that if the population is prepared entirely in the ground state ψ_g at $t \to -\infty$, then the whole population will return to the ground state for $t \to +\infty$, for *any* value of Δ. Now, we can regard the excited state population after the pulse is applied (generally present, but absent in our case) as a reflected wave in a scattering problem. Similarly, the ground state populations before and after the pulse can be regarded as the transmitted and incident waves, respectively (note the order). To formalize the analogy, we make the substitution $x = -t/\tau$, $u = \psi_g$, $v = -\psi_e$, $\lambda = \Delta\tau/2$. (Note that x is a *dimensionless* coordinate.) Now the dynamics of the system can be rewritten as a two-component spatial eigenvalue problem involving a 2×2 Hamiltonian, $\hat{H}$:

$$\begin{pmatrix} -i\frac{d}{dx} & \frac{1}{\cosh(x)} \\ -\frac{1}{\cosh(x)} & i\frac{d}{dx} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix} \quad (2)$$

We will classify the eigenstates by their wavevector k and by the eigenvalues of $\hat{H}$. For each k we have two eigenvalues $\lambda = \pm k$:

$$\begin{aligned} \mid \psi_k\rangle^{(\lambda=+k)} &\propto \begin{pmatrix} ik - \frac{\tanh(x)}{2} \\ \frac{-i}{2\cosh(x)} \end{pmatrix} e^{ikx} \\ \mid \psi_k\rangle^{(\lambda=-k)} &\propto \begin{pmatrix} \frac{-i}{2\cosh(x)} \\ ik - \frac{\tanh(x)}{2} \end{pmatrix} e^{ikx} \end{aligned} \quad (3)$$

We can see from the eigenstates that $\hat{H}$ is reflectionless. If one replaced the off-diagonal perturbation $\frac{1}{\cosh(x)}$ in (2) by a perturbation of a general position, the scattering state $\mid \psi_k\rangle^{(\lambda=+k)}$ (whose incident internal state is $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$) would show a reflected wave, $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)e^{-ikx}$, corresponding to the internal state $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. The peculiar property of the $\frac{1}{\cosh(x)}$ perturbation is exactly the absence of the reflected wave. The second scattering state, $\mid \psi_k\rangle^{(\lambda=-k)}$, shows the same phenomenon, with the internal states reversed.

Quantum-Mechanical Supersymmetry and Reflectionless Potentials.– Consider a version of quantum-mechanical supersymmetry that applies to non-Hermitian Hamiltonians. Let two Hamiltonians $\hat{H}_0$ and $\hat{H}_1$ be related by

$$\begin{aligned} \hat{H}_0 &= \hat{B}\hat{A} + \epsilon \\ \hat{H}_1 &= \hat{A}\hat{B} + \epsilon \end{aligned} \quad (4)$$

where ϵ is the factorization energy, and $\hat{A}$ and $\hat{B}$ are the SUSY factors. Their eigenstates will still be related in the usual way for supersymmetric partners:

$$\begin{aligned} \hat{A} \mid \psi_0\rangle &\propto \mid \psi_1\rangle \\ \hat{B} \mid \psi_1\rangle &\propto \mid \psi_0\rangle \end{aligned} \quad (5)$$

where $\mid \psi_0\rangle$ is an eigenstate of $\hat{H}_0$ and $\mid \psi_1\rangle$ is an eigenstate of $\hat{H}_1$ with the same energy.

A sequence of such supersymmetric relationships will generate a chain of Hamiltonians:

$$\begin{aligned} \hat{H}_0 &= \hat{B}_0\hat{A}_0 + \epsilon_0 \\ \hat{H}_1 &= \hat{A}_0\hat{B}_0 + \epsilon_0 = \hat{B}_1\hat{A}_1 + \epsilon_1 \\ &\vdots \\ \hat{H}_n &= \hat{A}_{n-1}\hat{B}_{n-1} + \epsilon_{n-1} \end{aligned} \quad (6)$$

