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Electromagnetic wave scattering by a thin layer in which many small particles are embedded

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Abstract

Scattering of electromagnetic (EM) waves by many small particles (bodies), embedded in a thin layer, is studied. Physical properties of the particles are described by their boundary impedances. The thin layer of depth of the order $O(a)$ with many embedded in it small particles of characteristic size a, is described by a boundary condition on the surface of the layer. The limiting interface boundary condition is obtained for the effective EM field in the limiting medium, in the limit $a \to 0$, where the number $M(a)$ of the particles tends to infinity at a suitable rate.

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1 Introduction

It is known (see, e.g., [2], [9]) that the light propagation through diffraction gratings may exibit strong resonances at certain frequencies. This is useful in applications. In this paper we study electromagnetic (EM) wave scattering by many small impedance particles D_m, $1 \le m \le M$, $M = M(a)$, embedded in a thin layer of the depth $h(a) \sim a$, where a is the characteristic dimension of a small particle. The shape of the particles may be fairly arbitrary, not necessarily spherical.

We assume that

$$\lim_{a\to 0} a/d(a) = 0, \qquad lim_{a\to 0} d(a) = 0, \qquad ka \ll 1, \tag{1}$$

where k is the wavenumber, and $d(a)$ is the distance between neighboring particles.

The thin layer is located on a smooth surface S. The permettivity ϵ_0, conductivity $\sigma_0 \geq 0$, $\epsilon' = \epsilon_0 + \frac{i\sigma_0}{\omega}$ and permeability μ_0 of the space are assumed constants, $k^2 = \omega^2 \epsilon' \mu_0$, ω is the frequency.

For example, one may assume that S is the plane $x_3 = 0$, but our arguments are valid for an arbitrary smooth S. The M particles on S are distributed according to the following law: in any open subset Δ of S there are $\mathcal{N}(\Delta)$ particles, where

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(s) ds [1 + o(1)], \quad a \to 0, \tag{2}$$

$N(s) \geq 0$ is a continuous function, vanishing outside of a finite domain $\Omega \subset S$ in which small particles (bodies) D_m are distributed, $\kappa \in (0, 1)$ is a number, and the boundary impedances of the small particles are defined by the formula

$$\zeta_m = \frac{h(s_m)}{a^\kappa}, \quad s_m \in D_m, \tag{3}$$

where $s_m \in S$ is a point inside $m-$th particle D_m, Re $h(s) \geq 0$, and $h(s)$ is a continuous function vanishing outside Ω.

We can choose κ and $h(s)$ as we wish.

Denote by $[E, H] = E \times H$ the cross product of two vectors, and by $(E, H) = E \cdot H$ the dot product of two vectors.

The impedance boundary condition on the surface S_m of the $m-$th particle D_m is $E^t = \zeta_m [H^t, N]$, where E^t (H^t) is the tangential component of E (H) on S_m, and N is the unit normal to S_m, pointing out of D_m. We define $E^t = [N, [E, N]] = E - N(E, N)$. This corresponds to the geometrical meaning of the tangential component of E, and differs from the definition $E^t = [N, E]$ that is used sometimes.

In this paper we use the methodology, developed in [7] and some results from [7].

2 EM wave scattering by many small particles

Electromagnetic (EM) wave scattering problem consists of finding vectors E and H satisfying the Maxwell equations:

$$\nabla \times E = i\omega\mu_0 H, \quad \nabla \times H = -i\omega\epsilon_0 E \quad \text{in } D := \mathbb{R}^3 \setminus \cup_{m=1}^M D_m, \tag{4}$$

the impedance boundary conditions:

$$[N,[E,N]] = \zeta_m [H,N] \text{ on } S_m,\ 1 \le m \le M, \tag{5}$$

and the radiation conditions:

$$E = E_0 + v_E, \quad H = H_0 + v_H, \tag{6}$$

where v_E and v_H satisfy the radiation condition, ζ_m is the impedance, defined in (3), N is the unit normal to S_m pointing out of D_m, E_0, H_0 are the incident fields satisfying equations (1) in all of $\mathbb{R}^3$. One often assumes that the incident wave is a plane wave, i.e., $E_0 = \mathcal{E}e^{ik\alpha\cdot x}$, $\mathcal{E}$ is a constant vector, $\alpha \in S^2$ is a unit vector, S^2 is the unit sphere in $\mathbb{R}^3$, $\alpha \cdot \mathcal{E} = 0$, v_E and v_H satisfy the radiation condition: $r(\frac{\partial v}{\partial r} - ikv) = o(1)$ as $r := |x| \to \infty$, $k = \omega\sqrt{\epsilon_0\mu_0}$.

