ERROR ESTIMATES IN HOROCYCLE AVERAGES ASYMPTOTICS: CHALLENGES FROM STRING THEORY

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Abstract. For modular functions of rapid decay, a classical result connects the error estimate in their long horocycle average asymptotic to the Riemann hypothesis. We study similar asymptotics, for modular functions with not that mild growing conditions, such as of polynomial growth and of exponential growth at the cusp. Hints on their long horocycle average are derived by translating the horocycle flow dynamical problem in string theory language. Results are then proved by designing an unfolding trick involving a Theta series, related to the spectral Eisenstein series by Mellin integral transform. We discuss how the string theory point of view leads to an interesting open question, regarding the behavior of long horocycle averages of a certain class of automorphic forms of exponential growth at the cusp.

1. Introduction

In this paper we use a new angle for obtaining insights on asymptotics of long horocycle averages of certain classes of $SL(2,\mathbb{Z})$-invariant automorphic functions. We focus on certain class of polynomial growing conditions, and on a certain class of exponential growing conditions. Modular functions with such growing conditions do appear in string theory, in perturbative (one-loop) closed string amplitudes. Remarkably, their horocycle averages count graded numbers[1] of physical particle-like excitations of a closed string [C1],[CC],[ACER].

The advantage of translating the dynamical problem in string theory terms is in the possibility of using consistency conditions from string theory to gain insights on the horocycle average asymptotic. For the two classes of modular forms we will focus on, the string theory perspective suggests a universal behavior of their long horocycle average. On the other hand, this universal behavior appears somehow surprising from the perspective of the theory of automorphic forms.

Once we obtain hints from string theory, we then devise an unfolding method to prove theorems. This unfolding trick involves a Theta series which is connected to the spectral Eisenstein series by Mellin integral transform. We shall compare this Theta-unfolding device with the classical Rankin-Selberg method involving the spectral Eisenstein series. We illustrate advantages of our methods for automorphic

[1]A genus one closed string amplitude $\mathcal{A}$ is given by the integral of a $SL(2,\mathbb{Z})$ invariant function f on the fundamental domain $\boldsymbol{\mathcal{D}} \simeq SL(2,\mathbb{Z})\backslash\boldsymbol{\mathcal{H}}$, $\mathcal{A} = \int_{\boldsymbol{\mathcal{D}}} d\mu f$. Numbers of closed string states are encoded in the expansion of the automorphic function f horocycle average, $\int_0^1 dx f(x,y) = \sum_{n=0}^{\infty}(d_n^B - d_n^F)e^{-\pi m_n^2 y}$, where $d_n^B(d_n^F)$ is the number of bosonic(fermionic) states corresponding to a (squared) mass m_n^2. Convergence of the long horocycle limit $y \to 0$ corresponds to a subtle cancelation between bosonic and fermionic particle-like closed string excitations. This cancelation was called asymptotic supersymmetry in [KS]. Quite interestingly, horocycle averages asymptotics as (1.2) when translated in closed string theory terms do show intriguing dependence of asymptotic supersymmetry on the Riemann hypothesis [C1], [CC], [ACER].

forms of not so mild growing conditions at the cusp. In particular, we reobtain by our methods results on analytic continuation of the Rankin-Selberg integral transform for automorphic functions of polynomial growth, derived by different methods in [Za2]. We then obtain asymptotics for long horocycle averages of modular functions of polynomial growth including the error estimate, which parallel the rapid decay case studied in [Za1].

Remarkably, for certain classes of automorphic forms a precise form of the error estimate in their long horocycle average asymptotic is equivalent to the Riemann hypothesis [Za1]. This result, originally proved for a certain class of modular functions of rapid decay in [Za1], is extended here to modular function of polynomial growth. When applied to modular function appearing in string theory, this result leads to fascinating connections between enumerative properties of closed string spectra and the Riemann hypothesis [C1],[CC],[ACER]. These connections extend to multi-loops closed string amplitudes [CC],[CC2], and results for measure rigidity of unipotent flows in homogenous spaces [Ra] are intertwined with properties of perturbative closed string theory.

Let $\boldsymbol{\mathcal{H}} = \{z = x + iy \in \mathbb{C} | y > 0\}$ be the upper complex plane, horocycles in $\boldsymbol{\mathcal{H}}$ are both circles tangent to the real axis in rational points (cusps), and horizonal lines, (which can be thought as circles tangent to the $z = i\infty$ cusp).

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ acts on $z \in \boldsymbol{\mathcal{H}}$ through the Möbius transformation $z \to \frac{az+b}{cz+d}$. The following one-parameter action of the upper triangular unipotent subgroup $\boldsymbol{U} \subset SL(2,\mathbb{R})$

$$\boldsymbol{g}_u := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \, |t| \le |u|, \right\}, \qquad u \in \mathbb{R} \tag{1.1}$$

generates motions along horizontal lines in $\boldsymbol{\mathcal{H}}$. Long horocycles in $\boldsymbol{\mathcal{H}}$ do not exhibit interesting dynamics, since the orbit $\boldsymbol{g}_u(x + iy) = \{x + t + iy, |t| \le |u|\}$ for $u \to \pm\infty$ just escapes to infinity.

However, $\boldsymbol{g}_u(x + iy)$ has an interesting dynamics in the quotient space $\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}$, $\boldsymbol{\Gamma} \simeq SL(2,\mathbb{Z})$. The horocycle $\boldsymbol{g}_{u=1}(x + iy)$ is a closed orbit in $\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}$ with length $1/y$, as measured by the hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$. Quite remarkably, in the long length limit $y \to 0$, the horocycle $\boldsymbol{g}_{u=1}(x + iy)$ tends to cover uniformly the modular domain $\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}$. Equidistribution of long horocycles in $\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}$ was first seen to follow by uniquely ergodicity of the horocycle flow [Fu][Da].

From a different angle, methods involving the theory of automorphic forms lead to interesting results for horocycle flow asymptotic. Quite remarkable is the relation between error estimates for asymptotics involving automorphic forms averages along long horocycles and the Riemann hypothesis. By using the Rankin-Selberg method, Zagier [Za1] has obtained the intriguing result

$$\int_0^1 dx f(x,y) \sim \frac{3}{\pi} \int_{\boldsymbol{\mathcal{D}}} d\mu f + O(y^{1-\frac{\Theta}{2}}), \qquad y \to 0 \tag{1.2}$$

when f is a smooth modular invariant function of rapid decay at the cusp $y \to \infty$, whose hyperbolic Laplacian Δf is of bounded polynomial growth at the cusp, (see section 2, proposition 3 for sufficient conditions on f for the asymptotic (1.2) to hold). μ is the hyperbolic $\boldsymbol{\mathcal{H}}$ measure, $d\mu = y^{-2}dxdy$, and the error estimate is governed by $\Theta = \mathrm{Sup}\{\Re(\rho)|\boldsymbol{\zeta}^*(\rho) = 0\}$, the superior of the real part of the non trivial zeros of the Riemann zeta function $\boldsymbol{\zeta}(s)$, $(\boldsymbol{\zeta}^*(s) = \pi^{-s/2}\boldsymbol{\Gamma}(s/2)\boldsymbol{\zeta}(s))$.

The error estimate for the convergence rate in (1.2) is remarkably linked to the Riemann hypothesis (RH)[2], indeed, RH is equivalent to the following condition

$$\int_0^1 dx f(x,y) \sim \frac{3}{\pi}\int_{\mathcal{D}} d\mu f + O(y^{3/4-\epsilon}), \qquad y \to 0 \tag{1.3}$$

for every $f \in C_{00}^{\infty}(\mathbf{\Gamma}\backslash\mathcal{H})$. Up to date, the error term can be estimated to be $O(y^{1/2+\epsilon})$, by using the bound $\Theta \leq 1$ on the real part of the Riemann zeta functions zeros ρ's in the critical strip $0 < \Re(\rho) < 1$.

Notation and Terminology

$\boldsymbol{\mathcal{H}} = \{z = x + iy \in \mathbb{C},\ y > 0\}$, the upper complex plane.

$\mathbf{\Gamma} \simeq SL(2,\mathbb{Z})$, the modular group.

$\boldsymbol{\mathcal{D}} \simeq \mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}$, the standard $SL(2,\mathbb{Z})$ fundamental domain with cusp at $z = i\infty$

$\mathbf{\Gamma}_\infty \subset \mathbf{\Gamma}$, the subgroup of upper triangular matrices.

$\boldsymbol{\zeta}(s) = \sum_{n\in\mathbb{N}} n^{-s}$, $\Re(s) > 1$, the Riemann zeta function.

$\boldsymbol{\zeta}^*(s) = \pi^{-s/2}\mathbf{\Gamma}(s/2)\boldsymbol{\zeta}(s)$.

$\mathbf{\Theta}_t(z) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} e^{-\pi t \frac{|mz+n|^2}{y}}$.

$\boldsymbol{E}_s(z) = \frac{1}{2}\sum_{(c,d)\in\mathbb{Z}^2,(c,d)=1} y^s|cz+d|^{-2s}$.

$\boldsymbol{E}_s^*(z) = \pi^{-s}\mathbf{\Gamma}(s)\sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} y^s|mz+n|^{-2s}$.

$\boldsymbol{\vartheta}_t(y) = \sum_{n\in\mathbb{N}_{>0}} e^{-\pi\frac{t}{y}n^2}$.

$\boldsymbol{\mathcal{M}}_y[\varphi](s) = \int_0^\infty dy y^{s-1}\varphi(y)$, the Mellin transform of the function φ.

$\mathcal{P}[\varphi](z) = \sum_{\gamma\in\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}} \varphi(\Im(\gamma(z)))$, the Poincaré series of the function $\varphi : \mathbb{R}_{>0} \to \mathbb{C}$.

$\boldsymbol{a}_0(y) = \int_0^1 dx f(x,y)$, the constant term of the modular invariant function $f(x,y) = \sum_{n\in\mathbb{Z}} \boldsymbol{a}_n(y)e^{2\pi i n x}$.

$\langle f, g\rangle_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}} = \int_{\boldsymbol{\mathcal{D}}} dx dy y^{-2}\bar{f}(z)g(z)$, the Petersson inner product of the modular invariant functions $f(z)$, $g(z)$.

$\langle \varphi, \xi\rangle_{\boldsymbol{U}\backslash\boldsymbol{\mathcal{H}}} = \int_0^\infty dy y^{-2}\bar{\varphi}(y)\xi(y)$, the inner product on the space of functions $\boldsymbol{U}\backslash\boldsymbol{\mathcal{H}} \simeq \mathbb{R}_{>0} \to \mathbb{C}$.

