Random Gaussian sums on trees

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Abstract

Let T be a tree with induced partial order $\preceq$. We investigate centered Gaussian processes $X = (X_t)_{t\in T}$ represented as

$$X_t = \sigma(t) \sum_{v\preceq t} \alpha(v)\xi_v$$

for given weight functions α and σ on T and with $(\xi_v)_{v\in T}$ i.i.d. standard normal. In a first part we treat general trees and weights and derive necessary and sufficient conditions for the a.s. boundedness of X in terms of compactness properties of (T, d). Here d is a special metric defined via α and σ, which, in general, is not comparable with the Dudley metric generated by X. In a second part we investigate the boundedness of X for the binary tree and for homogeneous weights. Assuming some mild regularity assumptions about α we completely characterize weights α and σ with X being a.s. bounded.

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1 Introduction

Let T be a finite or infinite tree with root $\mathbf{0}$ and let " $\preceq$ " be the induced partial order generated by the structure of T, i.e., it holds $t \preceq s$ or, equivalently $s \succeq t$, whenever t is situated on the branch connecting $\mathbf{0}$ with s. Suppose we are given two weight functions α and σ mapping T into $[0,\infty)$ with σ non–increasing, that is, $\sigma(t) \geq \sigma(s)$ whenever $t \preceq s$. If $(\xi_v)_{v\in T}$ denotes a family of independent standard normal random variables, then the centered Gaussian process $X = (X_t)_{t\in T}$ with

$$X_t := \sigma(t) \sum_{v\preceq t} \alpha(v)\xi_v\,, \quad t \in T\,, \tag{1.1}$$

is well defined. Its covariance function R_X is given by

$$R_X(t,s) = \sigma(t)\sigma(s) \sum_{v\preceq t\wedge s} \alpha(v)^2\,, \quad t,s \in T\,.$$

Fernique was probably the first to consider such summation schemes on trees in his constructions of majorizing measures [3]. More recently, they were extensively studied and applied in relation to various topics, see e.g. the literature on Derrida random energy model [1] or displacements in random branching walks [10], to mention just a few. Moreover, summation operators related to those processes have been recently investigated in [7], [8] and in [9]. Some of the ideas used there turned out to be useful as well for the study of Gaussian summation schemes on trees.

The basic question investigated in this paper is as follows: Given a tree T characterize weights α and σ such that X is a.s. bounded, i.e., that

$$\mathbb{P}\left(\sup_{t\in T} |X_t| < \infty\right) = 1\,. \tag{1.2}$$

In a first part we give necessary and sufficient conditions for the weights in order that (1.2) holds. These results are valid for arbitrary trees and they are based on covering properties of T by ε–balls with respect to a certain metric d first introduced in [8]. It is defined by

$$d(t,s) := \max_{t\prec r\preceq s} \sigma(r) \left(\sum_{t\prec v\preceq r} \alpha(v)^2\right)^{1/2}\,, \tag{1.3}$$

whenever $t \preceq s$ and we let $d(t,s) := d(t\wedge s,t) + d(t\wedge s,s)$, if t and s are not comparable. Here, as usual, $t\wedge s$ denotes the infimum of t and s in the induced partial order on T. Define the covering numbers of T by

$$N(T,d,\varepsilon) := \inf\left\{ n \geq 1 : T = \bigcup_{j=1}^{n} B_\varepsilon(t_j)\right\}$$

where $B_\varepsilon(t_j)$ are open ε–balls (w.r.t. the metric d) in T. Then the main result of the first part is as follows:

Theorem 1.1 *Suppose that X is defined by (1.1) with weights α and σ where σ is non–increasing. Let d be the metric on T given by (1.3). If*

$$\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}\,\mathrm{d}\varepsilon < \infty\,, \tag{1.4}$$

then X is a.s. bounded. Conversely, if X is a.s. bounded, then necessarily

$$\sup_{\varepsilon>0} \varepsilon \sqrt{\log N(T,d,\varepsilon)} < \infty \, . \tag{1.5}$$

It is worthwhile to mention that neither (1.4) nor (1.5) are direct consequences of the well–known conditions due to R.M. Dudley and V.N. Sudakov (cf. [2] and [11]), respectively. The latter results are based on compactness properties of (T, d_X) with so–called Dudley metric d_X defined by

$$d_X(t,s) := \left(\mathbb{E}\,|X_t - X_s|^2\right)^{1/2} \, , \quad t,s \in T \, , \tag{1.6}$$

and not on d as introduced in (1.3). We shall see below that, in general, the covering numbers w.r.t. d and to d_X may behave quite differently. The main advantage of Theorem 1.1 is that in many cases the covering numbers w.r.t. d are easier to handle than those defined by d_X (cf. [8] for concrete estimates of $N(T,d,\varepsilon)$ and also Corollary 3.4 below).

It is well–known that in general entropy estimates are too rough for deciding whether or not a given Gaussian process is bounded. Only majorizing measure techniques would work. But, unfortunately, majorizing measures are difficult to handle and we do not see how their use leads to a characterization of weights α and σ for which X is a.s. bounded. Therefore, in a second part, we investigate special trees and weights where a direct approach is possible. We suppose that T is a binary tree and that the weights are homogeneous, i.e. $\alpha(t)$ and $\sigma(t)$ only depend on the order $|t|$ (cf. (2.1) for the definition) of $t \in T$. Our results (cf. Theorems 6.1 and 6.3 below) imply the following:

Theorem 1.2 *Let T be a binary tree and suppose $\alpha(t) = \alpha_{|t|}$ and $\sigma(t) = \sigma_{|t|}$ for two sequences $(\alpha_k)_{k\geq 0}$ and $(\sigma_k)_{k\geq 0}$ of positive numbers with σ_k non–increasing.*

1. *If*

$$\sup_n \sup_{n\leq k\leq 2n} \frac{\alpha_k}{\alpha_n} < \infty \, , \tag{1.7}$$

then X defined by (1.1) is a.s. bounded if and only if

$$\sup_n \sigma_n \sum_{k=1}^{n} \alpha_k < \infty \, . \tag{1.8}$$

In particular, if the α_k are non–increasing, then (1.7) is always satisfied, hence in that case X is a.s. bounded if and only if (1.8) is valid.

2. *If the α_k are non–decreasing, then X is a.s. bounded if and only if*

$$\sup_n \sigma_n \sqrt{n} \left(\sum_{k=0}^{n} \alpha_k^2\right)^{1/2} < \infty \, .$$

The organization of this paper is a s follows. After a short introduction to trees, Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we thoroughly investigate the relation between the two metrics d and d_X. Here the main observation is that $N(T,d,\varepsilon)$ and $N(T,d_X,\varepsilon)$ may behave quite differently. Nevertheless, in view of Theorem 1.1 and the well–known results due to R.M. Dudley and to V.N. Sudakov, on the logarithmic level the covering numbers of these two metrics should be of similar order. We investigate this question in Section 5 more thoroughly.

In particular, we show that $\varepsilon^2 \log N(T, d, \varepsilon)$ is bounded if and only if $\varepsilon^2 \log N(T, d_X, \varepsilon)$ is so. In Section 6 we treat processes X indexed by a binary tree and with homogeneous weights. We prove slightly more general results than stated in Theorem 1.2. Finally, we give some interesting examples of bounded as well as unbounded processes indexed by a binary tree. In particular, these examples show that the boundedness of X may not be described by properties of the product $\alpha\,\sigma$ only.

2 Trees

Let us recall some basic notations related to trees which will be used later on. In the sequel T always denotes a finite or an infinite tree. We suppose that T has a unique root which we denote by $\mathbf{0}$ and that each element $t \in T$ has a finite number of offsprings. Thereby we do not exclude that some elements do not possess any offspring, i.e., the progeny of some elements may "die out". The tree structure leads in natural way to a partial order „$\preceq$" by letting $t \preceq s$, respectively $s \succeq t$, provided there are $t = t_0, t_1, \ldots, t_m = s$ in T such that for $1 \le j \le m$ the element t_j is an offspring of t_{j-1}. The strict inequalities have the same meaning with the additional assumption $t \neq s$. Two elements $t, s \in T$ are said to be comparable provided that either $t \preceq s$ or $s \preceq t$.

