Pencils of cubics with eight base points lying in convex position in $\mathbb{R}P^2$

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Abstract

To a generic configuration of eight points in convex position in the plane, we associate a *list* consisting of the following information: for all of the 56 conics determined by five of the points, we specify the position of each of the three remaining points, inside or outside. We prove that the number of possible lists, up to the action of D_8, is 49, and we give two possible ways of encoding these lists. A generic complex pencil of cubics has twelve singular (nodal) cubics and nine distinct base points, any eight of them determines the ninth one, hence the pencil. If the base points are real, exactly eight of these singular cubics are distinguished, that is to say real with a loop containing some base points. We call combinatorial cubic a topological type (cubic, base points), and combinatorial pencil the sequence of eight successive combinatorial distinguished cubics. We choose representants of the 49 orbits. In most cases, but four exceptions, a list determines a unique combinatorial pencil. We give a complete classification of the combinatorial pencils of cubics with eight base points in convex position.

1 Pencils of cubics

1.1 Preliminaries

Given nine generic points in $\mathbb{C}P^2$, there exists one single cubic passing through them. Given eight generic points in $\mathbb{C}P^2$, there exists a one-parameter family of cubics passing through them. We will call such a family a *pencil of cubics*. Let F_0 and F_1 be two cubics of a pencil $\mathcal{P}$. They intersect at a ninth point. As the other cubics of $\mathcal{P}$ are linear combinations of F_0 and F_1, they all pass through this ninth point. We call these nine points the *base points* of $\mathcal{P}$.If eight of the base points are real, the pencil is real, and hence the ninth base point is also real. A pencil of cubics is a line in the space $\mathbb{C}P^9$ of complex cubics. Let Δ be the discriminantal hypersurface of $\mathbb{C}P^9$, formed by the singular cubics. The hypersurface Δ is of degree 12. Hence, a generic pencil of cubics intersects Δ transversally at 12 regular points. Otherwise stated, a generic pencil $\mathcal{P}$ has exactly 12 singular (nodal) cubics. A non-generic pencil will be called *singular* *pencil*. Let $\mathcal{P}$ be a real pencil with nine real base points and denote the real part of $\mathcal{P}$ by $\mathbb{R}\mathcal{P}$. Let $n \leq 12$ be the number of real singular cubics of $\mathcal{P}$. Let C_3 be one of these cubics. The double point P of C_3 is *isolated* if the tangents to C_3 at P are non-real, otherwise P is *non-isolated*. If P is non-isolated, $C_3 \setminus P = \mathcal{J} \cup \mathcal{O}$, where $[\mathcal{J} \cup P] \neq 0$ and $[\mathcal{O} \cup P] = 0$ in $H_1(\mathbb{R}P^2)$. We say that $\mathcal{O}$ is the loop and $\mathcal{J}$ is the odd component of C_3. Notice that the loop $\mathcal{O}$ is convex. The estimation of n presented hereafter is due to V.Kharlamov, see [1].

One has: $n = n_1 + n_2 + n_3$, where n_1 is the number of cubics with an isolated double point, n_2 is the number of cubics with a loop containing no base points, and n_3 is the number of cubics with a loop containing some base points.To evaluate n, one recalculates the Euler characteristic of $\mathbb{R}P^2$, fibering $\mathbb{R}P^2$ with the cubics of $\mathcal{P}$. Each isolated double point, and each base point contributes by $+1$; and each non-isolated double point contributes by -1, so that one gets:

$$1 = \chi(\mathbb{R}P^2) = 9 + n_1 - (n - n_1)$$

So, $n - 2n_1 = 8$. Thus, $n = 8, 10$ or 12 and correspondingly, $n_1 = 0, 1$ or 2.