For example, if we start with $\hat{H}_0 = -\frac{d^2}{dx^2}$ representing kinetic energy (with $\hbar = 2m = 1$), such a sequence, with $\epsilon_n = -n^2$, generates the chain [10]

$$\hat{H}_n = -\frac{d^2}{dx^2} - \frac{n(n+1)}{\cosh^2(x)} \quad (7)$$

where, for any positive integer n, the potentials

$$V_n(x) = -\frac{n(n+1)}{\cosh^2(x)} \quad (8)$$

are reflectionless at all energies [20]. The case $n = 1$ is the famous Pöschl-Teller potential. The eigenstates of each potential are linked to the reflectionless eigenstates of $\hat{H}_0$ via the map

$$\mid \psi_n\rangle \propto \hat{A}_{n-1}\hat{A}_{n-2}\cdots\hat{A}_0 \mid \psi_0\rangle \quad (9)$$

Each of the operators $\hat{A}_m = \frac{d}{dx} + m\tanh(x)$ asymptotically becomes a differential operator with constant coefficients, as does their product. Therefore, the map between eigenstates locally converts plane waves to plane waves conserving the direction of momentum. This fact explains why every member of this SUSY chain is reflectionless at all energies. In general, if a Hamiltonian is linked to a potential-free Hamiltonian via a supersymmetric chain, it is reflectionless.

SUSY Decomposition of the $1/\cosh(t)$ *Pulse Hamiltonian.*– Using the known scattering solutions (3) as a guide, we found that the Hamiltonian $\hat{H}$ in the spatial-interpretation of the $1/\cosh(t)$ problem (2) is linked to a potential-free Hamiltonian $\hat{H}_0$ via a two-step supersymmetric chain:

$$\begin{aligned} \hat{H}_0 &= \hat{B}_0\hat{A}_0 - i/2 \\ \hat{H}_1 &= \hat{A}_0\hat{B}_0 - i/2 = \hat{B}_1\hat{A}_1 + i/2 \\ \hat{H} = \hat{H}_2 &= \hat{A}_1\hat{B}_1 + i/2 \quad , \end{aligned} \quad (10)$$

where

$$\begin{aligned}
\hat{H}_0 &= \begin{pmatrix} -i\frac{d}{dx} & 0 \\ 0 & i\frac{d}{dx} \end{pmatrix} \\
\hat{H}_1 &= \begin{pmatrix} -i\frac{d}{dx} & 0 \\ 2e^{-x} & i\frac{d}{dx} \end{pmatrix} \\
\hat{H} = \hat{H}_2 &= \begin{pmatrix} -i\frac{d}{dx} & \frac{1}{\cosh(x)} \\ -\frac{1}{\cosh(x)} & i\frac{d}{dx} \end{pmatrix} \\
\hat{A}_0 &= \begin{pmatrix} \frac{d}{dx} - \frac{1}{2} & 0 \\ ie^{-x} & 1 \end{pmatrix} \\
\hat{B}_0 &= \begin{pmatrix} -i & 0 \\ e^{-x} & i\frac{d}{dx} + \frac{i}{2} \end{pmatrix} \\
\hat{A}_1 &= \begin{pmatrix} 1 & -\frac{i}{2\cosh(x)} \\ -ie^{-x} & \frac{d}{dx} - \frac{\tanh(x)}{2} \end{pmatrix} \\
\hat{B}_1 &= \begin{pmatrix} -i\frac{d}{dx} - \frac{i\tanh(x)}{2} & \frac{1}{2\cosh(x)} \\ e^{-x} & i \end{pmatrix} \qquad (11)
\end{aligned}$$

Similar to the Eqn. (9), the map between the eigenstates of $\hat{H}_0$ and $\hat{H}$ is given by

$$\mid \psi\rangle \propto \hat{A}_1\hat{A}_0 \mid \psi_0\rangle$$

where

$$\hat{A}_1\hat{A}_0 = \begin{pmatrix} \frac{d}{dx} - \frac{\tanh(x)}{2} & -\frac{i}{2\cosh(x)} \\ -\frac{i}{2\cosh(x)} & \frac{d}{dx} - \frac{\tanh(x)}{2} \end{pmatrix} \qquad (12)$$

which is fully consistent with the known form of the eigenstates (3). Notice that each term in the map $\hat{A}_1\hat{A}_0$ asymptotically becomes a differential operator with constant coefficients, similar to (9). This ensures that the map preserves the reflectionless property of the plane wave eigenstates of $\hat{H}_0$.