For $t, s \in T$ with $t \preceq s$ the order interval $[t, s]$ is defined by

$$[t, s] := \{v \in T : t \preceq v \preceq s\}$$

and in a similar way we construct $(t, s]$ or (t, s) .

A subset $B \subseteq T$ is said to be a branch provided that all elements in B are comparable and, moreover, if $t \preceq v \preceq s$ with $t, s \in B$, then this implies $v \in B$ as well. Of course, finite branches are of the form $[t, s]$ for suitable $t \preceq s$.

For any $s \in T$ its order $|s| \ge 0$ is defined by

$$|s| := \#\{t \in T : t \prec s\} \ . \tag{2.1}$$

Let ρ be an arbitrary metric on the tree T. Given $\varepsilon > 0$ a set $\mathcal{O} \subseteq T$ is said to be an ε–order net w.r.t. ρ provided that for each $s \in T$ there is an $t \in \mathcal{O}$ with $t \preceq s$ and $\rho(t, s) < \varepsilon$. Let

$$\tilde{N}(T, \rho, \varepsilon) := \inf\{\#\{\mathcal{O}\} : \mathcal{O} \text{ is an } \varepsilon\text{–order net of } T\} \tag{2.2}$$

be the corresponding order covering numbers. Clearly, we have

$$N(T, \rho, \varepsilon) \le \tilde{N}(T, \rho, \varepsilon) \ .$$

As shown in [8, Proposition 3.3], for the metric d defined by (1.3) we also have a reverse estimate. More precisely, here it always holds

$$\tilde{N}(T, d, 2\varepsilon) \le N(T, d, \varepsilon) \ . \tag{2.3}$$

3 Proof of Theorem 1.1

Let T be an arbitrary tree and let α and σ be weights on T as before. Define $X = (X_t)_{t \in T}$ as in (1.1). Of course, whenever (1.2) holds, then we necessarily have

$$\sup_{t \in T} \left(\mathbb{E}|X_t|^2\right)^{1/2} = \sup_{t \in T} \sigma(t) \left(\sum_{v \preceq t} \alpha(v)^2\right)^{1/2} < \infty \ . \tag{3.1}$$

Thus let us assume that (3.1) is always satisfied.

In order to prove part one of Theorem 1.1 in a first step we replace the process X by a process $\hat{X}$ which is easier to handle.

To this end, if $k \in \mathbb{Z}$, define $I_k \subseteq T$ by

$$I_k := \left\{ t \in T : 2^{-k-1} < \sigma(t) \leq 2^{-k} \right\} \tag{3.2}$$

and a new weight $\hat{\sigma}$ by

$$\hat{\sigma} := \sum_{k \in \mathbb{Z}} 2^{-k} \mathbf{1}_{I_k} \ . \tag{3.3}$$

Let $\hat{X}$ be the process defined by α and $\hat{\sigma}$ via (1.1), i.e., it holds

$$\hat{X}_t = \hat{\sigma}(t) \sum_{v \preceq t} \alpha(v) \xi_v \ , \quad t \in T \ , \tag{3.4}$$

and let $\hat{d}$ denote the distance generated via α and $\hat{\sigma}$ as in (1.3). Then the following are valid.

Proposition 3.1

1. *If $t \preceq s$, then it holds*
$$d(t,s) \leq \hat{d}(t,s) \leq 2\, d(t,s) \ .$$
Consequently, it follows
$$\tilde{N}(T,d,\varepsilon) \leq \tilde{N}(T,\hat{d},\varepsilon) \leq \tilde{N}(T,d,\varepsilon/2) \ ,$$
where $\tilde{N}(T,d,\varepsilon)$ and $\tilde{N}(T,\hat{d},\varepsilon)$ are the order covering numbers corresponding to the respective metrics.

2. *The process X is a.s. bounded if and only if $\hat{X}$ is a.s. bounded.*

Proof: The first assertion follows easily by the definition of d and $\hat{d}$ while the second one is a direct consequence of

$$|X_t| \leq |\hat{X}_t| \leq 2\,|X_t| \ , \quad t \in T \ .$$

$\square$

As a consequence of the preceding proposition we conclude that it suffices to prove Theorem 1.1 in the case of non–increasing weights σ of the form

$$\sigma := \sum_{k \in \mathbb{Z}} 2^{-k} \mathbf{1}_{I_k} \tag{3.5}$$

The property that σ is non–decreasing reflects in the following properties of the partition $(I_k)_{k \in \mathbb{Z}}$ of T.

1. Whenever $B \subseteq T$ is a branch, then for each $k \in \mathbb{Z}$ either $B \cap I_k = \emptyset$ or it is an order interval in T.

2. If $l < k$, $t \in B \cap I_l$, $s \in B \cap I_k$, then this implies $t \prec s$.

3. $I_k = \emptyset$ whenever $k \leq k_0$ for a certain $k_0 \in \mathbb{Z}$.

Thus from now on we may suppose that the weight σ is as in (3.5) with a partition $(I_k)_{k\in\mathbb{Z}}$ of T possessing properties (1), (2) and (3) stated before.

In a second step of the proof of Theorem 1.1, first part, we define a process $Y := (Y_t)_{t\in T}$ which may be viewed as a localization of X. To this end let us write $t \equiv s$ provided there is a $k \in \mathbb{Z}$ such that $t, s \in I_k$. With this notation we set

$$Y_t := \sigma(t) \sum_{\substack{v \preceq t \\ v \equiv t}} \alpha(v)\xi_v \ , \quad t \in T \ . \tag{3.6}$$

It is an easy deal to relate the boundedness of X with that of Y.

Proposition 3.2 *The process Y is a.s. bounded if and only if X is a.s. bounded.*

Proof: Actually, we establish simple linear relations between Y and X, see (3.7) and (3.8) below. For any integers $\ell \le k$ and any $t \in I_k$ set $B_\ell(t) := [\mathbf{0}, t] \cap I_\ell$. Then we have

$$\begin{aligned} X_t &= 2^{-k} \sum_{\ell \le k} \sum_{v \in B_\ell(t)} \alpha(v)\xi_v \\ &= \sum_{\ell \le k} 2^{-(k-\ell)} \cdot 2^{-\ell} \sum_{v \in B_\ell(t)} \alpha(v)\xi_v \\ &= \sum_{\ell} 2^{-(k-\ell)} Y_{\lambda_\ell(t)}, \end{aligned} \tag{3.7}$$

where the last sum is taken over $\ell \le k$ such that $B_\ell(t) \ne \emptyset$ and $\lambda_\ell(t) := \max\{s : s \in B_\ell(t)\}$. It follows from (3.7) that the boundedness of Y yields that of X.

To prove the converse statement of Proposition 3.2, take an arbitrary $t \in T$ and consider two different cases.
If $t \equiv \mathbf{0}$ (recall that $\mathbf{0}$ denotes the root of T), then by the definition of Y we simply have $Y_t = X_t$.

Otherwise, if $t \not\equiv \mathbf{0}$, let

$$\lambda^-(t) = \max\{s : s \preceq t, s \not\equiv t\}.$$

By the definition of Y_t we obtain

$$\begin{aligned} Y_t &= \sigma(t) \sum_{\lambda^-(t) \prec v \preceq t} \alpha(v)\xi_v \\ &= \sigma(t) \left(\sum_{v \preceq t} \alpha(v)\xi_v - \sum_{v \preceq \lambda^-(t)} \alpha(v)\xi_v \right) \\ &= X_t - \frac{\sigma(t)}{\sigma(\lambda^-(t))} \, X_{\lambda^-(t)}. \end{aligned} \tag{3.8}$$

Since the weight σ is non–increasing, by $\lambda^-(t) \preceq t$ we get $\frac{\sigma(t)}{\sigma(\lambda^-(t))} \le 1$. It follows from (3.8) that if X is a.s. bounded this is also valid for Y as claimed. This completes the proof. □

In the next step we calculate the Dudley distance generated by Y and compare $N(T, d_Y, \varepsilon)$ with $\tilde{N}(T, d, \varepsilon)$. Recall that $\tilde{N}(T, d_Y, \varepsilon)$ and $\tilde{N}(T, d, \varepsilon)$ are the corresponding order covering numbers as introduced in (2.2).