Consider a motion in $\mathbb{R}\mathcal{P}$ starting from a cubic with an isolated double point. If we choose the direction of the motion properly, an oval appears, grows, and attaches itself to the odd component, forming a loop that contains no base point. Conversely, starting from a cubic with a loop containing no base point, one can move in the pencil so that there appears a cubic with an oval. As this oval lies inside of the loop, it shrinks when one moves further, and degenerates into an isolated double point. Thus, $n_2 = n_1$ and $n_3 = 8$ independently of n. Let us call the eight cubics of the third type the *distinguished cubics* of $\mathcal{P}$. We shall picture $\mathbb{R}\mathcal{P}$ by a circle, divided in eight portions by the eight distinguished cubics. Let us remark that this number $n_3 = 8$ is the Welschinger invariant W_3, see [7]. The number of real rational plane curves of degree d going through $3d - 1$ generic points of $\mathbb{R}P^2$ is always finite. Let c_1 be the number of such curves with an even number of isolated nodes, and c_2 be the number of such curves with an odd number of isolated nodes. Welschinger proved that the difference $W_d = c_1 - c_2$ does not depend on the choice of the $3d - 1$ points.

Pencils of cubics were applied in [4] to solve an interpolation problem, and in [2], [3] to study the isotopy types realizable by some real algebraic curves in $\mathbb{R}P^2$.

1.2 Singular pencils

Let $\mathcal{P} = \mathcal{P}(1, \ldots 8)$ be a real pencil of cubics, determined by eight generic points on $\mathbb{R}P^2$, say $1, \ldots 8$. Move these points till $\mathcal{P}$ degenerates into a singular pencil $\mathcal{P}_{sing}$. The degeneration is generic if and only if $\mathcal{P}_{sing}$ intersects Δ transversally at 10 regular points and

1. $\mathcal{P}_{sing}$ is tangent to Δ at one regular point, or
2. $\mathcal{P}_{sing}$ crosses transversally a stratum of codimension 1 of Δ

In both cases, two singular cubics C_3^1 and C_3^2 of $\mathcal{P}$ come together to yield one singular cubic C_3 of $\mathcal{P}_{sing}$. The cubic C_3 is necessarily real; the cubics C_3^1 and C_3^2 are either both real (case a), or complex conjugated (case b). Notice that, given a generic cubic F of Δ with node in some point p, one has:$T_F\Delta = \{F + G | G(p) = 0\}$. Therefore, $\mathcal{P}_{sing}$ satisfies the condition 1) if and only if $\mathcal{P}_{sing}$ has a double base point, at p. Otherwise stated, $\mathcal{P}_{sing}$ is obtained from $\mathcal{P}$ by letting two base points A and B of $\mathcal{P}$ come together. Move A towards B along the line (AB). For simplicity, we assume that B is the point $(0, 0)$ of the plane, and the direction is the x-axis. The condition that some cubic H passes through A and B becomes at the limit: $H(0) = 0$ and $\frac{\partial H}{\partial x}(0) = 0$. So a cubic of $\mathcal{P}_{sing}$ is either singular at $A = B$ or tangent to the prescribed direction at $A = B$, depending on whether the other partial derivative $\frac{\partial H}{\partial y}$ vanishes or not at $(0, 0)$. In case 2), the cubic C_3 must be reducible (product of a line and a conic), or have a cusp.If C_3 is reducible, the genericity imposes that three of the base points lie on the line, and the other six lie on the conic. Notice that if C_3 has a cusp, the cubics C_3^1 and C_3^2 are either complex conjugated, or non-distinguished real cubics, one with a loop, the other with an isolated double point.

Consider the generic degenerations of the form:

1. Two base points come together, say A and B, or

2. Three base points come onto a line, or equivalently the other six base points come onto a conic.

They split into four subcases 1a), 1b), 2a) and 2b), see Figure 1. In the right-hand part of the figure, the dotted crosses symbolize the non-real nodes of the cubics C_3^1 and C_3^2. Case 1a) deserves a supplementary explanation. Let us call *elementary arc* AB an arc connecting A to B and containing no other base point. As $\mathcal{P}$ can degenerate into $\mathcal{P}_{sing}$ letting A and B come together, some cubics of $\mathcal{P}$ must have an elementary arc AB. Start from such a cubic and move in any direction in $\mathbb{R}\mathcal{P}$. At some moment, the mobile arc AB must glue to another arc, and then disappear. Thus $\mathcal{P}$ has two distinguished cubics corresponding to the openings of the arc AB. We call *singular elementary arc* AB the non-smooth arc AB of either of these cubics. In case 1a), these two cubics are C_3^1 and C_3^2, they come together to yield the cubic C_3, which has a non-isolated double point at $A = B$. All three combinatorial cubics C_3, C_3^1 and C_3^2 are identical outside of a neighbourhood of $A \cup B$.