By impedance ζ_m we assume in this paper a constant, Re $\zeta_m \ge 0$, or a matrix function 2×2 acting on the tangential to S_m vector fields, such that

$$\mathrm{Re}(\zeta_m E^t, E^t) \ge 0 \quad \forall E^t \in T_m, \tag{7}$$

where T_m is the set of all tangential to S_m continuous vector fields such that $\mathrm{Div} E^t = 0$, where Div is the surface divergence, and E^t is the tangential component of E. Smallness of D_m means that $ka \ll 1$, where $a = 0.5 \max_{1 \le m \le M} \mathrm{diam} D_m$.

Lemma 1. *Problem* (4)-(7) *has at most one solution.*

Lemma 1 is proved in [7].

Let us note that problem (4)-(7) is equivalent to the problems (8), (9), (6), (7), where

$$\nabla \times \nabla \times E = k^2 E \text{ in } D, \qquad H = \frac{\nabla \times E}{i\omega\mu_0}, \tag{8}$$

$$[N,[E,N]] = \frac{\zeta_m}{i\omega\mu_0}[\nabla \times E, N] \text{ on } S_m,\ 1 \le m \le M. \tag{9}$$

This is the impedance boundary condition (see, e.g., [1], p. 301.) The expression $[N,[E,N]] = E - (E,N)N$ is the tangential component of the field E on the surface S_m, N is the unit normal to S_m pointing out of D_m.

Thus, we have reduced our problem to finding one vector $E(x)$. If $E(x)$ is found, then $H = \frac{\nabla \times E}{i\omega\mu_0}$.

Let us look for E of the form

$$E = E_0 + \sum_{m=1}^{M} \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt, \quad g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{10}$$

where $t \in S_m$, dt is an element of the area of S_m, and $\sigma_m(t) \in T_m$. This E for any continuous $\sigma_m(t)$ solves equation (8) in D because E_0 solves (8) and

$$\begin{aligned}\nabla \times \nabla \times \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt &= \nabla\nabla \cdot \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt \\ &- \nabla^2 \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt \\ &= k^2 \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt, \quad x \in D.\end{aligned} \tag{11}$$

Here we have used the known identity $divcurlE = 0$, valid for any smooth vector field E, and the known formula

$$-\nabla^2 g(x,y) = k^2 g(x,y) + \delta(x-y). \tag{12}$$

The integral $\int_{S_m} g(x,t)\sigma_m(t)dt$ satisfies the radiation condition. Thus, formula (10) gives $E(x)$ that solves problem (8), (9), and satisfies the radiation condition, if $\sigma_m(t)$ are chosen so that boundary conditions (9) are satisfied.

Define the effective field $E_e(x) = E_e^m(x) = E_e^{(m)}(x,a)$, acting on the $m-$th body D_m:

$$E_e(x) := E(x) - \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt := E_e^{(m)}(x), \tag{13}$$

where we assume that x is in a neigborhood of S_m. However, $E_e(x)$ is defined for all $x \in \mathbb{R}^3$.

Away from S, the field $E_e(x,a)$ tends to a limit $E(x) = E_e(x)$ as $a \to 0$, and $E_e(x)$ is a twice continuously differentiable function away from S, see [7]. To derive an integral equation for $\sigma_m = \sigma_m(t)$, substitute

$$E = E_e + \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt$$

into boundary condition (9), use the known formula (see, e.g., [3])

$$[N, \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt]_{\mp} = \int_{S_m} [N_s, [\nabla_s g(x,t), \sigma_m(t)]]dt \pm \frac{\sigma_m(t)}{2}, \tag{14}$$

where the -(+) signs denote the limiting values of the left-hand side of (14) as $x \to s$ from D (D_m), and get

$$\sigma_m(t) = A_m \sigma_m + f_m, \quad 1 \le m \le M. \tag{15}$$

Here A_m is a linear Fredholm-type integral operator, and f_m is a continuously differentiable function.