Proposition 3.3 *Suppose $\alpha(\mathbf{0}) = 0$, hence $Y_{\mathbf{0}} = 0$ a.s. Then it follows that*

$$N(T, d_Y, \varepsilon) \le \tilde{N}(T, d_Y, \varepsilon) \le \tilde{N}(T, d, \varepsilon) + 1 \,. \tag{3.9}$$

Proof: If $t \preceq s$, then we get

$$d_Y(t,s)^2 = \sigma(s)^2 \sum_{t \prec v \preceq s} \alpha(v)^2 = d(t,s)^2 \qquad \text{if} \quad t \equiv s$$

and

$$d_Y(t,s)^2 = \mathbb{E}|Y_t|^2 + \mathbb{E}|Y_s|^2 \quad \text{if} \quad t \not\equiv s \,.$$

Given $\varepsilon > 0$ let $\mathcal{O} \subseteq T$ be an ε–order net w.r.t. the metric d. Take $s \in T$ arbitrarily. Then there is a $t \in \mathcal{O}$ such that $t \preceq s$ and $d(t,s) < \varepsilon$. If $t \equiv s$, then this implies $d_Y(t,s) = d(t,s) < \varepsilon$ as well. But if $t \not\equiv s$, then we get

$$\begin{aligned} d_Y(\mathbf{0}, s) &= \left(\mathbb{E}|Y_s - Y_{\mathbf{0}}|^2\right)^{1/2} = \left(\mathbb{E}|Y_s|^2\right)^{1/2} = \sigma(s) \left(\sum_{\substack{v \preceq s \\ v \equiv t}} \alpha(v)^2 \right)^{1/2} \\ &\le \sigma(s) \left(\sum_{t \prec v \preceq s} \alpha(v)^2 \right)^{1/2} \le d(t,s) < \varepsilon \,. \end{aligned}$$

In different words, the set $\mathcal{O} \cup \{\mathbf{0}\}$ is an ε–order net of T w.r.t. d_Y. Of course, this implies the second inequality in (3.9), the first one being trivial. Thus the proof is complete. □

Proof of Theorem 1.1, first part: Without loosing generality we may assume $\alpha(\mathbf{0}) = 0$. Indeed, write

$$X_t = \sigma(t) \sum_{v \preceq t} \alpha(v) \xi_v = \sigma(t) \sum_{\mathbf{0} \prec v \preceq t} \alpha(v) \xi_v + \sigma(t) \alpha(\mathbf{0}) \xi_{\mathbf{0}}$$

and observe that $\sup_{t \in T} \sigma(t) < \infty$. Moreover, the metric d is independent of $\alpha(\mathbf{0})$. Note that this number never appears in the evaluation of $d(t,s)$ for arbitrary $t, s \in T$.

Thus let us assume now that (1.4) is valid. Then (2.3) implies

$$\int_0^\infty \sqrt{\log \tilde{N}(T, d, \varepsilon)} \, \mathrm{d}\varepsilon < \infty$$

as well. Hence Proposition 3.3 yields

$$\int_0^\infty \sqrt{\log N(T, d_Y, \varepsilon)} \, \mathrm{d}\varepsilon < \infty \,.$$

Consequently, Dudley's theorem (cf. [2] or [6], p.179) applies for Y and d_Y, hence Y possesses a.s. bounded paths. In view of Proposition 3.2 the paths of X are also a.s. bounded and this completes the proof of the first part of Theorem 1.1.

Proof of Theorem 1.1, second part: Take $\varepsilon > 0$. As proved in [8, Proposition 5.2] there are at least $m := N(T, d, 2\varepsilon) - 1$ disjoint order intervals $(t_i, s_i]$ in T with $d(t_i, s_i) \ge \varepsilon$. By the definition of d we find $t_i \prec r_i \preceq s_i$ such that

$$\sigma(r_i) \left(\sum_{t_i \prec v \preceq r_i} \alpha(v)^2 \right)^{1/2} \ge \varepsilon \,, \quad 1 \le i \le m \,.$$

Next, set

$$\eta_i := X_{r_i} - \frac{\sigma(r_i)}{\sigma(t_i)} X_{t_i}\,, \quad 1 \le i \le m\,. \tag{3.10}$$

Then it follows that

$$\eta_i = \sigma(r_i)\left[\sum_{v \preceq r_i} \alpha(v)\xi_v - \sum_{v \preceq t_i} \alpha(v)\xi_v\right] = \sigma(r_i)\left[\sum_{t_i \prec v \preceq r_i} \alpha(v)\xi_v\right]$$

and, consequently, the η_i are independent centered Gaussian with

$$(\mathbb{E}\,|\eta_i|^2)^{1/2} = \sigma(r_i)\left(\sum_{t_i \prec v \preceq r_i} \alpha(v)^2\right)^{1/2} \ge \varepsilon\,. \tag{3.11}$$

Since σ is assumed to be non–increasing, we get

$$\sup_{1 \le i \le m} |\eta_i| \le 2 \sup_{t \in T} |X_t|\;. \tag{3.12}$$

Suppose now that X is a.s. bounded. By Fernique's theorem (cf. [4] or [6], p.142) this implies

$$C := \mathbb{E} \sup_{t \in T} |X_t| < \infty\,,$$

hence (3.12) leads to

$$\mathbb{E} \sup_{1 \le i \le m} |\eta_i| \le 2\,C\,,$$

and by the choice of m the assertion follows by

$$c\,\varepsilon\,\sqrt{\log m} \le \mathbb{E} \sup_{1 \le i \le m} |\eta_i|$$

where we used (3.11) and the classical Fernique–Sudakov bound recalled below in (6.4). □

The main advantage of Theorem 1.1 is that there are quite general techniques to get precise estimates for $N(T,d,\varepsilon)$ (cf. [8]). For example, Theorem 1.1 implies the following.

Corollary 3.4 *Let T be a binary tree and suppose that*

$$\alpha(t)\sigma(t) \le c\,|t|^{-\gamma}\,, \quad t \in T\,,$$

for some $\gamma > 1$. Then X defined by (1.1) is a.s. bounded. Conversely, if

$$\alpha(t) \ge c\,|t|^{-\gamma}$$

for some $\gamma < 1$ and $\sigma(t) \equiv 1$, then the generated process X is a.s. unbounded.

Proof: As shown in [8], an estimate $\alpha(t)\sigma(t) \le c\,|t|^{-\gamma}$ implies $\log N(T,d,\varepsilon) \le c\,\varepsilon^{-2/(2\gamma-1)}$ for each $\gamma > 1/2$. Hence, if $\gamma > 1$, then (1.4) holds, hence Theorem 1.1 applies and completes the proof of the first part.

The second part follows by $\log N(T,d,\varepsilon) \ge c\,\varepsilon^{-2/(2\gamma-1)}$ whenever $\alpha(t) \ge c\,|t|^{-\gamma}$ for some $\gamma > 1/2$ and $\sigma(t) \equiv 1$ (cf. [8, Proposition 7.7]). Thus, by Theorem 1.1 the process X cannot be bounded if $\gamma < 1$. □

Remark: The second part of Corollary 3.4 does no longer hold for non–constant weights σ. In different words, an estimate $\alpha(t)\sigma(t) \geq c\,|t|^{-\gamma}$ with $1/2 < \gamma < 1$ does not always imply that X is unbounded (cf. the remark after Corollary 6.4 below).

Corollary 3.4 suggests that the boundedness of a process with weights $\alpha(t)$ and $\sigma(t)$ might be determined by the product $\sigma(t)\alpha(t)$. In other words, it is natural to ask what is the relation between the boundedness of this process and the process generated by the weights $\widetilde{\sigma}(t) :\equiv 1$ and $\widetilde{\alpha}(t) := \sigma(t)\alpha(t)$. It turns out that (only) a one–sided implication is valid.

To investigate this question, for a moment write $X^{\alpha,\sigma}$ for the process defined in (1.1).

Proposition 3.5 *If $X^{\alpha\sigma,1}$ is a.s. bounded, then this is also true for $X^{\alpha,\sigma}$.*

Proof: We only give a sketch of the proof.