Due to $\mathbf{\Gamma}_\infty = \boldsymbol{U} \cap SL(2,\mathbb{Z})$ invariance, a modular invariant function $f = f(x,y)$, can be decomposed in Fourier series in the x variable, $f(x,y) = \sum_{n\in\mathbb{Z}} \boldsymbol{a}_n(y)e^{2\pi i n x}$. The constant Fourier term $\boldsymbol{a}_0(y)$ then equals the f average along the horocycle $\boldsymbol{\mathcal{H}}_y := (\mathbb{R} + iy)/\mathbf{\Gamma}_\infty$

$$\boldsymbol{a}_0(y) = \int_0^1 dx f(x,y) = \frac{1}{L(\boldsymbol{\mathcal{H}}_y)}\int_{\boldsymbol{\mathcal{H}}_y} ds f, \tag{1.4}$$

where $L(\boldsymbol{\mathcal{H}}_y) = 1/y$ is the horocycle length, measured by the hyperbolic $\boldsymbol{\mathcal{H}}$ metric, $ds = y^{-1}\sqrt{dx^2 + dy^2}$.

[2] See also [Sa], [Ve] for a study of horocycle flows and Eisenstein series for more general quotients $\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}$, where $\mathbf{\Gamma} \subset SL(2,\mathbb{Z})$ is a lattice.

We focus on two classes of growing conditions for $SL(2,\mathbb{Z})$-invariant functions. Modular functions with polynomial growth at the cusp $y \to \infty$:
(1.5)

$$\boldsymbol{\mathcal{C}}_{TypeII} = \{f(x,y) \sim \sum_{i=1}^{l} \frac{c_i}{n_i!}\, y^{\alpha_i} \log^{n_i} y,\ y \to \infty,\ \ c_i, \alpha_i \in \mathbb{C}, \Re(\alpha_i) < 1/2, n_i \in \mathbb{N}_{\geq 0}\},$$

and modular functions with bounded exponential growth at the cusp, whose horocycle average $\boldsymbol{a}_0(y)$ grows polynomially at the cusp $y \to \infty$:

$$\boldsymbol{\mathcal{C}}_{Heterotic} = \{f(x,y) \sim y^{\alpha} e^{\pi\beta y} e^{2\pi i \kappa x}, y \to \infty; \beta < 1, \kappa \in \mathbb{Z}\backslash\{0\}, \Re(\alpha) < 1/2\}. \tag{1.6}$$

For modular functions in both $\boldsymbol{\mathcal{C}}_{TypeII}$ and $\boldsymbol{\mathcal{C}}_{Heterotic}$ we address questions related to the asymptotic of their long horocycle averages $\boldsymbol{a}_0(y),\ y \to 0$. The choices of symbols $\boldsymbol{\mathcal{C}}_{TypeII}$ and $\boldsymbol{\mathcal{C}}_{Heterotic}$, reflect the appearance of modular functions with such growing conditions respectively in type II string and heterotic string genus one closed string amplitudes, (with no tachyons in the spectrum). Bounds on α and on β in (1.5) and (1.6) are universal in string theory, and follow by consistency requirements (unitarity) of the quantum worldsheet conformal field theory.

String theory suggests that both automorphic functions with growing conditions in $\boldsymbol{\mathcal{C}}_{TypeII}$ and $\boldsymbol{\mathcal{C}}_{Heterotic}$ do have convergent horocycle average in the long limit $y \to 0$, and should exhibit asymptotic similar to (1.2). Those hints follow from the following considerations: the exponentially growing part for a modular function f in $\boldsymbol{\mathcal{C}}_{Heterotic}$ in string theory language corresponds to a "non-physical tachyon ", a tachyonic state which is not in the physical spectrum. Indeed, the exponentially growing part $f(x,y) \sim e^{2\pi i\kappa x} e^{2\pi\beta y}$, $y \to \infty$, $\kappa \in \mathbb{Z} \setminus \{0\}$ does not contribute to the f horocycle average, since

$$\int_0^1 dx\, e^{2\pi i \kappa x} e^{2\pi\beta y} = 0, \qquad \kappa \in \mathbb{Z} \setminus \{0\}.$$

Non-physical tachyonic states are expected not to influence the physical properties of the string. Therefore, one expects both Type II and Heterotic strings to have the same qualitative asymptotic behavior of the spectrum, i.e. both to enjoy asymptotic supersymmetry in the absence of *physical* tachyons in their spectra [KS]. This translates back in the expectation for modular functions in both $\boldsymbol{\mathcal{C}}_{TypeII}$ and $\boldsymbol{\mathcal{C}}_{Heterotic}$ to have the same asymptotic for their long horocycle average $\boldsymbol{a}_0(y)$ in the $y \to 0$ limit.

In this paper we prove theorems for long horocycle average asymptotic of automorphic functions in $\boldsymbol{\mathcal{C}}_{TypeII}$. We also prove some weaker results for $\boldsymbol{\mathcal{C}}_{Heterotic}$, and leave open the complete answer on long horocycle averages for automorhic functions in $\boldsymbol{\mathcal{C}}_{Heterotic}$. We believe this is an interesting open question, since peculiar features of the class of function $\boldsymbol{\mathcal{C}}_{Heterotic}$ and the bounds α and β do not seem to be directly suggested from the theory of automorphic functions. A complete answer on the horocycle average asymptotic for modular function in $\boldsymbol{\mathcal{C}}_{Heterotic}$ would probe the benefit one may actually gain by translating the homogenous dynamics horocycle problem in string theory terms.

In the rest of the introduction, we summarize our results and illustrate ideas and methods employed to derive them. We shall start with a brief illustration on how the asymptotic displayed in (1.2) for modular function of rapid decay is derived by the Rankin-Selberg method [Za2], (more material on that is presented in §2).

We then switch to modular functions of polynomial growth and discuss why their long horocycle asymptotic behavior cannot be derived by the *standard* Rankin-Selberg method. We then recall a result by Zagier [Za2], which extends the Rankin-Selberg method in the polynomial growth case, by designing an unfolding method for modular integrals on a truncated version of the fundamental domain $\boldsymbol{\mathcal{D}}$. We contrast Zagier's method with an alternative unfolding method we propose here, which relies on a unfolding trick employing the theta series $\boldsymbol{\Theta}_t(\tau)$. This theta series $\boldsymbol{\Theta}_t(\tau)$ is related to the spectral Eisenstein series $\boldsymbol{E}_s(\tau)$ by a Mellin transform. One of the advantages of our method is in avoiding complications with unfolding on a truncated version of the fundamental domain $\boldsymbol{\mathcal{D}}$.

1.1. **Modular functions of rapid decay and the Rankin-Selberg method.** Let us consider the Rankin-Selberg integral

$$\langle \boldsymbol{E}_s(z), f(z)\rangle_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}} = \int_{\boldsymbol{\mathcal{D}}} dxdyy^{-2}\boldsymbol{E}_s(z)\, f(x,y), \tag{1.7}$$

when $f = f(x,y)$ is a modular invariant function of rapid decay at the cusp $y \to \infty$.

The spectral Eisenstein series $\boldsymbol{E}_s(z)$ has a Poincaré series representation for $\Re(s) > 1$

$$\boldsymbol{E}_s(z) = \sum_{\gamma\in\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}} \Im(\gamma(z))^s, \qquad \Re(s) > 1, \tag{1.8}$$

where $\gamma(z) = \frac{az+b}{cz+d}$, with $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \boldsymbol{\Gamma}$.

The possibility of exchanging the series with the integration on the fundamental domain $\boldsymbol{\mathcal{D}}$

$$\begin{aligned}\int_{\boldsymbol{\mathcal{D}}} dxdyy^{-2}\boldsymbol{E}_s(z)\, f(x,y) &= \int_{\boldsymbol{\mathcal{D}}} dxdyy^{-2}\, f(x,y) \sum_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}} \Im(\gamma(z))^s \\ &= \sum_{\gamma\in\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}} \int_{\boldsymbol{\mathcal{D}}} dxdyy^{-2}\, f(x,y)\,\Im(\gamma(z))^s,\end{aligned} \tag{1.9}$$

amounts in being able to perform the unfolding trick. By using modular transformations $\gamma \in \boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}$, one unfolds the integration domain $\boldsymbol{\mathcal{D}} \simeq \boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}$ into the half-infinite strip $\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\mathcal{H}} \simeq [-1/2, 1/2) \times (0,\infty) \subset \boldsymbol{\mathcal{H}}$.

When $f = f(x,y)$ is of rapid decay at the cusp $y \to \infty$, follows from the polynomial behavior of $\boldsymbol{E}_s(z)$ at the cusp

$$\boldsymbol{E}_s(z) \sim y^s + \frac{\boldsymbol{\zeta}^*(2s-1)}{\boldsymbol{\zeta}^*(2s)} y^{1-s} + o(y^{-N}), \qquad y\to\infty, \qquad \forall N > 0, \tag{1.10}$$

eq. (1.9) follows by Lebesgue dominated convergence theorem on the sequence of products of partial sums of the series in (1.8) times the function $f(x,y)$.

This leads to connect the Rankin-Selberg integral to the Mellin transform of the function $\boldsymbol{a}_0(y)/y$

$$\int_0^\infty dy\, y^{s-2}\boldsymbol{a}_0(y) = \int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2}\boldsymbol{E}_s(z) f(x,y). \tag{1.11}$$

A relevant issue at this point is to determine analytic properties of the integral in the r.h.s. as a function of the complex variable s. Uniform convergence for $y \to \infty$ of the Rankin-Selberg integral with respect to the complex variable s, assures that the integral function in the r.h.s. $\boldsymbol{I}(s) := \langle \boldsymbol{E}_s(z), f(z)\rangle_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}}$ inherits analytic properties of $\boldsymbol{E}_s(z)$. In the present case f is of rapid decay, and uniform convergence of the integral function $\boldsymbol{I}(s)$ holds. Thus the Mellin transform in the l.h.s. of (1.11) inherits as a function of the variable $s \in \mathbb{C}$ the same analytic properties of the Eisenstein series $\boldsymbol{E}_s(z)$.

The spectral Eisenstein series $\boldsymbol{E}_s(z)$ has a simple pole in $s = 1$ with residue $\frac{1}{2\boldsymbol{\zeta}^*(2)} = \frac{3}{\pi}$, and poles in $s = \frac{\rho}{2}$, where ρ's are the non trivial zeros of the Riemann zeta function.

This leads to the following meromorphic continuation for the Mellin transform of the function $\boldsymbol{a}_0(y)/y$

$$\langle y^s, \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} = \int_0^\infty dy\, y^{s-2}\boldsymbol{a}_0(y) = \frac{C_0}{s-1} + \sum_{\boldsymbol{\zeta}^*(\rho)=0} \frac{C_\rho}{s-\rho/2}, \tag{1.12}$$

where

$$C_0 = Res_{s\to 1}\int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2}\boldsymbol{E}_s(z)f(z) = \frac{3}{\pi}\int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2} f(z),$$

and

$$C_\rho = Res_{s\to\rho/2}\int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2}\boldsymbol{E}_s(z)f(z),$$

(whenever ρ is a multiple non trivial zero of $\zeta(s)$, one has to raise the denominator in (1.12) to a power equal the order of this zero).