Let 9 be the ninth base point of $\mathcal{P}$, and C_3 be any cubic of $\mathcal{P}$. We call *combinatorial cubic* C_3, the topological type of $(C_3, 1, \dots, 9)$. We call *combinatorial pencil* $\mathcal{P}$ the list of the successive eight combinatorial distinguished cubics of $\mathcal{P}$. A combinatorial pencil undergoes a (generic) degeneration only if one of the cases 1a) 2a) takes place. Thus, when considering combinatorial pencils, we forget the strata of Δ formed by cuspidal cubics.

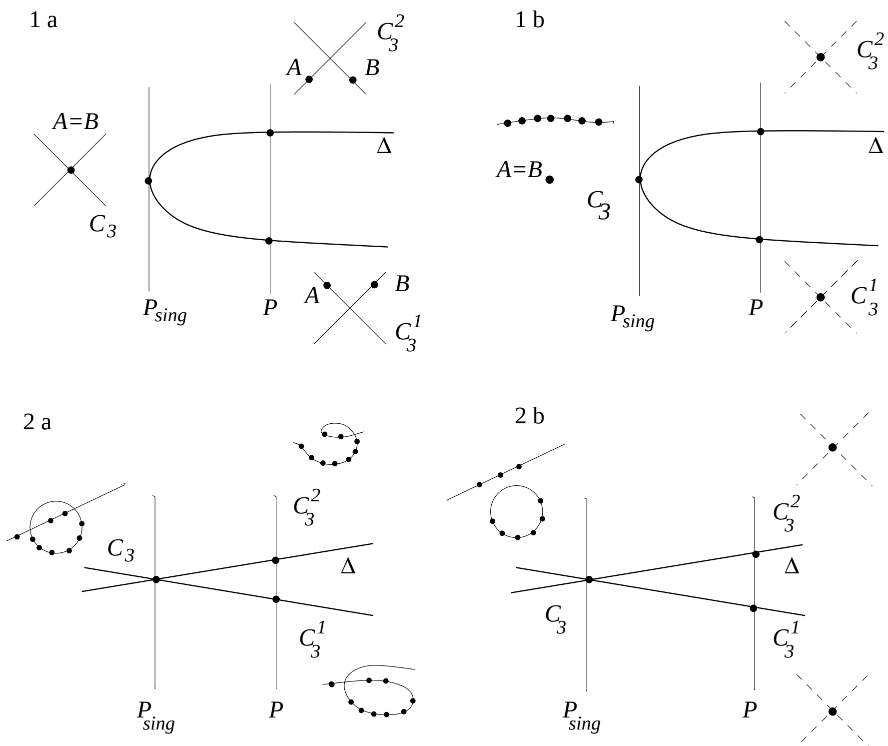

Figure 1: The degeneration of $\mathcal{P}$ into $\mathcal{P}_{sing}$

2 Configurations of 8 points lying in convex position in $\mathbb{R}P^2$

2.1 Mutual position of points and conics

We say that n generic points lie *in convex position in* $\mathbb{R}P^2$ if there exists a line $\Delta \subset \mathbb{R}P^2$ such that the n points lie in convex position in the affine plane $\mathbb{R}P^2 \setminus \Delta$. Notice that any set of $n \leq 5$ points in $\mathbb{R}P^2$ lie in convex position. Consider $n \geq 5$ generic points lying in strictly convex position in $\mathbb{R}P^2$, say $1, \dots, n$. Let $L(1, \dots, n)$ be the list of the C_n^5 conics through five of these points, enhanced for each conic, with the position of each of the remaining $n-5$ points (inside or outside).