1. Recall a general fact from the theory of Gaussian processes: If X and Y are two independent centered Gaussian processes, then $X + Y$ bounded yields X bounded. This is an immediate consequence of Anderson's inequality (cf. [6, p.135]).
2. By applying this fact we obtain: If two weights are related by $\alpha_1 \leq c\,\alpha_2$ and $X^{\alpha_2,1}$ is bounded, then $X^{\alpha_1,1}$ is bounded as well.
3. Suppose now that $X^{\alpha\sigma,1}$ is bounded, then $X^{\alpha\hat{\sigma},1}$ is bounded, where the binary weight $\hat{\sigma}$ is defined in (3.3). Set $X' := X^{\alpha\hat{\sigma},1}$.
4. Let now Y be the process constructed in the article associated to $X^{\alpha,\sigma}$. Since $Y_t = X'_t - X'_{\lambda^-(t)}$, we see that if X' is bounded, then Y is bounded.
5. Recall that we know that the boundedness of Y is equivalent to that of $X^{\alpha,\sigma}$.

□

Examples in Section 6 show that the statement of Proposition 3.5 cannot be reversed, i.e., in general the boundedness of $X^{\alpha,\sigma}$ does not yield that of $X^{\alpha\sigma,1}$.

4 Compactness properties of (T, d) versus those of (T, d_X)

The aim of this section is to compare the metric d on T defined in (1.3) and the Dudley distance d_X introduced in (1.6). For X defined by (1.1) the latter distance equals

$$d_X(t,s)^2 = |\sigma(t) - \sigma(s)|^2 \sum_{v \preceq t \wedge s} \alpha(v)^2 + \sigma(t)^2 \sum_{t \wedge s \prec v \preceq t} \alpha(v)^2 + \sigma(s)^2 \sum_{t \wedge s \prec v \preceq s} \alpha(s)^2 \,.$$

In particular, if $t \preceq s$, this reads as

$$d_X(t,s)^2 = |\sigma(t) - \sigma(s)|^2 \sum_{v \preceq t} \alpha(v)^2 + \sigma(s)^2 \sum_{t \prec v \preceq s} \alpha(v)^2 \,. \tag{4.1}$$

Recall, that for $t \preceq s$ we have

$$d(t,s) = \max_{t \prec r \preceq s} \sigma(r) \left(\sum_{t \prec v \preceq r} \alpha(v)^2 \right)^{1/2} \,. \tag{4.2}$$

Comparing (4.1) with (4.2), it is not clear at all how these two distances are related in general.

In a first result we show that the covering numbers w.r.t. d and to d_X may be of quite different order.

Proposition 4.1 *There are non–increasing weights α and σ on a tree T such that the generated process X is a.s. bounded and, moreover,*

$$\lim_{\varepsilon\to 0}\frac{N(T,d_X,\varepsilon)}{N(T,d,\varepsilon)}=\infty\ .$$

Proof: Take $T=\mathbb{N}_0=\{0,1,\ldots\}$ and let $\alpha(0)=\sigma(0)=1$. If $k\geq 1$ set

$$\alpha(k)=k^{-\nu}\quad\text{and}\quad\sigma(k)=k^{-\theta}$$

for some $\theta,\nu>0$, i.e.,

$$X_k=k^{-\theta}\left[\sum_{j=1}^{k}j^{-\nu}\,\xi_j+\xi_0\right]\ ,\quad k\geq 1\ . \tag{4.3}$$

The law of iterated logarithm tells us that the process X is a.s. bounded if and only if $\theta+\nu>1/2$. Thus let us assume that this is satisfied.

Take now any $1\leq k<l$. Then by (4.1) it follows

$$d_X(k,l)\geq k^{-\theta}-l^{-\theta}\geq k^{-\theta}-(k+1)^{-\theta}\geq c_\theta\,k^{-\theta-1}\ .$$

Hence, if $1\leq k<l\leq n$ for some $n\geq 2$, this implies

$$d_X(k,l)\geq c_\theta n^{-\theta-1}$$

which yields

$$N(T,d_X,\varepsilon)\geq c\,\varepsilon^{-1/(\theta+1)} \tag{4.4}$$

for some $c>0$ only depending on θ.

On the other hand, we have $\alpha(k)\sigma(k)=k^{-(\theta+\nu)}$. As shown in [8, Proposition 6.3] (apply this proposition with $q=2$, $H=0$ and $\gamma=2(\theta+\nu)$) a bound $\alpha(k)\sigma(k)\leq k^{-(\theta+\nu)}$ implies

$$N(T,d,\varepsilon)\leq c\,\varepsilon^{-1/(\theta+\nu)}\ . \tag{4.5}$$

Of course, if $\nu>1$, then (4.4) and (4.5) lead to

$$\lim_{\varepsilon\to 0}\frac{N(T,d_X,\varepsilon)}{N(T,d,\varepsilon)}=\infty \tag{4.6}$$

completing the proof. □

Let us state some interesting consequence of the preceding proposition. To this end recall a result due to M. Talagrand (cf. [12] and [5]). Suppose $X=(X_t)_{t\in T}$ is a centered Gaussian process on an arbitrary index set T and let d_X, as in (1.6), be the Dudley metric on T generated by X. If $N(T,d_X,\varepsilon)\leq\psi(\varepsilon)$ for a non–increasing function ψ satisfying

$$c_1\,\psi(\varepsilon)\leq\psi(\varepsilon/2)\leq c_2\,\psi(\varepsilon) \tag{4.7}$$

for certain $1<c_1<c_2$, then this implies

$$-\log\mathbb{P}\left(\sup_{t\in T}|X_t|<\varepsilon\right)\leq c\,\psi(\varepsilon) \tag{4.8}$$

for some $c>0$.

We claim now that in the case of processes X defined by (4.3) even holds

$$-\log \mathbb{P}\left(\sup_{k\geq 1}\left|k^{-\theta}\left[\sum_{j=1}^{k} j^{-\nu}\xi_j+\xi_0\right]\right|<\varepsilon\right)\approx \varepsilon^{-1/(\theta+\nu)} . \tag{4.9}$$

Indeed, if we apply Proposition 7.1 in [8] with $\varphi(x)=x^{-\gamma}$ where $\gamma=2(\theta+\nu)$, we see that estimate (4.5) is sharp, i.e., we obtain

$$N(T,d,\varepsilon)\approx\varepsilon^{-1/(\theta+\nu)} .$$

Consequently, (4.9) follows by Proposition 9.1 in [8].

Comparing (4.9) with (4.4) shows that for $\nu>1$ estimate (4.8) cannot lead to sharp estimates while, as seen above, the use of $N(T,d,\varepsilon)$ does so. In some sense this observation proves that the metric d fits better to those processes X than d_X does.

One may ask now whether or not there are examples of trees and weights such that the quotient in (4.6) tends to zero, i.e., whether there are examples with

$$\lim_{\varepsilon\to 0}\frac{N(T,d,\varepsilon)}{N(T,d_X,\varepsilon)}=\infty . \tag{4.10}$$

Although we do not know the answer to this question let us shortly indicate why such examples are hardly to construct provided they exist. Indeed, if $N(T,d,\varepsilon)\approx\varepsilon^{-a}\left|\log\varepsilon\right|^{b}$ for some $a>0$ and $b\geq 0$, then by Proposition 9.1 in [8] this implies

$$-\log\mathbb{P}\left(\sup_{t\in T}|X_t|<\varepsilon\right)\approx\varepsilon^{-a}\left|\log\varepsilon\right|^{b} .$$

Consequently, whenever $N(T,d_X,\varepsilon)\approx\psi(\varepsilon)$ with ψ satisfying (4.7), then by (4.8) we get

$$N(T,d,\varepsilon)\leq c\,\psi(\varepsilon)\leq c'\,N(T,d_X,\varepsilon) ,$$

hence in that situation examples satisfying (4.10) cannot exist.

In spite of this observation we will show now that $d_X(t,s)$ may become arbitrarily small while $d(t,s)\geq C>0$. Hence an estimate $d(t,s)\leq c\,d_X(t,s)$ cannot be valid in general. Recall that in view of Proposition 4.1 a relation $d_X(t,s)\leq c\,d(t,s)$ is impossible as well.