Finally, one obtains the $y \to 0$ behavior of $\boldsymbol{a}_0(y)$ displayed in (1.2) by using the meromorphic continuation given in (1.12), and, (whenever the inverse Mellin transform exists), by using the following proposition:

Proposition 1. *Let $\varphi = \varphi(y)$ be a function $\varphi : (0,\infty) \to \mathbb{C}$, of rapid decay for $y \to \infty$, with Mellin transform $\boldsymbol{\mathcal{M}}[\varphi](s)$.*

Suppose, that $\boldsymbol{\mathcal{M}}[\varphi](s)$ can be analytically continued to the meromorphic function

$$\boldsymbol{\mathcal{M}}[\varphi](s) = -\sum_{i=1}^{l} \frac{1}{(\alpha_i - s)^{n_i+1}}, \qquad \alpha_i \in \mathbb{C}, \quad n_i \in \mathbb{N}_{\geq 0}, \tag{1.13}$$

then the following asymptotic holds true

$$\varphi(y) \sim \sum_{i=1}^{l} \frac{1}{n_i!}\, y^{-\alpha_i}\log^{n_i} y + o(y^N) \qquad y \to 0, \qquad \forall N > 0.$$

Therefore, if one supplies extra conditions on f, which guarantee convergence of the inverse Mellin transform integral, (discussion on this matter is postponed to section §2), then from eq. (1.12) and proposition 1, one can prove the asymptotic eq. (1.2).

In section §2 extra material on the rapid decay case is provided. There, we also contrast horocycle average asymptotic of f of rapid decay with asymptotic and error estimate of the rate of uniform distribution of the horocycle itself $\boldsymbol{\Gamma}_\infty\backslash(\mathbb{R} + iy)$ in $\boldsymbol{\mathcal{D}}$ in the limit $y \to 0$.

1.2. **Modular functions of not-so-mild growing conditions.** Let us start by discussing what does not go through in the analysis presented in the previous section when one considers modular functions which decay slower at the cusp then those of rapid decay.

When f is in $\boldsymbol{\mathcal{C}}_{TypeII}$ (1.5), the Rankin-Selberg integral in (1.7) is convergent for $Max\{\alpha_i\} < \Re(s) < 1 - Max\{\alpha_i\}$, but it is not uniformly convergent. When $\min\{\alpha_i\} > 0$ this domain of convergence is disjointed from the strip $\Re(s) > 1$ of convergence of $\boldsymbol{E}_s(z)$ as the Poincaré series (1.8). This implies that one cannot use Lebesgue dominate convergence theorem for proving the unfolding trick (1.9), and thus one cannot reach eq. (1.11).

Moreover, for $f \in \boldsymbol{\mathcal{C}}_{TypeII}$, the Rankin-Selberg integral is not uniformly convergent for $y \to \infty$ with respect to the complex parameter s. This leads to the expectation that $\boldsymbol{I}(s)$ does not inherits *only* analytic properties of $\boldsymbol{E}_s(z)$, but that $\boldsymbol{I}(s)$ had singularities also depending on α_i, n_i.

Zagier [Za2] has designed a Rankin-Selberg method for automorphic functions of polynomial behavior at the cusp by devising an unfolding trick for modular integral restricted to a truncated version of the fundamental domain $\boldsymbol{\mathcal{D}}_T := \{x + iy \in \boldsymbol{\mathcal{D}} | y \leq T, T > 1\}$. In this way, He connects analytic properties of the Rankin-Selberg integral on $\boldsymbol{\mathcal{D}}_T$, to various quantities involving the modular function $f(x, y)$, and its constant term $\boldsymbol{a}_0(y)$. Then by studying the $T \to \infty$ limit, He obtains analytic properties of the following Rankin-Selberg integral transform $\boldsymbol{R}^*(f, s)$,

$$\boldsymbol{R}^*(f, s) := \boldsymbol{\zeta}^*(2s) \int_0^\infty dy y^{s-2} \left(\boldsymbol{a}_0(y) - \varphi(y)\right), \tag{1.14}$$

where

$$\varphi(y) := \sum_{i=1}^{l} \frac{c_i}{n_i!} y^{\alpha_i} \log^{n_i} y$$

is the leading polynomial growing part for the $f(x, y)$ $y \to \infty$ asymptotic.

$\boldsymbol{R}^*(f, s)$ is the relevant integral transform for the polynomial growth case, which parallels the Mellin transform (1.11) of the rapid decay case.

Analytic continuation of $\boldsymbol{R}^*(f, s)$ is given by the following theorem:

Theorem 1. *(**Zagier**, [Za2]) Let f be a modular invariant function of polynomial growth at the cusp*

$$f(x, y) \sim \sum_{i=1}^{l} \frac{c_i}{n_i!} y^{\alpha_i} \log^{n_i} y + o(y^{-N}), \qquad y \to \infty, \qquad \forall N > 0,$$

then the Rankin-Selberg transform (1.14) can be analytically continued to the meromorphic function

$$\boldsymbol{R}^*(f, s) = \sum_{i=1}^{l} c_i \left(\frac{\boldsymbol{\zeta}^*(2s)}{(1 - s - \alpha_i)^{n_i+1}} + \frac{\boldsymbol{\zeta}^*(2s - 1)}{(s - \alpha_i)^{n_i+1}} + \frac{\text{entire function of s}}{s(s - 1)} \right). \tag{1.15}$$

Eq. (1.15) parallels eq. (1.12) of the rapid decay case.

Below we present our methods which allow also to prove theorem 1 by a distinct route. Our route avoids to use unfolding tricks on truncated versions of $\boldsymbol{\mathcal{D}}$ as in [Za2]. With our route we will also prove various results of this paper. In order to illustrate our methods, and in the polynomial growing case, to contrast it with those in [Za2], we start by introducing the following

***Lattices series magic square*:**

$$\begin{array}{ccc} \boldsymbol{\Theta}_t(z) & \xrightarrow{\boldsymbol{\mathcal{M}}_t} & \boldsymbol{E}^*_s(z) \\ \uparrow \mathcal{P}_y & & \uparrow \mathcal{P}_y \\ \boldsymbol{\vartheta}_t(\Im(z)) & \xrightarrow{\boldsymbol{\mathcal{M}}_t} & \boldsymbol{\mathcal{E}}^*_s(z) \end{array} \tag{1.16}$$

relating four functions of great relevance in analytic number theory. In the upper vertexes of the square sit two 2-dimensional lattices series, the dressed spectral Eisenstein series $\boldsymbol{E}^*_s(z)$, and the 2-lattice theta series $\boldsymbol{\Theta}_t(z)$,

$$\boldsymbol{E}^*_s(z) := \pi^{-s}\boldsymbol{\Gamma}(s) \sum_{\omega\in\Lambda_z} \left(\frac{|\omega|^2}{\Im(z)}\right)^{-s},$$

$$\boldsymbol{\Theta}_t(z) := \sum_{\omega\in\Lambda_z} e^{-\pi t\left(\frac{|\omega|^2}{\Im(z)}\right)},$$

with $\Lambda_z := \{mz + n \in \mathbb{C}, (m,n) \in \mathbb{Z}^2 \setminus \{0\}, z \in \boldsymbol{\mathcal{H}}\}$ a two dimensional lattice, with modular parameter z. These two 2-lattice series are related by Mellin integral transform $\boldsymbol{\mathcal{M}}$

$$\boldsymbol{E}^*_s(z) := \int_0^\infty dt\, t^{s-1}\boldsymbol{\Theta}_t(z) = \int_0^\infty dt\, t^{s-1} \sum_{\omega\in\Lambda_z} e^{-\pi t\left(\frac{|\omega|^2}{\Im(z)}\right)}.$$

In the lower vertices of the magic square sit two 1-dimensional lattice $\mathbb{N}_{>0}$ series, that are the homologous of the two dimensional ones

$$\boldsymbol{\mathcal{E}}^*_s(\Im(z)) := \pi^{-s}\boldsymbol{\Gamma}(s) \sum_{n\in\mathbb{N}_{>0}} \left(\frac{n^2}{\Im(z)}\right)^{-s} = \Im(z)^s\boldsymbol{\zeta}^*(2s),$$

$$\boldsymbol{\vartheta}_t(\Im(z)) := \sum_{n\in\mathbb{N}_{>0}} e^{-\pi t\left(\frac{n^2}{\Im(z)}\right)}.$$

The above two 1-dimensional lattice series are also related by a Mellin integral transform

$$\boldsymbol{\mathcal{E}}^*_s(\Im(z)) := \int_0^\infty dt\, t^{s-1}\boldsymbol{\vartheta}_t(\Im(z)). \tag{1.17}$$

The vertical arrows in the magic square uplift one dimensional lattice series to two dimensional lattice series. This works through the relation $\Lambda_z = \mathbb{N}_{>0} \otimes \tilde{\Lambda}_z$, where $\tilde{\Lambda}_z := \{cz+d|(c,d) \in \mathbb{Z}^2, (c,d) = 1\}$ is the co-primed 2-lattice. The $\tilde{\Lambda}_z$ modular group is $\mathbf{\Gamma} \sim SL(2,\mathbb{Z})$ identified by the $\mathbb{N}_{>0}$ left action, i.e. $\mathbf{\Gamma}_\infty \backslash \mathbf{\Gamma}$. Therefore

$$\begin{aligned} \boldsymbol{E}^*_s(z) &= \pi^{-s}\mathbf{\Gamma}(s) \sum_{\omega\in\Lambda_z} \left(\frac{|\omega|^2}{\Im(z)}\right)^{-s}, \\ &= \pi^{-s}\mathbf{\Gamma}(s) \sum_{n\in\mathbb{N}_{>0}} \sum_{\tilde{\omega}\in\tilde{\Lambda}_z} \left(\frac{|\tilde{\omega}|^2}{\Im(z)}\right)^{-s} \\ &= \sum_{\gamma\in\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}} \mathcal{E}^*_s(\Im(\gamma(z))), \end{aligned}$$

and by applying a reasoning as above

$$\mathbf{\Theta}^*_t(z) = \sum_{\gamma\in\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}} \boldsymbol{\vartheta}_t(\Im(\gamma(z))). \tag{1.18}$$

Given a modular invariant function $f = f(x,y)$, for certain classes of growing conditions at the cusp $y \to \infty$, to be discussed below in this paper, by taking inner products defined $\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}$ and for functions defined on $\boldsymbol{U}\backslash\boldsymbol{\mathcal{H}} \simeq \mathbb{R}_{>0}$ with functions appearing in diagram (1.16), one finds a set of relations displayed by the following

***Inner products magic square*:**

$$\begin{array}{ccc} \langle\mathbf{\Theta}_t(z), f(z)\rangle_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}} & \xrightarrow{\boldsymbol{\mathcal{M}}_t} & \langle\boldsymbol{E}^*_s(z), f(z)\rangle_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}} \\ \downarrow Unfolding & & \downarrow Unfolding \\ \langle\boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} & \xrightarrow{\boldsymbol{\mathcal{M}}_t} & \boldsymbol{\zeta}^*(2s)\langle y^s, \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}}. \end{array} \tag{1.19}$$