Additionally, there exists another supersymmetric chain linking the Hamiltonians $\hat{H}_0$ and $\hat{H}$. It can be obtained if one reverses the order of the factorization energies $\mp i/2$ and uses another set of the SUSY factors represented by $\hat{A}' = \sigma_1\hat{A}\sigma_1$ and $\hat{B}' = -\sigma_1\hat{B}\sigma_1$, where σ_1 is the first Pauli matrix.

$1/\cosh(t)$ *Pulse Hamiltonian and sine-Gordon equation.*– We now explore another parallel between the Schrödinger equation and our two-level time-dependent problem. It is well known [4, 5] that reflectionless potentials $V(x)$ for the Schrödinger operator $-(d^2/dx^2)+V(x)$ produce multi-soliton solutions of the Korteweg-deVries equation if used as initial states. Additionally, all of the known reflectionless potentials for the Schrödinger operator can be obtained via SUSY chains originating from the potential-free Hamiltonian $-(d^2/dx^2)$ [10–13]. In particular, the first ($n = 1$) member of the chain (8) produces a single-soliton solution of the Korteweg-deVries equation. The subsequent members of the chain produce n-soliton solutions, where at $t = 0$ all solitons are localized at the origin. In general, for any initial configuration of the solitons, the corresponding initial state can be obtained via a SUSY procedure.

Consider now the following trivial substitution: $\zeta = -x$, $\psi_1 = v$, $\psi_2 = u$. Replace the off-diagonal perturbation $1/\cosh(x)$ in Eqn. (2) by a derivative of a function of two variables, $\Phi(\zeta, \eta)$. Eqn. (2) then becomes

$$\begin{aligned}
\frac{d}{d\zeta}\psi_1 &= +i\lambda\psi_1 + \frac{i}{2}\left(\frac{\partial}{\partial\zeta}\Phi\right)\psi_2 \\
\frac{d}{d\zeta}\psi_2 &= -i\lambda\psi_2 + \frac{i}{2}\left(\frac{\partial}{\partial\zeta}\Phi\right)\psi_1 \qquad (13)
\end{aligned}$$

This is nothing else but the first member of the Lax pair for the sine-Gordon equation

$$\frac{\partial^2}{\partial\zeta\partial\eta}\Phi = \sin(\Phi) \qquad (14)$$

(see [19]). Now notice that, exactly as in the Korteweg-deVries case [10–13], the lowest relevant member of a supersymmetric chain (10),(11) produces the single soliton solution,

$$\Phi(\zeta, \eta) = 4\arctan(\exp(\zeta + \eta)) \qquad (15)$$

of the sine-Gordon equation. Indeed, in this case $\left(\frac{\partial}{\partial\zeta}\Phi\right)\Big|_{\eta=0} = 1/\cosh(\eta)$, leading directly to system (2).

Summary and outlook.– In this Letter we use quantum-mechanical supersymmetry to explain why a two level atom cannot change its internal state under a $1/\cosh(t)$ pulse for any value of the laser detuning. At the same time, we show that the problem trivially maps to the first member of the Lax pair for the sine-Gordon equation [19], where the $1/\cosh(t)$ pulse corresponds to a single kink soliton.

We conjecture that, by analogy with the Korteweg-deVries equation [10–13], the multi-soliton solutions of the sine-Gordon equation can also be obtained using supersymmetric chains. This constitutes one of our projects for the near future.

The striking similarity between the reflectionless wavefunctions (3) for the $1/\cosh(t)$ pulse and the scattering solutions of the Bogoliubov-de Gennes equations for a Bose condensate in the presence of a bright soliton may also indicate a SUSY mechanism for the absence of reflection in the latter case [16, 17]. Identifying this chain constitutes another direction of future research.

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