Proposition 4.2 *There are weights α and σ on $T=\mathbb{N}_0$ such that the corresponding process X is a.s. bounded and such that $\lim_{k\to\infty}d_X(\mathbf{0},k)=0$ while $d(\mathbf{0},k)=C>0$ for all $k\geq 1$.*

Proof: For $k\in\mathbb{N}_0$ choose $\sigma(k)=2^{-k}$ while $\alpha(0)=0$ and $\alpha(k)=k^{-1}$ for $k\geq 1$. Of course, the generated process X is a.s. bounded. Moreover, if $k\geq 1$, then it follows that

$$d_X(\mathbf{0},k)=2^{-k}\left(\sum_{v=1}^{k}v^{-2}\right)^{1/2} .$$

In particular, $d_X(\mathbf{0},k)\to 0$ quite rapidly as $k\to\infty$. On the other hand,

$$d(\mathbf{0},k)=2^{-1}\alpha(1)=2^{-1}$$

and this completes the proof with $C=2^{-1}$. $\square$

5 More about the relation between processes and their entropy

So far, we came out with a somewhat messy set of relations between the processes and their entropy. Let us try to rearrange it again and to put the things in order. We have three consecutively generated processes $X \to \hat{X} \to Y$. In this section, we will not identify or replace X by $\hat{X}$, as we did sometimes before.

Before proceeding further, let us make a useful and well–known identification. Suppose $X = (X_t)_{t\in T}$ is an arbitrary Gaussian process (in fact we only need that it is a process with finite second moments) modeled over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then we may regard X as subset of the Hilbert space $L_2(\Omega, \mathcal{A}, \mathbb{P})$, i.e., we identify X with $\{X_t : t \in T\}$ and the induced distance equals

$$\|X_t - X_s\|_2 = \left(\mathbb{E}|X_t - X_s|^2\right)^{1/2} = d_X(t, s) \ .$$

In particular, we may also build the absolutely convex hull of X in $L_2(\Omega, \mathcal{A}, \mathbb{P})$ which we denote by $\mathrm{aco}(X)$.

Suppose now that the processes X, $\hat{X}$ and Y are defined as in (1.1), (3.4) and (3.6), respectively, where for the construction of $\hat{X}$ and Y we use the partition $(I_k)_{k\in\mathbb{Z}}$ given by (3.2). First we show that $\mathrm{aco}(X)$, $\mathrm{aco}(\hat{X})$, and $\mathrm{aco}(Y)$ are the same sets up to a numeric constant. Namely, the following is valid.

Proposition 5.1 *We have*

$$\mathrm{aco}(X) \subseteq \mathrm{aco}(\hat{X}) \subseteq 2\,\mathrm{aco}(Y) \subseteq 4\,\mathrm{aco}(\hat{X}) \subseteq 8\,\mathrm{aco}(X). \tag{5.1}$$

Proof: By the definition of $\hat{X}$ it follows that

$$\mathrm{aco}(X) \subseteq \mathrm{aco}(\hat{X}) \subseteq 2\,\mathrm{aco}(X). \tag{5.2}$$

On the other hand, (3.7) yields

$$\hat{X} \subseteq \sum_{m=0}^{\infty} 2^{-m} Y, \tag{5.3}$$

hence

$$\mathrm{aco}(\hat{X}) \subseteq \sum_{m=0}^{\infty} 2^{-m} \mathrm{aco}(Y) = 2\,\mathrm{aco}(Y), \tag{5.4}$$

while (3.8) implies

$$Y \subseteq \hat{X} - [0,1] \cdot \hat{X},$$

hence

$$\mathrm{aco}(Y) \subseteq 2\,\mathrm{aco}(\hat{X}). \tag{5.5}$$

By combining the inclusions (5.2), (5.4), (5.5), claim (5.1) follows. □

Remark: Clearly, (5.1) means that the three processes are either all bounded or all are unbounded. But, certainly, it contains even more information.

Now we move to covering numbers in order to clarify the role of the distance d defined in (1.3). We will show now that on the *logarithmic* level there is no much difference between $N(X, ||\cdot||_2, \varepsilon) = N(T, d_X, \varepsilon)$ and $N(T, d, \varepsilon)$.

Theorem 5.2 *We have*

$$\int_0^\infty \sqrt{\log N(T,d_X,u)}\,\mathrm{d}u < \infty \quad \Leftrightarrow \quad \int_0^\infty \sqrt{\log N(T,d,u)}\,\mathrm{d}u < \infty\,.$$

and

$$\sup_{\varepsilon>0} \varepsilon^2 \log N(T,d_X,\varepsilon) < \infty \quad \Leftrightarrow \quad \sup_{\varepsilon>0} \varepsilon^2 \log N(T,d,\varepsilon) < \infty\,.$$

Proof: We first give the lower bounds for $N(T,d_X,\varepsilon)$. By (3.10) and (3.11) it follows that

$$N(X-[0,1]\cdot X, ||\cdot||_2, \frac{\varepsilon}{\sqrt{2}}) \geq N(T,d,2\varepsilon) - 1. \tag{5.6}$$

Our next task is to replace $X-[0,1]\cdot X$ by X in (5.6) by using the following trivial fact.

Lemma 5.3 *Let X be a subset of a normed space and $M_X := \sup_{x\in X} ||x||$. Then*

$$N([0,1]\cdot X, \|\cdot\|, 2\varepsilon) \leq N(X,\|\cdot\|,\varepsilon)\,\frac{M_X}{\varepsilon} \quad \textit{and} \tag{5.7}$$

$$N(X-[0,1]\cdot X, \|\cdot\|, 3\varepsilon) \leq N(X,\|\cdot\|,\varepsilon)^2\,\frac{M_X}{\varepsilon}\,. \tag{5.8}$$

Proof of the lemma: Let B be an ε–net for X and set

$$C = \left\{ \frac{j\varepsilon}{M_X}\, y \;:\; y \in B,\ j\in\mathbb{N},\, 1 \leq j \leq \frac{M_X}{\varepsilon} \right\}.$$

Clearly,

$$\#\{C\} \leq \#\{B\}\,\frac{M_X}{\varepsilon}\,.$$

Take any $z=\theta x \in [0,1]\cdot X$ with $x\in X$, $\theta\in[0,1]$. Find $y\in B$ and a positive integer $j \leq \frac{M_X}{\varepsilon}$ such that

$$||x-y|| < \varepsilon, \qquad \left| j - \frac{\theta M_X}{\varepsilon}\right| \leq 1\,.$$

Then $z' := \frac{j\varepsilon}{M_X}\, y \in C$ and observe that

$$\begin{aligned}
||z-z'|| &= \left\|\theta x - \frac{j\varepsilon}{M_X}\, y\right\| \\
&\leq \theta||x-y|| + \left|\theta - \frac{j\varepsilon}{M_X}\right|\ ||y|| \\
&< \varepsilon + \frac{\varepsilon}{M_X}\, M_X = 2\varepsilon.
\end{aligned}$$

Hence, C is a 2ε–net for $[0,1]\cdot X$ and the first claim of the lemma is proved. The second one follows immediately. □

We may proceed now with the proof of Theorem 5.2. Combining (5.8) with (5.6) leads to

$$N(T,d_X,\varepsilon)^2 \geq \frac{\varepsilon}{M_X}\left[N(T,d,6\sqrt{2}\varepsilon) - 1\right].$$

Hence, we conclude

$$\int_0^\infty \sqrt{\log N(T,d_X,u)}\,\mathrm{d}u < \infty \quad \Rightarrow \quad \int_0^\infty \sqrt{\log N(T,d,u)}\,\mathrm{d}u < \infty$$

and

$$\sup_{\varepsilon>0} \varepsilon^2 \log N(T, d_X, \varepsilon) < \infty \quad \Rightarrow \quad \sup_{\varepsilon>0} \varepsilon^2 \log N(T, d, \varepsilon) < \infty \, .$$

Conversely, we will move now towards an upper bound for $N(T, d_X, \varepsilon)$. By Proposition 3.3 and Proposition 3.2 of [8] we have

$$N(T, d_Y, \varepsilon) \leq \tilde{N}(T, \hat{d}, \varepsilon) + 1 \leq \tilde{N}(T, d, \varepsilon/2) + 1 \leq N(T, d, \varepsilon/4) + 1. \tag{5.9}$$