The inner product on $\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}$ corresponds to the Petersson inner product between two modular invariant functions. Given two modular functions f and g, it is defined as follows

$$\langle f, g\rangle_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}} := \int_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}} dxdy\, y^{-2}\bar{f}(z)g(z), \tag{1.20}$$

where $\bar{f}$ is the complex conjugate of f. The inner product on $\boldsymbol{U}\backslash\boldsymbol{\mathcal{H}}$ for a pair of functions φ and ξ on $\mathbb{R}_{>0}$ with values in $\mathbb{C}$ is defined as

$$\langle\varphi, \xi\rangle_{\boldsymbol{U}\backslash\boldsymbol{\mathcal{H}}} := \int_0^\infty dy\, y^{-2}\bar{\varphi}(y)\xi(y). \tag{1.21}$$

Vertical arrows in the diagram (1.19) correspond to the following unfolding trick, which allows to identify the constant map $\boldsymbol{a}_0$ as the adjoint map of the Poincaré map $\mathcal{P}$ with respect to the inner products (1.20) and (1.21)

$$\begin{aligned}
\langle \mathcal{P}[\varphi], f\rangle_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}} &= \int_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}} dxdy\, y^{-2}\mathcal{P}[\varphi](z)f(z),\\
&= \int_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}} dxdy\, y^{-2} f(z) \sum_{\gamma\in\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}} \varphi(\Im(\gamma(z)))\\
&= \sum_{\gamma\in\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}} \int_{\Gamma\backslash\boldsymbol{\mathcal{H}}} dxdy\, y^{-2} f(z)\, \varphi(\Im(\gamma(z)))\\
&= \int_0^\infty dy\, y^{-2}\bar{\varphi}(y) \int_0^1 dx f(z)\\
&= \langle \varphi, \boldsymbol{a}_0[f]\rangle_{U\backslash\boldsymbol{\mathcal{H}}},
\end{aligned} \tag{1.22}$$

where $\boldsymbol{a}_0[f]$ is the constant map,

$$\boldsymbol{a}_0[f](y) := \int_0^1 dx f(z).$$

The constant map $\boldsymbol{a}_0$ in geometrical terms gives the horocycle average of the modular invariant function f.

The above unfolding trick is equivalent of being able to exchange in the inner product $\langle \mathcal{P}[\varphi], f\rangle_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}}$ the series over modular transformations in $\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}$, with integration on the fundamental domain $\boldsymbol{\mathcal{D}} \simeq \mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}$. This possibility depends on the behavior at the cusp of the product of the modular function $f(z)$ with the Poincaré series $\sum_{\gamma\in\mathbf{\Gamma}_\infty\backslash\mathbf{\Gamma}} \varphi(\Im(\gamma(z)))$.

In the rest of this introduction, we discuss and contrast the classical Rankin-Selberg method, which we introduced in §1.1, and it corresponds to moving along the *right* column of diagram 1.19 in the direction of the arrow, to a Theta unfolding method. This latter method corresponds to moving along the *left* column of diagram 1.19 in the direction of the vertical arrow, and then by using the horizontal lower arrow. For various classes of growing conditions at the cusp, we shall contrast unfolding of a modular integral of the product of a function f with the spectral Eisenstein series $\mathbf{E}^*_s(z)$, with unfolding by using the double theta series $\mathbf{\Theta}_t(z)$. Discussions and results of this paper should illustrate advantages of using the double theta series $\mathbf{\Theta}_t(z)$ for the unfolding trick, when one considers modular invariant functions which have not-so-mild growing conditions at the cusp. Whether $\boldsymbol{E}^*_s(z)$ grows polynomially at the cusp, a subseries of terms of $\mathbf{\Theta}_t(z)$ decay exponentially at the cusp, and they are indeed those terms which allow to perform the unfolding trick. This unfolding trick allows a better control for modular functions with not-so-mild growing condition at the cusp.

Our Theta method corresponds to the following route

$$\begin{array}{lcl}
\langle \mathbf{\Theta}_t(z), f(z)\rangle_{\mathbf{\Gamma}\backslash\boldsymbol{\mathcal{H}}} & & \\
\quad\downarrow Unfolding & & \\
\langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} & \xrightarrow{\boldsymbol{\mathcal{M}}_t} & \boldsymbol{\zeta}^*(2s)\langle y^s, \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} = \boldsymbol{R}^*(f,s).
\end{array} \tag{1.23}$$

The advantage of this route is that it does not require truncations of the domain of integration $\boldsymbol{\mathcal{D}}$. Unfolding of the integration domain in the modular integral

$$\langle \boldsymbol{\Theta}_t(z), f(z)\rangle_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}} = \int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2} f(x,y) \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} e^{-\frac{\pi}{y}|mz+n|^2}, \tag{1.24}$$

follows from decomposition for the theta series $\boldsymbol{\Theta}_t(z)$

(1.25)

$$\boldsymbol{\Theta}_t(z) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} e^{-\pi t\frac{(mx+n)^2+m^2y^2}{y}} = 1+\boldsymbol{\vartheta}_t(\Im(z))+\sum_{\gamma\in\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}'} \boldsymbol{\vartheta}_t(\Im(\gamma(z))), \qquad \boldsymbol{\Gamma}' := \boldsymbol{\Gamma}\setminus\{\mathbb{I}\},$$

with $\gamma(z) := \frac{az+b}{cz+d}$.

One uses modular transformations from contributions from the third term on the r.h.s.. where terms of this subseries have the form

$$\boldsymbol{\vartheta}_t(\Im(\gamma(z))) = \sum_{r\neq 0} e^{-\pi\frac{r^2t}{y}((cx+d)^2+c^2y^2)} \qquad c,d\in\mathbb{Z}, c\neq 0, (c,d)=1,$$

and correspond to the $m\neq 0$ subseries in (1.25), whose terms decay exponentially for $y\to\infty$. Thus, for f in $\boldsymbol{\mathcal{C}}_{TypeII}$, by dominate convergence theorem one can unfold the modular integral $\langle \boldsymbol{\Theta}_t(z), f(z)\rangle_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}}$ in the upper vertex of the triangular diagram 1.23, and obtain the quantity in the left lower vertex $\langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}}$. This corresponds to prove the vertical arrow of the triangular diagram 1.23 to hold for functions in the class $\boldsymbol{\mathcal{C}}_{TypeII}$.

As a next step, in section 3, we estimate both the $t\to 0$ and the $t\to\infty$ asymptotics of the function

$$\boldsymbol{i}(t) := \langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}}, \tag{1.26}$$

which appears in the left lower vertex of 1.23. Due to the arrow in the lower side of the triangular diagram 1.23, knowledge of $t\to 0$ and $t\to\infty$ asymptotics of the function $\boldsymbol{i}(t)$ allows to reconstruct meromorphic expansion of its Mellin transform in the right lower vertex of 1.23. Since the function in the right lower corner coincides with the Rankin-Selberg transform of the constant term $\boldsymbol{a}_0(y)$, this allows to prove theorem 1. Moreover, the lower row of diagram (1.23) shows a simple connection between the two functions $\boldsymbol{i}(t)$ and $\boldsymbol{a}_0(y)$. This allows to obtain the $y\to 0$ asymptotic of $\boldsymbol{a}_0(y)$ by having proved the $\boldsymbol{i}(t)$ asymptotic. By this route we shall prove the following

Theorem 2. *For a given $f = f(x,y)$ modular invariant function with polynomial behavior at the cusp*

$$f(x,y) \sim \sum_{i=1}^{l} \frac{c_i}{n_i!}\, y^{\alpha_i}\log^{n_i} y + o(y^{-N}), \qquad y\to\infty \qquad \forall N>0,$$

for $c_i, \alpha_i\in\mathbb{C}$, $\Re(\alpha_i)<1/2$, $n_i\in\mathbb{N}_{\geq 0}$, the following asymptotic holds true

$$\boldsymbol{a}_0(y) \sim C_0 + \sum_{\boldsymbol{\zeta}^*(\rho)=0} C_\rho y^{1-\frac{\rho}{2}} + \sum_{i=1}^{l}\frac{c_i}{n_i!}\,\frac{\boldsymbol{\zeta}^*(2\alpha_i-1)}{\boldsymbol{\zeta}^*(2\alpha_i)} y^{1-\alpha_i}\log^{n_i} y + o(y^N), \quad y\to 0, \quad \forall N>0,$$

where

$$C_0 = \frac{3}{\pi}\int_{\boldsymbol{\mathcal{D}}} dxdy y^{-2} f(z).$$

We now sketch how we do prove asymptotics for the function $\boldsymbol{i}(t)$. This is done in two steps, first we need the following lemma

Lemma 1. *Given a modular invariant function $f = f(x, y)$ with finite integral on $\boldsymbol{\mathcal{D}}$, $C_0 :=< 1, f >_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}}$. Let $\boldsymbol{a}_0(y)$ the f constant Fourier term, then the following relation holds true*

$$\langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} = \frac{1}{t}\langle \boldsymbol{\vartheta}_{1/t}(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} + \frac{C_0}{t} - C_0.$$

Lemma 1 then allows to prove the following lemma on asymptotics of the function $\boldsymbol{i}(t)$

Lemma 2. *Let $f = f(x, y)$ a modular invariant function with polynomial behavior at the cusp*

$$f(x, y) \sim \sum_{i=1}^{l} \frac{c_i}{n_i!} y^{\alpha_i} \log^{n_i} y + o(y^{-N}), \qquad y \to \infty \qquad \forall N > 0$$

where $\alpha_i, c_i \in \mathbb{C}$, $\Re(\alpha_i) < 1/2$, $n_i \in \mathbb{N}_{\geq 0}$.

Then, for the function $\boldsymbol{i}(t) := \langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}}$ the following asymptotics hold true

$$i) \qquad \boldsymbol{i}(t) \sim \sum_{i=1}^{l} \frac{c_i}{n_i!}\boldsymbol{\zeta}^*(2\alpha_i - 1)t^{\alpha_i - 1} \log^{n_i} t + O\left(t^{A-1}\log^{N-1} t\right), \qquad t \to \infty$$

$$ii) \qquad \boldsymbol{i}(t) \sim -C_0 + \frac{C_0}{t} - \sum_{i=1}^{l} \frac{c_i}{n_i!}\boldsymbol{\zeta}^*(2\alpha_i - 1)t^{-\alpha_i} \log^{n_i} t + O\left(t^{-a}\log^{n-1} t\right), \qquad t \to 0$$

where $A := max\{\Re(\alpha_i)\}$, $a := min\{\Re(\alpha_i)\}$, $N := max\{n_i\}$, $n := min\{n_i\}$, and

$$C_0 = \int_{\boldsymbol{\mathcal{D}}} dxdy y^{-2} f(z).$$

We then prove Zagier expansion (1.15) for the Rankin-Selberg transform $\boldsymbol{R}^*(f, s)$ in theorem 1. Our proof uses lemma 2, the horizontal arrow in diagram 1.23 and proposition 1. Thereafter, with all the collected results we prove theorem 2, on the long horocycle average asymptotic of functions in $\boldsymbol{\mathcal{C}}_{TypeII}$.