Let us specify A_m and f_m. One has

$$f_m = 2[N_s, f_e(s)], \quad f_e(s) := [N_s, [E_e(s), N_s]] - \frac{\zeta_m}{i\omega\mu_0}[\nabla \times E_e, N_s]. \tag{16}$$

Condition (9) and formula (14) yield

$$\begin{aligned} f_e(s) + \frac{1}{2}[\sigma_m(s), N_s] + [\int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt, N_s] \\ - \frac{\zeta_m}{i\omega\mu_0}[\nabla \times \nabla \times \int_{S_m} g(x,t)\sigma_m(t)dt, N_s]|_{x\to s} = 0 \end{aligned} \tag{17}$$

Using the known formula $\nabla \times \nabla = graddiv - \nabla^2$, the relation

$$\begin{aligned} \nabla_x \nabla_x \cdot \int_{S_m} g(x,t)\sigma_m(t)dt &= \nabla_x \int_{S_m} (-\nabla_t g(x,t), \sigma_m(t))dt \\ &= \nabla_x \int_{S_m} g(x,t)\text{Div}\sigma_m(t)dt = 0, \end{aligned} \tag{18}$$

where Div is the surface divergence, and a consequence of formula (12)

$$-\nabla_x^2 \int_{S_m} g(x,t)\sigma_m(t)dt = k^2 \int_{S_m} g(x,t)\sigma_m(t)dt, \quad x \notin S, \tag{19}$$

one gets from (17) the following equation

$$[N_s, \sigma_m(s)] + 2f_e(s) + 2B\sigma_m = 0. \tag{20}$$

Here

$$B\sigma_m := [\int_{S_m} [N_s, [\nabla_s g(s,t), \sigma_m(t)]]dt, N_s] + \zeta_m i\omega\epsilon_0[\int_{S_m} g(s,t)\sigma_m(t)dt, N_s]. \tag{21}$$

Take cross product of N_s with the left-hand side of (20) and use the formulas $N_s \cdot \sigma_m(s) = 0$, $f_m := f_m(s) := 2[N_s, f_e(s)]$, and

$$[N_s, [N_s, \sigma_m(s)]] = -\sigma_m(s), \tag{22}$$

to get from (20) equation (15):

$$\sigma_m(s) = 2[N_s, f_e(s)] + 2[N_s, B\sigma_m] := A_m\sigma_m + f_m, \tag{23}$$

where $A_m\sigma_m = 2[N_s, B\sigma_m]$. The operator A_m is linear and compact in the space $C(S_m)$, so that equation (23) is of Fredholm type. Therefore, equation (23) is solvable for any $f_m \in T_m$ if the homogeneous version of (23) has only the trivial solution $\sigma_m = 0$. In this case the solution σ_m to equation (23) is of the order of the right-hand side f_m, that is, $O(a^{-\kappa})$ as $a \to 0$, see formula (16). Moreover, it follows from equation (23) that the main term of the asymptotics of σ_m as $a \to 0$ does not depend on $s \in S_m$.

Lemma 2. *Assume that $\sigma_m \in T_m$, $\sigma_m \in C(S_m)$, and $\sigma_m(s) = A_m\sigma_m$. Then $\sigma_m = 0$.*

Lemma 2 is proved in [7].