Next, we trivially obtain from (5.3) that

$$\begin{aligned} N(T, d_{\hat{X}}, 2\varepsilon) &\leq N\Big(T, d_{\hat{X}}, \Big(\sum_{m=0}^{\infty}(m+1)^{-2}\Big)\,\varepsilon\Big) \\ &\leq \prod_{m=0}^{\infty} N(T, d_{2^{-m}Y}, (m+1)^{-2}\varepsilon) \\ &= \prod_{m=0}^{\infty} N(T, d_Y, 2^m(m+1)^{-2}\varepsilon). \\ &\leq \prod_{m=0}^{\infty} N_*(T, d, 2^{m-2}(m+1)^{-2}\varepsilon), \end{aligned}$$

where we used (5.9) on the last step and

$$N_*(T, d, r) := \begin{cases} N(T, d, r) + 1 & : \; N(T, d_Y, 4r) > 1, \\ 1 & : \; N(T, d_Y, 4r) = 1. \end{cases}$$

It follows that

$$\log N(T, d_{\hat{X}}, 2\varepsilon) \leq \sum_{\{m\geq 0:\, 2^m(m+1)^{-2}\varepsilon \leq M_Y\}} \log\big(N(T, d, 2^{m-2}(m+1)^{-2}\varepsilon) + 1\big)\,, \tag{5.10}$$

where $M_Y := \sup_{t\in T} ||Y_t||_2$. For the Dudley integral this implies

$$\begin{aligned} \int_0^\infty \sqrt{\log N(T, d_{\hat{X}}, 2\varepsilon)}\,\mathrm{d}\varepsilon &\leq \sum_{m=0}^{\infty} \int_0^{\frac{M_Y}{2^m(m+1)^{-2}}} \sqrt{\log\left(N(T, d, 2^{m-2}(m+1)^{-2}\varepsilon) + 1\right)}\,\mathrm{d}\varepsilon \\ &\leq \sum_{m=0}^{\infty} \frac{(m+1)^2}{2^{m-2}} \int_0^\infty \sqrt{\log\left(N(T, d, u) + 1\right)}\,\mathrm{d}u \\ &= C \int_0^\infty \sqrt{\log\left(N(T, d, u) + 1\right)}\,\mathrm{d}u \, . \end{aligned}$$

Hence,

$$\int_0^\infty \sqrt{\log N(T, d, u)}\,\mathrm{d}u < \infty \quad \Rightarrow \quad \int_0^\infty \sqrt{\log N(T, d_{\hat{X}}, u)}\,\mathrm{d}u < \infty \, .$$

Moreover, (5.10) yields

$$\sup_{\varepsilon>0} \varepsilon^2 \log N(T, d, \varepsilon) < \infty \quad \Rightarrow \quad \sup_{\varepsilon>0} \varepsilon^2 \log N(T, d_{\hat{X}}, \varepsilon) < \infty \, .$$

The final passage goes from $\hat{X}$ to X. Since $X \subseteq [0,1]\cdot\hat{X}$, by applying (5.7) to $\hat{X}$ we obtain

$$\log N(T, d_X, 2\varepsilon) \leq \log N([0,1]\cdot\hat{X}, ||\cdot||_2, 2\varepsilon) \leq \log N(T, d_{\hat{X}}, \varepsilon) + \log\left(\frac{M_{\hat{X}}}{\varepsilon}\right).$$

Hence

$$\int_0^\infty \sqrt{\log N(T, d_{\hat{X}}, u)}\,\mathrm{d}u < \infty \quad \Rightarrow \quad \int_0^\infty \sqrt{\log N(T, d_X, u)}\,\mathrm{d}u < \infty$$

as well as

$$\sup_{\varepsilon>0} \varepsilon^2 \log N(T, d_{\hat{X}}, \varepsilon) < \infty \quad \Rightarrow \quad \sup_{\varepsilon>0} \varepsilon^2 \log N(T, d_X, \varepsilon) < \infty.$$

By combining the preceding estimates we finish the proof. □

6 The binary tree with homogeneous weights

Before investigating Gaussian processes on binary trees let us shortly recall some basic facts about suprema of Gaussian sequences.

Let $(X_1, \ldots, X_n)$ be a centered Gaussian random vector. Introduce the following notations:

$$\sigma_1^2 := \min_j \mathbb{E}X_j^2, \quad \sigma_2^2 := \max_j \mathbb{E}X_j^2, \quad S := \max_j X_j\ ,$$

and let m_S be a median of S. Then the following is well known.

- It is true that

  $$m_S \le \mathbb{E}S. \tag{6.1}$$

  See [6], p.143.

- The following concentration principle is valid:

  $$\mathbb{P}(S > m_S + r) \le \hat{\Phi}(r/\sigma_2) \le \exp(-r^2/2\sigma_2^2), \qquad \forall r > 0,$$

  where

  $$\hat{\Phi}(r) = \frac{1}{\sqrt{2\pi}} \int_r^\infty e^{-\frac{u^2}{2}}\,\mathrm{d}u$$

  is the standard Gaussian tail. See [6], p.142. By combining this with (6.1) we also have

  $$\mathbb{P}(S > \mathbb{E}S + r) \le \exp(-r^2/2\sigma_2^2), \qquad \forall r > 0. \tag{6.2}$$

- It is true that

  $$\mathbb{E}S \le \sqrt{2\log n}\ \sigma_2\,. \tag{6.3}$$

  See [6], p.180.

- If $X_1, \ldots, X_n$ are independent, then

  $$\mathbb{E}S \ge c\sqrt{\log n}\ \sigma_1\,. \tag{6.4}$$

  with $c = 0.64$. See [6], p.193–194.

Remark that the same properties hold true for

$$S' := \max_{j\le n} |X_j| = \max_{j\le n} \max\{X_j, -X_j\}.$$

Let T be a binary tree and suppose that the weights depend only on the level numbers, i.e. $\alpha(t) = \alpha_{|t|}$ and $\sigma(t) = \sigma_{|t|}$ for some sequences $(\alpha_k)_{k\ge0}$ and $(\sigma_k)_{k\ge0}$ of positive numbers with $(\sigma_k)_{k\ge0}$ non–increasing. The following two theorems give, with a certain overlap, necessary and sufficient conditions for the boundedness of $(X_t)_{t\in T}$ in that case.

Theorem 6.1 *a) If $X = (X_t)_{t\in T}$ is a.s. bounded, then*

$$G := \sup_n \ \sigma_n \sum_{k=1}^{n} \alpha_k < \infty. \tag{6.5}$$

b) Moreover, if $(\alpha_k)_{k\geq 0}$ satisfies the regularity assumption

$$Q := \sup_n \ \sup_{n\leq k\leq 2n} \frac{\alpha_k}{\alpha_n} < \infty, \tag{6.6}$$

then X is a.s. bounded if and only if (6.5) holds.

Proof: a) Let us construct a random sequence $(t_n)_{n\geq 0}$ in T and a sequence of random variables $(\zeta_n)_{n\geq 1}$ by the following inductive procedure. Let $t_0 = \mathbf{0}$. Next, assuming that t_n is constructed, let t' and t'' be the two offsprings of t_n. We let

$$\zeta_{n+1} := \max\{\xi_{t'}, \xi_{t''}\}, \qquad t_{n+1} := \operatorname{argmax}\{\xi_{t'}, \xi_{t''}\}.$$

It is obvious that (ζ_n) are i.i.d. random variables with strictly positive expectation. Our construction yields

$$X_{t_n} = \sigma_n \left(\alpha_0 \xi_{\mathbf{0}} + \sum_{j=1}^{n} \alpha_j \ \zeta_j \right), \qquad n \geq 1.$$

It follows that

$$\mathbb{E} \sup_{t\in T} X_t \geq \sup_{n\geq 1} \mathbb{E} X_{t_n} = C \ \sup_{n\geq 1} \sigma_n \sum_{j=1}^{n} \alpha_j,$$

where $C := \mathbb{E}\zeta_j > 0$. Since the assumption "$(X_t)_{t\in T}$ is a.s. bounded" implies $\mathbb{E}\sup_{t\in T} X_t < \infty$, we obtain (6.5).