1.3. **String inspired class of modular functions of exponential growth at the cusp.**

Section 4 contains proofs for the class of modular function with growing conditions in $\boldsymbol{\mathcal{C}}_{Heterotic}$ (1.6). Examples of functions with such exponentially growing conditions do appear in one-loop amplitudes in heterotic string theory. We are able to prove much weaker results on the $y \to 0$ behavior of their horocycle average. However, string theory suggests better converging behavior then what we managed to prove in this paper. We leave string theory suggestions as open question at the end of section §4. By following the route given by the arrows in diagram (1.23), we are able to prove the following bound on the growing of the long horocycle average for modular functions in $\mathcal{C}_{Heterotic}$:

Theorem 3. *Let $f = f(x, y)$ a modular invariant function with growing conditions in the class $\mathcal{C}_{Heterotic}$ defined by eq. (1.6), then its constant term $\boldsymbol{a}_0(y)$ in the $y \to 0$ grows slower then any function of the form $e^{C/y}$, $\Re(C) > 0$*

$$\boldsymbol{a}_0(y) \sim o(e^{C/y}) \qquad y \to 0, \qquad \forall c \in \mathbb{C},\, \Re(C) > 0. \tag{1.27}$$

However, as discussed at the beginning of this introductive section, string theory suggests a much stronger result, namely that in the $y \to 0$ limit $\boldsymbol{a}_0(y)$ be convergent and to have asymptotic as in theorem 2. This leads to the following open question:

Open Problem 1. *(Prove or disprove the following statement): Given $f = f(x, y)$ modular invariant function in the class $\mathcal{C}_{Heterotic}$ (1.6), the following asymptotic holds true*

$$\boldsymbol{a}_0(y) \sim C_0 + \sum_{\boldsymbol{\zeta}^*(\rho)=0} C_\rho y^{1-\frac{\rho}{2}} + \frac{\boldsymbol{\zeta}^*(2\alpha - 1)}{\boldsymbol{\zeta}^*(2\alpha)} y^{1-\alpha} + o(y^N), \qquad \forall N > 0, \qquad y \to 0,$$

where

$$C_0 = \frac{3}{\pi} \int_{\mathcal{D}} dy \int dx y^{-2} f(z).$$

where this integral is meant in the conditional sense, with integration along the real axis performed first.

Besides discussing string theory connections and hints for the question in (1), at the end of section 4 we also discuss the possibility of having a sort of rigidity in the way the constant term $\boldsymbol{a}_0(y)$ may grow in the $y \to 0$ limit. The following result related to this issue is given at the end of section 4:

Proposition 2. *Given a $SL(2, \mathbb{Z})$ invariant function f which grows as $f(x, y) \sim e^{2\pi\beta y} e^{2\pi i\kappa x}$ for $y \to \infty$ for a certain non-zero integer $\kappa \in \mathbb{Z} \setminus \{0\}$. Then*

$$\boldsymbol{a}_0(y) + \sum_{r \in \mathbb{Z} \setminus \{0\}} \boldsymbol{a}_r(y) e^{2\pi i r \frac{a}{c}} \sim e^{-2\pi i \kappa \frac{d}{c}} e^{2\pi\beta \frac{c^2}{y}}, \qquad y \to 0, \tag{1.28}$$

for every pairs of Farey fractions $\frac{a}{c}$, $\frac{d}{c}$, $a, c, d \in \mathbb{Z}$, $(a, c) = 1$, $|a| < c$, $(d, c) = 1$ $|d| < c$, $c > 0$. $\boldsymbol{a}_r(y)$ are the Fourier modes in the expansion $f(x, y) = \sum_{r \in \mathbb{Z}} \boldsymbol{a}_r(y) e^{2\pi i r x}$.

We end up section 4 by discussing the possibility that proposition 2 together with the bound given by theorem 3 may be of help in addressing the open question raised in 1.

2. Rapid decay case: the Rankin-Selberg method and Zagier connection to RH

This section contains a review in some details of the Rankin-Selberg method [R-S] for automorphic functions of rapid decay, (some of the material contained in this section overlaps with §1.1). We review in details, Zagier proof [Za1] of the dependence of the error estimate in the horocycle average asymptotic of modular function of rapid decay on the Riemann hypothesis, displayed in eq. (1.2) of the introduction. Most of the material is contained in [Za1], although we have expanded some of the discussions in [Za1]. Material reported in this section is introductory for our proofs in sections 3 and section 4.

Given $f = f(x, y)$ a modular invariant function of rapid decay at the cusp $y \to \infty$, the Rankin-Selberg integral is the following modular integral

$$\boldsymbol{I}(s) = \int_{\boldsymbol{\mathcal{D}}} dxdy y^{-2} f(z) \boldsymbol{E}_s(z), \tag{2.1}$$

on the $SL(2, \mathbb{Z})$ fundamental domain $\boldsymbol{\mathcal{D}}$, where

$$\boldsymbol{E}_s(z) = \sum_{\gamma \in \boldsymbol{\Gamma}_\infty \backslash \boldsymbol{\Gamma}} \Im(\gamma(z))^s = \frac{1}{2} \sum_{c,d \in \mathbb{Z}, (c,d)=1} \frac{y^s}{|cz + d|^{2s}},$$

is the spectral Eisenstein series. This series is convergent for $\Re(s) > 1$, and can be analytically continued to the full plane s except for a simple pole in $s = 1$ with residue $\frac{3}{\pi}$, and poles in $s = \rho/2$, where ρ's are the non trivial zeros of the Riemann zeta function, $\boldsymbol{\zeta}^*(\rho) = 0$.

2.1. **Unfolding and analytic heritage.** The sequence of partial sums of $\boldsymbol{E}_s(z)$ times the function $f(z)$ is dominated by $\boldsymbol{E}_s(z)|f(z)|$, a integrable function on $\boldsymbol{\mathcal{D}}$, for $\Re(s) > 1$. Thus, by dominated convergence Lebesgue theorem, one can exchange the series with the integral, which amounts to use the unfolding trick for enlarging the integration domain to half-infinite strip $[-1/2, 1/2) \times (0, \infty) \subset \boldsymbol{\mathcal{H}}$

$$\begin{aligned} \boldsymbol{I}(s) &= \int_{\boldsymbol{\mathcal{D}}} dxdy y^{-2} f(z) \sum_{\gamma \in \boldsymbol{\Gamma}_\infty \backslash \boldsymbol{\Gamma}} \Im(\gamma(z))^s \\ &= \sum_{\gamma \in \boldsymbol{\Gamma}_\infty \backslash \boldsymbol{\Gamma}} \int_{\boldsymbol{\Gamma} \backslash \boldsymbol{\mathcal{H}}} dxdy y^{s-2} f(z) \\ &= \int_0^\infty dy y^{s-2} \boldsymbol{a}_0(y). \end{aligned}$$

The integral function $\boldsymbol{I}(s)$ inherits analytic properties of $\boldsymbol{E}_s(z)$, since the modular integral (2.1) is uniformly convergent for $y \to \infty$ in the complex parameter s. In fact $\boldsymbol{E}_s(z)$ grows polynomially for $y \to \infty$

$$\boldsymbol{E}_s(z) \sim y^s + \frac{\boldsymbol{\zeta}^*(2s-1)}{\boldsymbol{\zeta}^*(2s)} y^{1-s} \qquad y \to \infty,$$

while $f(x, y)$ is of rapid decay for $y \to \infty$.

Uniform convergence of the modular integral for the complex parameter s on a set $\boldsymbol{A}$ for $z \to i\infty$ means that given $\epsilon > 0$ there exists a corresponding neighborhood $\mathcal{U}_\epsilon$ of the cusp $z = i\infty$ such that

$$\left| \int_{\mathcal{U}_\epsilon} dxdy y^{-2} f(z) \partial_s^n \boldsymbol{E}_s(z) \right| < \epsilon \qquad \forall s \in \boldsymbol{A}, \qquad \forall n.$$

In this case $\mathcal{U}_\epsilon = \{z \in \boldsymbol{\mathcal{D}} | \Im(z) > M_\epsilon\}$.

2.2. **Poles and Residues of $\boldsymbol{E}_s(z)$.** $\boldsymbol{E}_s(z)$ has a simple pole in $s=1$ with residue $3/\pi$. In fact

$$\boldsymbol{E}^*_s(z) = \boldsymbol{\zeta}^*(2s)\boldsymbol{E}_s(z) = \frac{1}{2}\pi^{-s}\boldsymbol{\Gamma}(s)\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}}\frac{y^s}{|mz+n|^{2s}},$$

is the Mellin transform with respect to the variable t of the function $\boldsymbol{\Theta}_t(z)$

$$\boldsymbol{E}^*_s(z) = \frac{1}{2}\int_0^\infty dt\, t^{s-1}\boldsymbol{\Theta}_t(z). \tag{2.2}$$

Double Poisson summation gives

$$\boldsymbol{\Theta}_t(z) = -1 + \frac{1}{t} + \frac{1}{t}\boldsymbol{\Theta}_{1/t}(z),$$

thus

$$\boldsymbol{\Theta}_t(z) \sim -1 + \frac{1}{t} \qquad t\to 0,$$

while $\boldsymbol{\Theta}_t(z)$ is of rapid decay for $t\to\infty$. Therefore by proposition 1, $\boldsymbol{E}^*_s(z)$ has a pole in $s=0$ with residue $-1/2$ and pole in $s=1$ with residue $1/2$.

Thus it follows that

$$\boldsymbol{E}_s(z) = \frac{\boldsymbol{E}^*_s(z)}{\boldsymbol{\zeta}^*(2s)}$$

has a pole in $s=1$ with residue $\frac{1}{2\boldsymbol{\zeta}^*(s)} = 3/\pi$ and poles in $\rho/2$, where ρ's are the zeros of the Riemann zeta function $\boldsymbol{\zeta}^*(\rho)=0$.

2.3. **Zagier's result on $\boldsymbol{a}_0(y)$, $y\to 0$ asymptotic.** A sufficient condition for the following $y\to 0$ asymptotic to hold, (displayed in (1.2))

$$\boldsymbol{a}_0(y) \sim C + \sum_{\boldsymbol{\zeta}^*(\rho)=0} C_\rho y^{1-\rho/2} \qquad y\to 0, \tag{2.3}$$

is f of rapid decay at the cusp $y\to\infty$, plus some degree of smoothness of the function $f(x,y)$, and suitable $y\to\infty$ growing conditions for Δf, (where $\Delta := y^2(\partial_x^2+\partial_y^2)$ is the hyperbolic Laplacian). We make this precise, and derive a sufficient condition for (2.3) to occur.