> *How many different possibilities can be realized by $L(1, \dots, n)$ when one lets the points $1, \dots, n$ move?*

For $n = 6$, $L(1, \dots, 6)$ is determined e.g. by the position of the point 6 with respect to the conic 12345. (Note that the six conics of the list intersect pairwise at four points among $1, \dots, 6$).Thus, the number of possibilities realized by the lists $L(1, \dots, 6)$ is 2, see Figure 2. We write shortly: $\natural L(1, \dots, 6) = 2$. Consider now a configuration of seven points $1, \dots, 7$. There are two possibilities for $L(1, \dots, 6)$. For either of them, 7 may be placed in seven different ways with respect to the set of conics passing through five points among $1, \dots, 6$. One checks easily that the data $L(1, \dots, 6)$, *position of* 7 determines the list $L(1, \dots, 7)$. Thus, $\natural L(1, \dots, 7) = 14$. We may reprove this another way round: let C_3 be a cubic passing through the points $1, \dots, 7$. By abuse of language we will also call *cubic* C_3 the topological type of $(C_3, 1, \dots, 7)$. Let F be the equation of C_3 and p be a point among $1, \dots, 7$. The condition $F(p) = 0$ is a linear equation in the coefficients of F. If p is a singular point of C_3, one gets two supplementary linear equations: $\frac{\partial F}{\partial x}(p) = 0$ and $\frac{\partial F}{\partial y}(p) = 0$.Thus,as $1, \dots, 7$ are generic, there exists exactly one real nodal cubic passing through $1, \dots, 7$ and having p as double point. Let $S(1, \dots, 7)$ be the list of the seven nodal cubics passing through the points $1, \dots, 7$, one of them being the double point.

Proposition 1 *The list $S(1, \dots, 7)$ can realize fourteen possibilities, denoted by $1\pm, 2\pm, \dots, 7\pm$.*

This proposition is proved in [4]. The lists $1+$ and $1-$ are shown in Figure 3; the other lists $n\pm$ are obtained from $1\pm$ performing on $1, \dots, 7$ the cyclic permutation that replaces 1 by n. (The double point of the last cubic in each list may be an isolated node or a crossing.) Note that any of the five non-extreme cubics of the list $S(1, \dots, 7)$ determines the whole of this list.

In what follows $k < C_2$ means *k lies inside of the conic C_2*. Let us get back to our lists $L(1, \dots, n)$. For $n = 7$, let $C_2 = ijklm$ be one of the 21 conics in consideration, and let C_3 be a cubic from the list $S(1, \dots, 7)$, having its double point at one of the i, j, k, l, m. Applying Bezout's theorem between C_2 and C_3, one deduces the position of either of the remaining two points with respect

$L(1,\dots 6)$				
C_2	in	out	in	out
12345	6			6
12346		5	5	
12356	4			4
12456		3	3	
13456	2			2
23456		1	1	