b) Let us assume that $G < \infty$, $Q < \infty$ and prove that $(X_t)_{t\in T}$ is a.s. bounded. For any $m \geq 0$ set $B_m = [2^m, 2^{m+1})$ and $J_m := \{t \in T : |t| \in B_m\}$. For any $M \geq 0$ and $t \in J_M$ write

$$\sum_{v\preceq t} \alpha(v)\xi_v = \sum_{m=0}^{M} \sum_{\substack{v\preceq t \\ v\in J_m}} \alpha(v)\xi_v \leq \sum_{m=0}^{M} U_m, \tag{6.7}$$

where

$$U_m := \sup_{u\in J_m} \left| \sum_{\substack{v\preceq u \\ v\in J_m}} \alpha(v)\xi_v \right|.$$

By using that $(\sigma_k)_{k\geq 0}$ is non–increasing, we infer from (6.7) for any $M \geq 0$ and $t \in J_M$

$$\begin{aligned} X_t &= \sigma_t \sum_{v\preceq t} \alpha(v)\xi_v \leq \sigma_{2^M} \sum_{m=0}^{M} U_m \\ &= \sigma_{2^M} \sum_{m=0}^{M} (\mathbb{E}U_m + (U_m - \mathbb{E}U_m)) \\ &\leq \sigma_{2^M} \sum_{m=0}^{M} (\mathbb{E}U_m + (U_m - \mathbb{E}U_m)_+) \\ &\leq \sigma_{2^M} \sum_{m=0}^{M} \mathbb{E}U_m + \sum_{m=0}^{\infty} \sigma_{2^m} (U_m - \mathbb{E}U_m)_+. \end{aligned}$$

Hence,

$$\sup_{t\in T} X_t \le \sup_{M\ge 0} \sigma_{2^M} \sum_{m=0}^{M} \mathbb{E}U_m + \sum_{m=0}^{\infty} \sigma_{2^m}(U_m - \mathbb{E}U_m)_+. \tag{6.8}$$

We will use now standard Gaussian techniques in order to evaluate the quantities on the r.h.s. Note that on the binary tree

$$\#\{J_m\} \le \#\{t : |t| < 2^{m+1}\} \le 2^{2^{m+1}}.$$

Moreover, we have

$$h_m^2 := \sup_{u\in J_m} \sum_{\substack{v\preceq u\\ v\in J_m}} \alpha(v)^2 \le \sum_{k\in B_m} \alpha_k^2.$$

Assuming (6.6) to hold, we obtain

$$h_m^2 \le \sum_{k\in B_m} \alpha_k^2 \le Q^2\ 2^m\ \alpha_{2^m}^2.$$

Using (6.6) again we arrive at

$$h_m \le Q\ 2^{m/2}\ \alpha_{2^m} \le Q^2\ 2^{1-m/2} \sum_{k\in B_{m-1}} \alpha_k. \tag{6.9}$$

Now by (6.3) it follows that

$$\mathbb{E}U_m \le \sqrt{\log(2\#\{J_m\})}\, h_m \le 4Q^2 \sum_{k\in B_{m-1}} \alpha_k.$$

Hence, for any M we get

$$\sigma_{2^M} \sum_{m=0}^{M} \mathbb{E}U_m \le \sigma_{2^M} 4Q^2 \sum_{m=0}^{M} \sum_{k\in B_{m-1}} \alpha_k = \sigma_{2^M} 4Q^2 \sum_{k=0}^{2^M-1} \alpha_k \le 4Q^2 G.$$

On the other hand, by the Gaussian concentration principle (6.2),

$$\mathbb{E}(U_m - \mathbb{E}U_m)_+ = \int_0^\infty \mathbb{P}(U_m - \mathbb{E}U_m > r)\, \mathrm{d}r \le \int_0^\infty \exp(-r^2/2h_m^2)\, \mathrm{d}r \le 2h_m.$$

From (6.9) it follows that

$$\begin{aligned}
\sigma_{2^m}\mathbb{E}(U_m - \mathbb{E}U_m)_+ &\le 2\sigma_{2^m} h_m \\
&\le 2\sigma_{2^m} Q^2\ 2^{1-m/2} \sum_{k\in B_{m-1}} \alpha_k. \\
&\le 2^{2-m/2} Q^2 \sigma_{2^m} \sum_{k\le 2^m} \alpha_k \\
&\le 2^{2-m/2} Q^2 G.
\end{aligned}$$

By plugging this into (6.8), we arrive at

$$\mathbb{E}\sup_{t\in T} X_t \le \sup_{M\ge 0} \sigma_{2^M} \sum_{m=0}^{M} \mathbb{E}U_m + \sum_{m=0}^{\infty} \mathbb{E}\sigma_{2^m}(U_m - \mathbb{E}U_m)_+ \le 4Q^2 G + Q^2 G \sum_{m=0}^{\infty} 2^{2-m/2} < \infty$$

and $(X_t)_{t\in T}$ is a.s. bounded. $\square$

Let us start with a first example where Theorem 6.1 applies. Take the binary tree T and suppose that either $\alpha(t) = (|t|+1)^{-1}$ and $\sigma(t) \equiv 1$ or that $\alpha(t) \equiv 1$ and $\sigma(t) = (|t|+1)^{-1}$. Note these weights lead to critical cases, namely, we have $\log N(T,d,\varepsilon) \approx \varepsilon^{-2}$ for both pairs of weights.

Corollary 6.2 *The process*

$$X'_t := (|t|+1)^{-1} \sum_{v \preceq t} \xi_v\,, \quad t \in T\,,$$

is a.s. bounded while

$$X''_t := \sum_{v \preceq t} (|v|+1)^{-1} \xi_v\,, \quad t \in T\,,$$

is a.s. unbounded.

Proof: In the first case (6.5) and (6.6) are satisfied while in the second one (6.5) fails. Thus both assertions follow by Theorem 6.1. $\square$

Remark: The preceding corollary is of special interest because $\alpha(t)\sigma(t) = (|t|+1)^{-1}$ in both cases. Consequently, the boundedness of the process X cannot be described by the behavior of $\alpha\sigma$. This is in contrast to the main results about metric entropy in [8] which only depend on this product behavior.

Theorem 6.1 does not apply in the case of rapidly increasing sequences $(\alpha_k)_{k\geq 0}$ because (6.6) fails for them. The next theorem fills this gap.

Theorem 6.3 *a) If $X = (X_t)_{t\in T}$ is a.s. bounded, then*

$$G_1 := \sup_n \sup_{m \leq n} \ \sigma_n \sqrt{m} \left(\sum_{k=m}^{n} \alpha_k^2 \right)^{1/2} < \infty. \tag{6.10}$$

b) If

$$G_2 := \sup_n \ \sigma_n \sqrt{n} \left(\sum_{k=0}^{n} \alpha_k^2 \right)^{1/2} < \infty, \tag{6.11}$$

then $(X_t)_{t\in T}$ is a.s. bounded.

c) Moreover, if $(\alpha_k)_{k\geq 0}$ is non–decreasing, then the conditions (6.10) and (6.11) are equivalent, thus X is a.s. bounded if and only if either of them holds.

Proof: a) Let us fix a pair of integers $m \leq n$. Take any mapping $L : \{t : |t| = m\} \to \{t : |t| = n\}$ such that $t \preceq L(t)$ for all t. Consider

$$Y_t := \sigma_n \sum_{t \preceq s \preceq L(t)} \alpha_{|s|} \xi_s, \qquad |t| = m.$$

Notice that the $(Y_t)_{|t|=m}$ are independent and that

$$\mathbb{E} Y_t^2 = \sigma_n^2 \sum_{m \leq k \leq n} \alpha_k^2\,.$$

By (6.4) it follows

$$\mathbb{E}\max_{|t|=m} Y_t \ge c\sqrt{\log(2^m)}\ \sigma_n \left(\sum_{m\le k\le n} \alpha_k^2\right)^{1/2} = \tilde{c}\ \sqrt{m}\ \sigma_n \left(\sum_{m\le k\le n} \alpha_k^2\right)^{1/2}.$$

On the other hand

$$Y_t = X_{L(t)} - \frac{\sigma_n}{\sigma_m} X_t\,,$$

hence

$$\max_{|t|=m} Y_t \le 2\sup_{t\in T}|X_t|.$$

We arrive at

$$2\ \mathbb{E}\sup_{t\in T}|X_t| \ge \tilde{c}\ \sqrt{m}\ \sigma_n \left(\sum_{m\le k\le n} \alpha_k^2\right)^{1/2},$$

and achieve the proof of a) by taking the supremum over m and n.