The starting point is the Rankin-Selberg integral

$$\boldsymbol{I}(s) = \int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2}\boldsymbol{E}_s(z)f(z). \tag{2.4}$$

Since the integral function $\boldsymbol{I}(s)$ inherits analytic properties of $\boldsymbol{E}_s(z)$, $\boldsymbol{I}(s)$ has a meromorphic continuation with poles in $s=1$, and $s=\rho/2$, with ρ's such that $\boldsymbol{\zeta}^*(\rho)=0$. Define $\Theta := Sup\{\Re(\rho)|\boldsymbol{\zeta}^*(\rho)=0\}$, $1/2\le\Theta<1$, then $\boldsymbol{I}(s)-\frac{C}{s-1}$ is defined on $\Re(s)>\Theta/2$.

Since $\boldsymbol{I}(s) = \boldsymbol{\mathcal{M}}[y^{-1}\boldsymbol{a}_0(y)](s)$, a way to obtain (2.3) is to use an inverse Mellin transform argument. The Mellin inverse-transform of $\boldsymbol{I}(s)$ is

$$\boldsymbol{\mathcal{M}}^{-1}[\boldsymbol{I}(s)](y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\, y^{-s}\boldsymbol{I}(s) = \frac{y^{-\sigma}}{2\pi i}\int_{-\infty}^{\infty} dt\, y^{-it}\boldsymbol{I}(\sigma+it), \tag{2.5}$$

wherever $\boldsymbol{I}(\sigma+it)$ falls off as $o(1/t)$ for $t\to\pm\infty$.

If $f(z)$ is twice differentiable, then one can use $\Delta \boldsymbol{E}_s(z) = s(s-1)\boldsymbol{E}_s(z)$ and by integration by parts one finds

$$\boldsymbol{I}(s) = -\frac{1}{s(s-1)} \int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2} \boldsymbol{E}_s(z) \Delta f(z). \tag{2.6}$$

This shows that $I(\sigma + it)$ falls off as t^{-2} for $t \to \pm\infty$, whenever the integral r.h.s. of (2.6) is convergent. For our purposes one has to check that this integral is convergent in $\sigma = \frac{\Theta}{2} + \epsilon$. For $y \to \infty$, the Eisenstein series goes as $E_z(s) \sim y^s + \frac{\boldsymbol{\zeta}^*(2s-1)}{\boldsymbol{\zeta}^*(2s)} y^{1-s}$, and since $1/4 < \sigma = \frac{\Theta}{2} + \epsilon < 1/2$, indeed $E_{\frac{\Theta}{2}+\epsilon}(z) \sim y^{1-\frac{\Theta}{2}-\epsilon}$. Thus the integral in (2.6) is convergent if $\Delta f(z)$ respects and upper bound for its polynomial growing $y \to \infty$, namely $\Delta f(z) \lesssim O(y^{1/4})$.

Alltogether, we have the following sufficient condition:

Proposition 3. *Given $f = f(x,y)$ a modular invariant function of rapid decay $y \to \infty$. If f is twice differentiable and $\Delta f \lesssim O(y^{1/4})$ for $y \to \infty$, then the following holds true*

$$\boldsymbol{a}_0(y) \sim C + O(y^{1-\frac{\Theta}{2}}) \qquad y \to 0,$$

with $\Theta := Sup\{\Re(\rho) | \boldsymbol{\zeta}^*(\rho) = 0\}$.

2.4. **Rate of uniform distribution of long horocycles.** For the rate of uniform distribution of horocycles $\boldsymbol{\mathcal{H}}_y := (\mathbb{R} + iy)/\boldsymbol{\Gamma}_\infty \subset \boldsymbol{\mathcal{D}}$, in the modular surface $\boldsymbol{\mathcal{D}} \simeq \boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}$, one can prove that

$$\frac{L(\boldsymbol{\mathcal{H}}_{1/y} \cap \boldsymbol{\mathcal{U}})}{L(\boldsymbol{\mathcal{H}}_{1/y})} \sim \frac{A(\boldsymbol{\mathcal{U}})}{A(\boldsymbol{\mathcal{D}})} + O(y^{1/2}), \qquad y \to 0 \tag{2.7}$$

for every open set $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{D}}$. L indicates hyperbolic length, ($L(\gamma) = \int_\gamma y^{-1}\sqrt{dx^2 + dy^2}$ for a given curve $\gamma \subset \boldsymbol{\mathcal{H}}$), and A hyperbolic area $A(\boldsymbol{\mathcal{U}}) = \int_{\boldsymbol{\mathcal{U}}} dxdyy^{-2}$. Eq. (2.7) shows that for every open set $\boldsymbol{\mathcal{U}}$ contained in $\boldsymbol{\mathcal{D}}$, the portion of horocycle $\boldsymbol{\mathcal{H}}_y$ contained in $\boldsymbol{\mathcal{U}}$ in the limit $y \to 0$ tends to become proportional to the ratio between the area $A(\boldsymbol{\mathcal{U}})$ of $\boldsymbol{\mathcal{U}}$, and the area $A(\boldsymbol{\mathcal{D}}) = \pi/3$ of $\boldsymbol{\mathcal{D}}$.

The missing presence of $\Theta = Sup\{\Re(\rho)|\boldsymbol{\zeta}^*(\rho) = 0\}$ and thus the missing link with the Riemann hypothesis in the error estimate of (2.7) is due to the fact that some of the arguments used to prove proposition (3) do not go through in the present case. In fact, one has

$$\frac{L(\boldsymbol{\mathcal{H}}_{1/y} \cap \boldsymbol{\mathcal{U}})}{L(\boldsymbol{\mathcal{H}}_{1/y})} = \int_0^1 dx\; \boldsymbol{\chi}_{\boldsymbol{\mathcal{U}}}(x + iy), \tag{2.8}$$

where $\boldsymbol{\chi}_{\boldsymbol{\mathcal{U}}}(z)$ is the characteristic function of $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{D}}$. Also, by using the Rankin-Selberg method

$$I_{\boldsymbol{\chi}}(s) := \boldsymbol{\mathcal{M}}\left(\frac{1}{y}\frac{L(\boldsymbol{\mathcal{H}}_{1/y} \cap \boldsymbol{\mathcal{U}})}{L(\boldsymbol{\mathcal{H}}_{1/y})}\right)(s) = \int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2} \boldsymbol{\chi}_{\boldsymbol{\mathcal{U}}}(z) \boldsymbol{E}_z(s). \tag{2.9}$$

Since $\boldsymbol{\chi}_{\boldsymbol{\mathcal{U}}}(z)$ is not smooth, one cannot use the Laplacian $\boldsymbol{\Delta}$ argument as it was done for deriving proposition 3. Thus, the inverse-Mellin argument does not go through, and there is no connection between the rate of uniform distribution of long horocycles in the modular surface $\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}$ and the Riemann hypothesis.

3. Modular functions of polynomial growth

For a modular invariant function f of polynomial growth at the cusp

$$f(z) \sim \varphi(y) + o(y^{-N}), \qquad y \to \infty \qquad \forall N > 0$$

where

$$\varphi(y) := \sum_{i=1}^{l} \frac{c_i}{n_i!}\, y^{\alpha_i}\, \log^{n_i} y, \tag{3.1}$$

and

$$\alpha_i, c_i \in \mathbb{C}, \Re(\alpha_i) < 1/2, n_i \in \mathbb{N}_{\geq 0}.$$

Zagier [Za2] has proved analytic continuation and functional equation of the following Rankin-Selberg integral transform

$$\begin{aligned}
\boldsymbol{R}^*(f,s) : &= \boldsymbol{\zeta}^*(2s) \int_0^\infty dy y^{s-2}(\boldsymbol{a}_0(y) - \varphi(y)) \\
&= \sum_{i=1}^{l} c_i \left(\frac{\boldsymbol{\zeta}^*(2s)}{(1-s-\alpha_i)^{n_i+1}} + \frac{\boldsymbol{\zeta}^*(2s-1)}{(s-\alpha_i)^{n_i+1}} + \frac{\text{entire function of } s}{s(s-1)} \right).
\end{aligned} \tag{3.2}$$

Eq. (3.2) is obtained in [Za2] by a method which in terms of the following diagram

$$\begin{array}{ccc}
\langle \boldsymbol{\Theta}_t(z), f(z) \rangle_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}} & \xrightarrow{\boldsymbol{\mathcal{M}}_t} & \langle \boldsymbol{E}^*_s(z), f(z) \rangle_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}} \\
\downarrow Unfolding & & \downarrow Unfolding \\
\langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y) \rangle_{U\backslash\boldsymbol{\mathcal{H}}} & \xrightarrow{\boldsymbol{\mathcal{M}}_t} & \boldsymbol{\zeta}^*(2s) \langle y^s, \boldsymbol{a}_0(y) \rangle_{U\backslash\boldsymbol{\mathcal{H}}},
\end{array} \tag{3.3}$$

corresponds in considering the Rankin-Selberg integral in the right upper vertex of diagram 3.3, albeit with a regularization in the integration domain given by a cutoff $T > 1$, $\boldsymbol{\mathcal{D}}_T = \{z \in \boldsymbol{\mathcal{D}} | y < T, \}$. This truncation allows to apply a version of the unfolding trick devised for truncated domains $\boldsymbol{\mathcal{D}}_T$, and to move along the right column of this diagram. The obtained unfolded T-dependent quantity comprises several terms, and a careful analysis of the $T \to \infty$ limit [Za2] allows to extract information on $\boldsymbol{R}^*(f,s) = \boldsymbol{\zeta}^*(2s)\langle y^s, \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}}$, in the lower right corner of the diagram 3.3. This leads to prove equation (3.2) for the meromorphic continuation of $\boldsymbol{R}^*(f,s)$, plus additional results on functional equation for the Rankin-Selberg transform $\boldsymbol{R}^*(f,s)$ [Za2].