Let us write (10) as

$$E(x) = E_0(x)+\sum_{m=1}^{M}[\nabla_x g(x,x_m), Q_m]+\sum_{m=1}^{M}\nabla\times\int_{S_m}(g(x,t)-g(x,x_m))\sigma_m(t)dt, \tag{24}$$

where

$$Q_m := \int_{S_m}\sigma_m(t)dt. \tag{25}$$

Since $\sigma_m = O(a^{-\kappa})$, one has $Q_m = O(a^{2-\kappa})$. We want to prove that the second sum in (24) is negligible compared with the first sum. One has

$$j_1 := |[\nabla_x g(x,x_m), Q_m]| \leq O\left(\max\left\{\frac{1}{d^2}, \frac{k}{d}\right\}\right)O(a^{2-\kappa}), \tag{26}$$

$$j_2 := |\nabla\times\int_{S_m}(g(x,t)-g(x,x_m))\sigma_m(t)dt| \leq aO\left(\max\left\{\frac{1}{d^3}, \frac{k^2}{d}\right\}\right)O(a^{2-\kappa}), \tag{27}$$

and

$$\left|\frac{j_2}{j_1}\right| = O\left(\max\left\{\frac{a}{d}, ka\right\}\right) \to 0, \qquad \frac{a}{d} = o(1), \qquad a \to 0. \tag{28}$$

Thus, one may neglect the second sum in (26), and write

$$E(x) = E_0(x) + \sum_{m=1}^{M}[\nabla_x g(x,x_m), Q_m] \tag{29}$$

with an error that tends to zero as $a \to 0$. Let us estimate Q_m asymptotically, as $a \to 0$. Integrate equation (23) over S_m to get

$$Q_m = 2\int_{S_m}[N_s, f_e(s)]ds + 2\int_{S_m}[N_s, B\sigma_m]ds. \tag{30}$$

It follows from (16) that

$$[N_s, f_e] = [N_s, E_e] - \frac{\zeta_m}{i\omega\mu_0}[N_s, [\nabla \times E_e, N_s]]. \tag{31}$$

If E_e tends to a finite limit as $a \to 0$, then formula (31) implies

$$[N_s, f_e] = O(\zeta_m) = O\left(\frac{1}{a^\kappa}\right), \quad a \to 0. \tag{32}$$

By Lemma 2 the operator $(I - A_m)^{-1}$ is bounded, so $\sigma_m = O\left(\frac{1}{a^\kappa}\right)$, and

$$Q_m = O\left(a^{2-\kappa}\right), \quad a \to 0, \tag{33}$$

because integration over S_m adds factor $O(a^2)$. As $a \to 0$, the sum (29) converges to the integral (see [8], Lemma 1)

$$E = E_0 + \nabla \times \int_S g(x,s)N(s)Q(s)ds, \tag{34}$$

where $N(s)$ is the function from (2), and $Q(s)$ is the function such that

$$Q_m = Q(x_m)a^{2-\kappa}. \tag{35}$$

The function $Q(y)$ can be expressed in terms of E:

$$Q(y) = -\frac{8\pi}{3} i\omega\epsilon_0 h(s)(\nabla \times E)(s), \tag{36}$$

see [7]. Here the factor $\frac{8\pi}{3}$ appears if D_m are balls. Otherwise a tensorial factor c_m, depending on the shape of S_m, should be used in place of $\frac{8\pi}{3}$. The factor c_m is defined by the formula $\int_{S_m} \nabla \times E_e(s)ds = a^2 c_m \nabla \times E_e(x_m)$, where $x_m \in S$ is a point in D_m.

Thus, equation (36) takes the form

$$E(x) = E_0(x) - \frac{8\pi}{3} i\omega\epsilon_0 \nabla \times \int_S g(x,s)\nabla \times E(s)h(s)N(s)ds. \tag{37}$$

It follows from equation (37) that the limiting field $E(x)$ satisfies equation (8) away from S, and a transmission boundary condition on S:

$$[N_s, E_-(s) - E_+(s)] = -\frac{8\pi}{3} i\omega\epsilon' h(s)N(s). \tag{38}$$

3 Conclusions

It is proved that a distribution of many small impedance particles in a thin layer on a smooth surface S can be described by a transmission boundary condition (38). This condition shows that the equivalent surface currents on S are calculated analytically in the limit $a \to 0$ in terms of boundary impedance function h and the distribution density function $N(s)$. Therefore, these currents can be controlled.

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