b) Let $S_n := \max_{|t|=n} X_t$. By (6.3) we have

$$\mathbb{E}S_n \le \sqrt{2\log(2^n)}\ \sigma_n \left(\sum_{k=0}^{n} \alpha_k^2\right)^{1/2} \le 2G_2. \tag{6.12}$$

We also have

$$\mathbb{E}X_t^2 = \sigma_n^2 \sum_{k=0}^{n} \alpha_k^2 \le \frac{G_2^2}{n}, \qquad |t| = n. \tag{6.13}$$

Since $\sup_{t\in T} X_t = \sup_n S_n$, for any $r > 0$ it follows that

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in T} X_t > 2G_2 + r\right) &\le \sum_{n=0}^{\infty} \mathbb{P}\left(S_n \ge 2G_2 + r\right) \\
&\le \sum_{n=0}^{\infty} \mathbb{P}\left(S_n \ge \mathbb{E}S_n + r\right) \qquad (\text{by (6.12)}) \\
&\le \mathbb{P}\left(S_0 \ge \mathbb{E}S_0 + r\right) + \sum_{n=1}^{\infty} \exp\left(-\frac{r^2 n}{2G_2^2}\right) \qquad (\text{by (6.13) and (6.2)}) \\
&= \mathbb{P}\left(X_{\mathbf{0}} \ge r\right) + \frac{\exp\left(-\frac{r^2}{2G_2^2}\right)}{1-\exp\left(-\frac{r^2}{2G_2^2}\right)} \to 0, \qquad \text{as } r\to\infty.
\end{aligned}$$

It follows that $(X_t)_{t\in T}$ is a.s. bounded. Thus assertion b) is proved.

c) The inequality $G_1 \le G_2$ is obvious for any $(\alpha_k)_{k\ge 0}$. We only need to show that a bound in the opposite direction holds, too. Let

$$m_n := \begin{cases} \frac{n}{2} & : \ n \text{ even} \\ \frac{n+1}{2} & : \ n \text{ odd.} \end{cases}$$

Assuming that $(\alpha_k)_{k\ge 0}$ is non-decreasing, we have

$$\sum_{m_n\le k\le n} \alpha_k^2 \ge \sum_{0\le k< m_n} \alpha_k^2,$$

hence

$$2 \sum_{m_n \le k \le n} \alpha_k^2 \ge \sum_{0 \le k \le n} \alpha_k^2.$$

It follows that

$$G_1 \ge \sup_n \ \sigma_n \sqrt{m_n} \left(\sum_{m_n \le k \le n} \alpha_k^2 \right)^{1/2} \ge \frac{1}{2} \sup_n \ \sigma_n \sqrt{n} \sum_{0 \le k \le n} \alpha_k^2 = \frac{G_2}{2}.$$

□

Corollary 6.4 *Let $\alpha_k = k^b \, 2^k$ for some $b \in \mathbb{R}$. Then $(X_t)_{t \in T}$ is a.s. bounded if and only if*

$$\sup_n \ \sigma_n \, n^{1/2+b} \, 2^n < \infty.$$

Remark: Note that criterion (6.5) from Theorem 6.1 fails to work in that case. Moreover, letting $b = -\gamma$ with $1/2 < \gamma < 1$ and $\sigma_n = 2^{-n}$, by Corollary 6.4 the corresponding process is bounded although $\alpha(t)\sigma(t) \ge |t|^{-\gamma}$ for $t \in T$. This shows that the second part of Corollary 3.4 is no longer valid for non–constant weights σ.

Another example where Theorem 6.1 does not apply is as follows.

Corollary 6.5 *Let $\alpha_k^2 = \exp((\log k)^\beta)$ with $\beta > 1$. Then $(X_t)_{t \in T}$ is a.s. bounded if and only if*

$$\sup_n \ \sigma_n \ \frac{n}{(\log n)^{\frac{\beta-1}{2}}} \ \exp((\log n)^\beta/2) < \infty.$$

Proof: Easy calculation shows that

$$\begin{aligned} \sum_{k=0}^{n} \alpha_k^2 \ &\sim \ \int_1^n \exp((\log u)^\beta) du = \int_0^{(\log n)^\beta} \exp(z + z^{1/\beta}) \frac{dz}{\beta z^{1-1/\beta}} \\ &\sim \ \frac{n}{\beta (\log n)^{\beta-1}} \ \exp((\log n)^\beta). \end{aligned}$$

An application of Theorem 6.3 yields the result. □

Our message is that Theorems 6.1 and 6.3 should *jointly* cover any reasonable case. Let us illustrate this by the following example. Recall that by the first part of Corollary 3.4, if T is the binary tree and $\alpha(t)\sigma(t) \le c\,|t|^{-\gamma}$ for some $\gamma > 1$, then the generated process X is a.s. bounded. For homogeneous (level–dependent) weights this means that $\alpha_k \sigma_k \le c\, k^{-\gamma}$ for some $\gamma > 1$ yields the a.s. boundedness of X. Let us see how this fact is related to Theorems 6.1 and 6.3.

Essentially, we have the following

- If $(\alpha_k)_{k \ge 0}$ is decreasing, then

$$\sigma_n \sum_{k \le n} \alpha_k \le \sigma_n \sum_{k \le n} \frac{c \ k^{-\gamma}}{\sigma_k} \le \sum_{k \le n} c \ k^{-\gamma} \le c \ \sum_{k=1}^{\infty} k^{-\gamma},$$

  hence (6.5) and (6.6) hold and Theorem 6.1 yields the boundedness.

- If $(\alpha_k)_{k\geq 0}$ is increasing, then

$$\sigma_n\sqrt{n}\left(\sum_{k\leq n}\alpha_k^2\right)^{1/2} \leq \sigma_n\sqrt{n}\left(n\alpha_n^2\right)^{1/2} = \sigma_n\alpha_n\ n \leq c\ n^{1-\gamma},$$

thus (6.11) holds even for $\gamma \geq 1$, and Theorem 6.3 yields the boundedness.

Finally let us relate the results in Theorems 6.1 and 6.3 to those about compactness properties of (T,d) with d defined in (1.3). Here we have the following partial result.

Proposition 6.6 *The expression G_1 in (6.10) is finite if and only if there is a constant $c>0$ such that*

$$d(t,s) \leq c\,|t|^{-1/2} \tag{6.14}$$

for all $t,s\in T$ with $t \prec s$.

Proof: First note that in the case of homogeneous weights we get

$$d(t,s) = \max_{|t|<l\leq|s|} \sigma_l \left(\sum_{k=|t|+1}^{l} \alpha_k^2\right)^{1/2}.$$

Next we remark that $G_1<\infty$ if and only if there is a constant $c>0$ such that

$$\sigma_n\left(\sum_{k=m+1}^{n}\alpha_k^2\right)^{1/2} \leq c\,m^{-1/2} \tag{6.15}$$

for all $0\leq m<n<\infty$.

Suppose now that (6.14) holds and take integers $m<n$. Next choose two elements $t,s\in T$ with $t\prec s$ such that $m=|t|$ and $n=|s|$. Note that (6.14) implies

$$\sigma_n\left(\sum_{k=m+1}^{n}\alpha_k^2\right)^{1/2} \leq d(t,s) \leq c\,|t|^{-1/2} = c\,m^{-1/2}$$

which proves (6.15).

Conversely, assume (6.15) and take any two elements $t\prec s$ in T. Furthermore, let $v\in(t,s]$ be a node where

$$d(t,s) = \sigma_{|v|}\left(\sum_{k=|t|+1}^{|v|}\alpha_k^2\right)^{1/2}.$$

Applying (6.15) with $m:=|t|$ and $n:=|v|$ leads to

$$d(t,s) \leq c\,m^{-1} = c\,|t|^{-1/2}$$

as claimed. This completes the proof. □

Remark: Clearly (6.14) implies $\log N(T,d,\varepsilon) \leq c\,\varepsilon^{-2}$ as we already know by combining Theorems 1.1 and 6.3. But it says a little bit more. Namely, an ε–net giving this order may be chosen as $\{t \in T : |t| \le c\,\varepsilon^{-1/2}\}$ for a certain $c > 0$. Of course, this heavily depends on the fact that we deal with homogeneous weights.

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