Here we employ an alternative method which leads us to prove (3.2). This method allows us to obtain results on the long horocycle average of functions with growing conditions given in (3.1), i.e. functions in $\boldsymbol{\mathcal{C}}_{TypeII}$. Our method comprises the following two steps in the diagram

$$\langle \boldsymbol{\Theta}_t(z), f(z)\rangle_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}}$$

$$\downarrow Unfolding \tag{3.4}$$

$$\langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} \overset{\boldsymbol{\mathcal{M}}_t}{\longrightarrow} \boldsymbol{\zeta}^*(2s)\langle y^s, \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} = \boldsymbol{R}^*(f,s).$$

The advantage of this route is that it does not require regularization (truncations) of the domain of integration $\boldsymbol{\mathcal{D}}$. In order to perform the unfolding of the integration domain in the modular integral

$$\langle \boldsymbol{\Theta}_t(z), f(z)\rangle_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}} = \int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2} f(x,y) \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} e^{-\frac{\pi}{y}|mz+n|^2}, \tag{3.5}$$

from the decomposition for the theta series $\boldsymbol{\Theta}_t(z)$

$$\boldsymbol{\Theta}_t(z) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} e^{-\pi t\frac{(mx+n)^2+m^2y^2}{y}} = 1 + \boldsymbol{\vartheta}(t/y) + \sum_{\gamma\in\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}'} \boldsymbol{\vartheta}_t(1/\Im(\gamma(z))),$$

one uses contributions from the third term on the r.h.s., where $\boldsymbol{\Gamma}' := \boldsymbol{\Gamma} \setminus \{\mathbb{I}\}$ is the set of modular transformations minus the identity $\mathbb{I}$. Each term in this series has the form

$$\boldsymbol{\vartheta}_t(1/\Im(\gamma(z))) = \sum_{r\neq 0} e^{-\pi\frac{r^2}{y}((cx+d)^2+c^2y^2)} \qquad c,d\in\mathbb{Z}, c\neq 0, (c,d)=1$$

and corresponds to the $m \neq 0$ subseries in (3.5), whose terms decay exponentially for $y \to \infty$. Thus by dominate convergence theorem one can unfold the modular integral $\langle \boldsymbol{\Theta}_t(z), f(z)\rangle_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}}$ in the left upper entry of (3.4) and prove the vertical arrow connecting the left upper entry with the left lower entry $\langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}}$.

The unfolding trick is doable without using a truncated domain, since the integral in the left upper corner of the diagram is convergent, under the assumptions $\Re(\alpha_i) < 1/2$, for the growing term $\varphi(y)$ in (3.1). Indeed, by Poisson summation one can check that $\boldsymbol{\Theta}_t(z) \sim \sqrt{y}$ for $y \to \infty$. Moreover, $\boldsymbol{\Theta}_t(z)$ has series representation convergent for every $t > 0$. Thus we have the following proposition for Theta-unfolding of a modular invariant function f with growing conditions in $\boldsymbol{\mathcal{C}}_{TypeII}$ (1.5):

Proposition 4. *Let $f = f(x,y)$ a modular invariant function of polynomial growth at the cusp $y \to \infty$*

$$f(x,y) \sim \sum_{i=1}^{l} \frac{c_i}{n_i!} y^{\alpha_i} \log^{n_i} y + o(y^{-N}), \qquad \forall N \geq 0, \qquad y \to \infty$$

with

$$\alpha_i, c_i \in \mathbb{C}, \Re(\alpha_i) < 1/2, n_i \in \mathbb{N}_{\geq 0}.$$

Then, the following Theta-unfolding relation holds true

$$\int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2} f(x,y)\boldsymbol{\Theta}_t(z) = \int_0^\infty dy\, y^{-2}\boldsymbol{a}_0(y)\boldsymbol{\vartheta}_t(y). \tag{3.6}$$

Proposition 4 states that the vertical arrow in diagram (3.4) holds true for modular functions of polynomial growth class $\boldsymbol{\mathcal{C}}_{TypeII}$.

The horizontal arrow in diagram (3.4) indicates that due to the relation between the functions

$$\boldsymbol{i}(t) := \langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y) \rangle_{U \backslash \boldsymbol{\mathcal{H}}} \tag{3.7}$$

and the function $\boldsymbol{\zeta}^*(2s)\langle y^s, \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}}$ through Mellin transform, knowledge of the $t \to \infty$ and $t \to 0$ asymptotics for $\boldsymbol{i}(t)$ implies knowledge of the meromorphic continuation with orders and locations of poles of the function $\boldsymbol{\zeta}^*(2s)\langle y^s, \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}}$ of complex variable s. We therefore prove $\boldsymbol{i}(t)$ asymptotics in two steps, by the two following lemmas.

Lemma. 1. *Let $f = f(x, y)$ a modular invariant functions with growing conditions as in proposition 4. Let $C_0 :=< 1, f >_{\boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}}$, its integral over the fundamental domain $\boldsymbol{\mathcal{D}}$, and let $\boldsymbol{a}_0(y)$ be the f constant term.*

Then, the following relation holds true

$$\langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} = \frac{1}{t}\langle \boldsymbol{\vartheta}_{1/t}(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}} + \frac{C_0}{t} - C_0$$

Proof. By double Poisson summation one finds $\boldsymbol{\Theta}_t(z) = \frac{1}{t}\boldsymbol{\Theta}_{1/t}(z) - \frac{1}{t} - 1$. The thesis then follows by applying the Theta-unfolding in proposition 4, which corresponds of using the left column in diagram 3.4.

□

By previous lemma, we are now in the position of proving the following lemma on the asymptotics $t \to \infty$ and $t \to 0$ of the function $\boldsymbol{i}(t)$ defined by (3.7), (which appears in the left lower entry of diagram 3.4):

Lemma. 2. *Let $f = f(x, y)$ a modular invariant function with polynomial behavior at the cusp*

$$f(z) \sim \sum_{i=1}^{l} \frac{c_i}{n_i!} y^{\alpha_i} \log^{n_i} y + o(y^{-N}), \qquad y \to \infty \qquad \forall N > 0$$

where $\alpha_i, c_i \in \mathbb{C}$, $\Re(\alpha_i) < 1/2$, $n_i \in \mathbb{N}_{\geq 0}$.

Then, for the function $\boldsymbol{i}(t) := \langle \boldsymbol{\vartheta}_t(y), \boldsymbol{a}_0(y)\rangle_{U\backslash\boldsymbol{\mathcal{H}}}$ the following asymptotics hold true

$$i) \qquad \boldsymbol{i}(t) \sim \sum_{i=1}^{l} \frac{c_i}{n_i!}\boldsymbol{\zeta}^*(2\alpha_i - 1)t^{\alpha_i - 1} \log^{n_i} t + O\left(t^{A-1}\log^{N-1} t\right), \qquad t \to \infty$$

$$ii) \qquad \boldsymbol{i}(t) \sim -C_0 + \frac{C_0}{t} - \sum_{i=1}^{l} \frac{c_i}{n_i!}\boldsymbol{\zeta}^*(2\alpha_i - 1)t^{-\alpha_i} \log^{n_i} t + O\left(t^{-a}\log^{n-1} t\right), \qquad t \to 0$$

where $A := max\{\Re(\alpha_i)\}$, $a := min\{\Re(\alpha_i)\}$, $N := max\{n_i\}$, $n := min\{n_i\}$, and

$$C_0 = \int_{\boldsymbol{\mathcal{D}}} dxdyy^{-2} f(z).$$

Proof. We start by proving i), the function $\boldsymbol{i}(t)$ is the following integral function

$$\boldsymbol{i}(t) = \int_0^\infty dyy^{-2}\boldsymbol{a}_0(y) \sum_{r \in \mathbb{Z}\backslash\{0\}} e^{-\pi r^2 \frac{t}{y}},$$

by change of integration variable $y \to ty$ one finds

$$\boldsymbol{i}(t) = \frac{1}{t}\int_0^\infty dy y^{-2}\boldsymbol{a}_0(yt)\sum_{r\in\mathbb{Z}\setminus\{0\}} e^{-\pi\frac{r^2}{y}}.$$

Therefore for $t \to \infty$

$$\begin{aligned}\boldsymbol{i}(t) &\sim \sum_{i=1}^{l}\frac{c_i}{n_i!}t^{\alpha_i-1}\int_0^\infty dy y^{-2+\alpha_i}\left(\log y+\log t\right)^{n_i}\sum_{r\in\mathbb{Z}\setminus\{0\}} e^{-\pi\frac{r^2}{y}},\\ &\sim \sum_{i=1}^{l}\frac{c_i}{n_i!}\boldsymbol{\zeta}^*(2\alpha_i-1)t^{\alpha_i-1}\log^{n_i} t + O\left(t^{A-1}\log^{N-1} y\right).\end{aligned}$$

In order to prove ii), we use lemma 3 which allows to rewrite $\boldsymbol{i}(t)$ in the following form

$$\begin{aligned}\boldsymbol{i}(t) &= \frac{1}{t}\int_0^\infty dy y^{-2}\boldsymbol{a}_0(y)\boldsymbol{\vartheta}_{1/t}(y) + \frac{C_0}{t} - C_0\\ &= \int_0^\infty dy y^{-2}\boldsymbol{a}_0(y/t)\sum_{r\in\mathbb{Z}\setminus\{0\}} e^{-\pi\frac{r^2}{y}} + \frac{C_0}{t} - C_0,\end{aligned}$$

also for $t \to 0$

$$\begin{aligned}&\int_0^\infty dy y^{-2}\boldsymbol{a}_0(y/t)\sum_{r\in\mathbb{Z}\setminus\{0\}} e^{-\pi\frac{r^2}{y}}\\ \sim &\sum_{i=1}^{l}\frac{c_i}{n_i!}\int_0^\infty dy y^{-2-\alpha_i}\left(\log y-\log t\right)^{n_i}\sum_{r\in\mathbb{Z}\setminus\{0\}} e^{-\pi\frac{r^2}{y}}\\ \sim &-\sum_{i=1}^{l}\frac{c_i}{n_i!}\boldsymbol{\zeta}^*(2\alpha_i-1)t^{-\alpha_i}\log^{n_i} t + O\left(t^{-a}\log^{n-1} t\right)\end{aligned}$$

□

In order to prove Zagier theorem 1 on the analytic continuation of the Rankin-Selberg transform, from lemma 3 and from the lower row of diagram 3.3, we also need proposition 1 on standard properties of Mellin transforms. Due to the lower row in diagram 1.23, by applying proposition 1 on the asymptotics in lemma 3, we obtain analytic continuation of the Rankin-Selberg transform, as in theorem 1.

From lower row of diagram 1.23, lemma 3 and proposition 1, we also prove the following theorem on the asymptotic of the long horocycle average of a modular function in $\boldsymbol{\mathcal{C}}_{TypeII}$:

Theorem. 2. *Let $f = f(x, y)$ a modular invariant function of polynomial growth at the cusp*

$$f(x,y) \sim \sum_{i=1}^{l}\frac{c_i}{n_i!}y^{\alpha_i}\log^{n_i} y + o(y^{-N}), \qquad y\to\infty \qquad \forall N>0$$

where $\alpha_i, c_i \in \mathbb{C}$, $\Re(\alpha_i) < 1/2$, $n_i \in \mathbb{N}_{\geq 0}$.

The long length limit of the f horocycle average has the following asymptotic

$$\boldsymbol{a}_0(y) \sim C_0 + \sum_{\boldsymbol{\zeta}^*(\rho)=0} C_\rho y^{1-\frac{\rho}{2}} + \sum_{i=1}^{l} \frac{c_i}{n_i!} \frac{\boldsymbol{\zeta}^*(2\alpha_i - 1)}{\boldsymbol{\zeta}^*(2\alpha_i)} y^{1-\alpha_i} \log^{n_i} y + O\left(y^{1-A} \log^{n-1} y\right) \qquad y \to 0,$$

where $A := max\{\Re(\alpha_i)\}$*,* $n := min\{n_i\}$*, and*

$$C_0 = \frac{3}{\pi} \int_{\boldsymbol{\mathcal{D}}} dxdy\, y^{-2} f(z).$$

4. Modular functions of exponential growth

We now turn to discuss modular invariant functions in the class of growing conditions $\boldsymbol{\mathcal{C}}_{Heterotic}$, defined in (1.6). Proofs are obtained by using same methods we employed in previous sections for the $\boldsymbol{\mathcal{C}}_{TypeII}$ case, which follow arrows in diagram 3.4.

We start by proving a bound on the growing of the long horocycle average for a modular function in $\boldsymbol{\mathcal{C}}_{Heterotic}$. Some of the ideas contained in the proof of theorem 3 are taken with some degree of re-elaboration them from [KS].

Theorem. 3. *Let* $f = f(x,y)$ *be a modular invariant function with the following growing condition*

$$f(x,y) \sim y^\alpha e^{2\pi i \kappa x} e^{\pi \beta y} \qquad y \to \infty \qquad \kappa \in \mathbb{Z} \setminus \{0\}, \qquad \beta < 1, \qquad \alpha \in \mathbb{C},\ \Re(\alpha) < 1/2.$$

Then the f *long horocycle average* $\boldsymbol{a}_0(y)$ *satisfies the following bound*

$$\boldsymbol{a}_0(y) \lesssim o(e^{C/y}) \qquad y \to 0, \qquad \forall C \in \mathbb{C},\ \Re(C) > 0.$$

Proof. We consider the following Theta-integral on the modular domain $\boldsymbol{\mathcal{D}}$

$$\boldsymbol{I}(t) := \int_{\boldsymbol{\mathcal{D}}} dxdy y^{-2} f(z) \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} e^{-\frac{\pi t}{y}|mx+n|^2}, \tag{4.1}$$

which corresponds to Petersson inner product of the theta series $\boldsymbol{\Theta}_t(z)$ with the function f which appears in the left upper entry of diagram 3.4. Due to the f growing conditions for $y \to \infty$, the function $\boldsymbol{I}(t)$, for small t, has to be understood as the result of an integration over $\boldsymbol{\mathcal{D}}$, with integration along the real axis performed first. In fact, the modular integral is only conditionally convergent for $z \to i\infty$.

We employ the following decomposition for the theta series $\boldsymbol{\Theta}_t(z)$

$$\boldsymbol{\Theta}_t(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} e^{-\frac{\pi t}{y}|mz+n|^2} = \sum_{\mathbb{Z}\setminus\{0\}} e^{-\pi t \frac{r^2}{y}} + \sum_{\boldsymbol{\Gamma}_\infty\setminus\boldsymbol{\Gamma}'} \sum_{\mathbb{Z}\setminus\{0\}} e^{-\pi t \frac{r^2}{\Im(\gamma(z))}}, \tag{4.2}$$

where $\boldsymbol{\Gamma}' = \boldsymbol{\Gamma} \setminus \{\mathbb{I}\}$ is the modular group $\boldsymbol{\Gamma}$ minus the identity $\mathbb{I}$.

One has for $\gamma \in \boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}'$

$$e^{-\pi t \frac{r^2}{\Im(\gamma(z))}} \sim e^{-\pi t m^2 y}, \qquad y \to \infty,$$

with $m = cr$ and $c \neq 0$ is the third entry of the modular transformation γ. Modular transformations in $\gamma \in \boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}'$ allow to unfold integration domain $\boldsymbol{\mathcal{D}} \simeq \boldsymbol{\Gamma}\backslash\boldsymbol{\mathcal{H}}$ in $\boldsymbol{I}(t)$

into the half-infinite strip $\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\mathcal{H}}$. From Lebesgue dominated convergence theorem, for $t>1$ one finds

$$\int_{\boldsymbol{\mathcal{D}}} dxdyy^{-2}f(z)\sum_{\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}'}\sum_{\mathbb{Z}\backslash\{0\}} e^{-\pi t\frac{r^2}{\Im(\gamma(z))}} = \sum_{\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}'}\sum_{\mathbb{Z}\backslash\{0\}}\int_{\boldsymbol{\mathcal{D}}} dxdyy^{-2}f(z)e^{-\pi t\frac{r^2}{\Im(\gamma(z))}}. \tag{4.3}$$

This leads to the following Theta-unfolding relation ($t>1$)

$$\sum_{\boldsymbol{\Gamma}_\infty\backslash\boldsymbol{\Gamma}'}\sum_{\mathbb{Z}\backslash\{0\}}\int_{\boldsymbol{\mathcal{D}}} dxdyy^{-2}f(z)e^{-\pi t\frac{r^2}{\Im(\gamma(z))}} = \int_0^\infty dyy^{-2}\int_{-1/2}^{1/2} dxf(x,y)\sum_{\mathbb{Z}\backslash\{0\}} e^{-\pi t\frac{r^2}{y}}. \tag{4.4}$$

Therefore, for $t>1$, the following unfolding relation holds

$$\boldsymbol{I}(t) = \int_0^\infty dyy^{-2}\boldsymbol{a}_0(y)\sum_{\mathbb{Z}\backslash\{0\}} e^{-\pi t\frac{r^2}{y}}. \tag{4.5}$$

Moreover, one can prove that the function $\boldsymbol{I}(t)$ in her original incarnation (4.1), is analytic on a strip $t\in(0,\infty)\times(-\delta_\beta,\delta_\beta)\subset\mathbb{C}$, where $\delta_\beta := 1-\beta>0$. Proof of this statement follows by Poisson summation

$$\boldsymbol{I}(t) = \frac{1}{\sqrt{t}}\int_{\boldsymbol{\mathcal{D}}} dxdyy^{-3/2}f(z)\sum_{(m,n)\in\mathbb{Z}^2\backslash\{0\}} e^{-\pi y\left(\frac{m^2}{t}+n^2t\right)}e^{2\pi imnx}, \tag{4.6}$$

and by the growing assumption we make on $f(x,y)$ for $y\to\infty$.

Due to analyticity of the l.h.s. in (4.4) on the strip $t\in(0,\infty)\times(-\delta_\beta,\delta_\beta)\subset\mathbb{C}$, where $\delta_\beta = 1-\beta>0$, the r.h.s. cannot be divergent on this strip. This rules out the following behavior

$$\boldsymbol{a}_0(y) = \int_0^1 dxf(x,y)\sim e^{C/y} \qquad y\to 0 \qquad C\in\mathbb{C},\ \Re(C)>0,$$

since such a growing condition would make the integral function in the r.h.s. of (4.4) to diverge for $0<t<\Re(C)$. □

As remarked already few times in the text, string theory suggests a much stronger result than theorem 4, namely that in the $y\to 0$ limit $\boldsymbol{a}_0(y)$ be convergent and to have asymptotic as in theorem 3.

This leads to the following open question:

Open Problem. 1. *Given $f=f(x,y)$ modular invariant function in the class $\boldsymbol{\mathcal{C}}_{Heterotic}$ (1.6), prove or disprove that the following asymptotic holds true*

$$\boldsymbol{a}_0(y)\sim C_0+\sum_{\boldsymbol{\zeta}^*(\rho)=0} C_\rho y^{1-\frac{\rho}{2}}+\frac{\boldsymbol{\zeta}^*(2\alpha-1)}{\boldsymbol{\zeta}^*(2\alpha)}y^{1-\alpha}+o(y^N), \qquad \forall N>0, \qquad y\to 0,$$

$$C_0 = \frac{3}{\pi}\int_{\boldsymbol{\mathcal{D}}} dy\int dxy^{-2}f(z),$$

where this integral is meant in the conditional sense, with integration along the real axis first performed.

Finally, we would like to add few remarks, that may be relevant to address the question raised in 1. We consider the possibility that there may be some kind of rigidity in the ways the horocycle average can grow in the long length limit, for a modular invariant function f with growing conditions in $\boldsymbol{\mathcal{C}}_{Heterotic}$. Rigidity on the way $\boldsymbol{a}_0(y)$ grows in the $y \to 0$ limit under growing conditions on f in $\boldsymbol{\mathcal{C}}_{Heterotic}$, may arise by proposition 2 below.

By using the following standard formulae for transformations of the real and imaginary part of $z \in \boldsymbol{\mathcal{H}}$ under a $SL(2,\mathbb{Z})$ modular transformation

$$\gamma(z) = \frac{az+b}{cz+d},$$

with $c \neq 0$

One has

$$\Re(\gamma(z)) = \frac{a}{c}\frac{(x+b/a)(x+d/c)+y^2}{(x+d/c)^2+y^2},$$

$$\Im(\gamma(z)) = \frac{1}{c}\frac{y}{(x+d/c)^2+y^2},$$

one can prove the following proposition for modular functions in $\boldsymbol{\mathcal{C}}_{Heterotic}$

Proposition. 2. *Given a $SL(2,\mathbb{Z})$ invariant function f which grows as $f(x,y) \sim e^{2\pi\beta y}e^{2\pi i\kappa x}$ for $y \to \infty$, $\kappa \in \mathbb{Z}\setminus\{0\}$, Then*

$$\boldsymbol{a}_0(y) + \sum_{r\in\mathbb{Z}\setminus\{0\}} \boldsymbol{a}_r(y)e^{2\pi i r\frac{a}{c}} \sim e^{-2\pi i\kappa\frac{d}{c}}e^{2\pi\beta\frac{c^2}{y}}, \qquad y \to 0 \tag{4.7}$$

for every pair of Farey fractions $\frac{a}{c}$, $\frac{d}{c}$, $a,c,d \in \mathbb{Z}$, $(a,c)=1$, $|a|<c$, $(d,c)=1$ $|d|<c$, $c>0$. $\boldsymbol{a}_r(y)$ are the Fourier modes in the expansion $f(x,y) = \sum_{r\in\mathbb{Z}} \boldsymbol{a}_r(y)e^{2\pi i r x}$.

Perhaps proposition 2 together with theorem 3 turn out to be sufficient to address the open question 1. Another possibility is that question 1 holds true in the following way. It may be that all the modular functions in the growing class $\boldsymbol{\mathcal{C}}_{Heterotic}$ split as the sum of a modular invariant function in $\boldsymbol{\mathcal{C}}_{TypeII}$ plus a cusp function in $\boldsymbol{\mathcal{C}}_{Heterotic}$, (a function whose constant term $\boldsymbol{a}_0(y)$ is identically vanishing) [Za3]. We are not able to provide answers on this latter possibility, which may give a way to reconcile string theory suggestions with results in the automorphic function domain.

Finally, it could be as well possible that question raised in 1 has an answer in the negative. This latter possibility would open some interesting questions in string theory, related a asymmetry in the ultraviolet limit between Type II and Heterotic closed strings asymptotics for very massive graded number of states.

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