Figure 2: The two lists $L(1,\dots 6)$

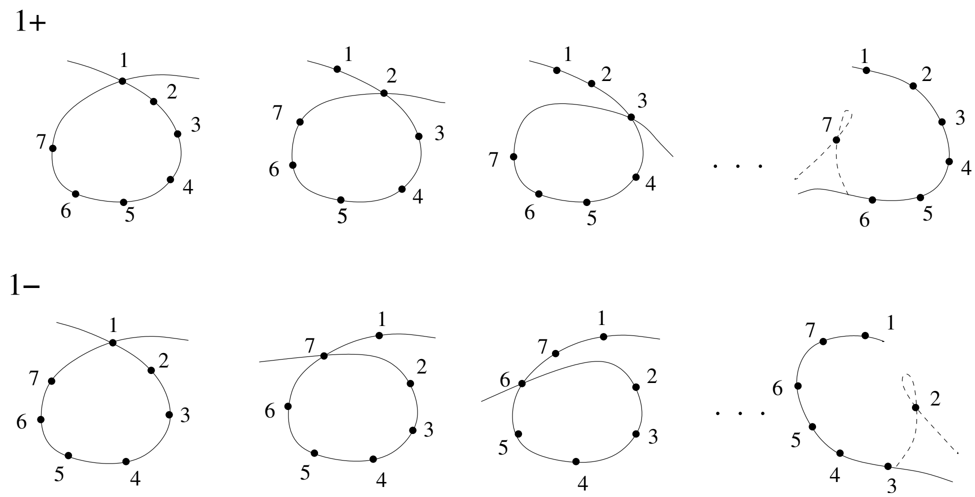

Figure 3: The lists 1+ and 1−

15	26	37	48	(+1)(15)	(+1)(26)	(+1)(37)	(+1)(48)
$1 \leftrightarrow 1$	$1 \leftrightarrow 3$	$1 \leftrightarrow 5$	$1 \leftrightarrow 7$	$1 \leftrightarrow 2$	$1 \leftrightarrow 4$	$1 \leftrightarrow 6$	$1 \leftrightarrow 8$
$2 \leftrightarrow 8$	$2 \leftrightarrow 2$	$2 \leftrightarrow 4$	$2 \leftrightarrow 6$	$3 \leftrightarrow 8$	$2 \leftrightarrow 3$	$2 \leftrightarrow 5$	$2 \leftrightarrow 7$
$3 \leftrightarrow 7$	$4 \leftrightarrow 8$	$3 \leftrightarrow 3$	$3 \leftrightarrow 5$	$4 \leftrightarrow 7$	$5 \leftrightarrow 8$	$3 \leftrightarrow 4$	$3 \leftrightarrow 6$
$4 \leftrightarrow 6$	$5 \leftrightarrow 7$	$6 \leftrightarrow 8$	$4 \leftrightarrow 4$	$5 \leftrightarrow 6$	$6 \leftrightarrow 7$	$7 \leftrightarrow 8$	$4 \leftrightarrow 5$
$5 \leftrightarrow 5$	$6 \leftrightarrow 6$	$7 \leftrightarrow 7$	$8 \leftrightarrow 8$				

Figure 4: Action of D_8

to C_2. So, $S(1,\dots,7)$ determines $L(1,\dots,7)$. It turns out that the 14 lists $S(1,\dots,7)$ give rise to distinct lists $L(1,\dots,7)$. Thus, the combinatorial data $L(1,\dots,7)$ and $S(1,\dots,7)$ are equivalent, and $\natural L(1,\dots,7) = 14$, see Figures 30-31 in section 6. (For convenience, we have gathered some of the auxiliary tabulars at the end.)

All of these lists are equivalent up to the action of the dihedral group D_7. Move the seven points keeping them in strictly convex position. Both lists $L(1,\dots,7)$ and $S(1,\dots,7)$ change if and only if six points cross one conic. For the non-generic configuration of points $1,\dots,7$ with six of them on a conic, we say that the lists are *non-generic*. Let A, B be two consecutive points in a configuration $1,\dots,7$ realizing a generic list, and let C_2 be the conic through the five other points. By definition, the *distance* $A \to B$ is 0 if A, B lie both inside or both outside of C_2; $+1$ if $A < C_2$ and $B > C_2$; -1 if $A > C_2$ and $B < C_2$. For the list $1+$, one has thus $1 \to 2 = \dots = 6 \to 7 = 0$, and $7 \to 1 = +1$. For $n = 8$, denote by $\hat{i}$ the list determined by all of the points but i. The data $L(1,\dots,8)$ is equivalent to $(\hat{1},\hat{2},\dots,\hat{8})$ with $\hat{i} = k\pm, k \in \{1,\dots,\hat{i},\dots 8\}$. Indeed, these data consist of the same set of elementary pieces of information *position of some point with respect to some conic*.

Consider the group D_8 of symmetries of the octagon, generated by the cyclic permutation $a = +1$ and the symmetry with respect to the axis $\sigma = 15$, see Figure 4.

$$D_8 = \{a, \sigma | a^8 = id, \sigma^2 = id, a\sigma = \sigma a^{-1}\}$$

Proposition 2 *Up to the action of* D_8, $\natural L(1,\dots,8) = 49$. *The total number of generic lists is* $784 = 49 \times 16$.

This proposition will be proved in two steps, in sections 2.2 (restriction part) and 4.1 (construction part).

2.2 Admissible lists

Let $1,\dots 8$ lie in convex position and $\{A,\dots G\} = \{1,\dots 7\}$. Move 8, leaving the other points fixed and preserving the convex position. Consider an event of the form 8 *crosses a conic* $ABCDE$, it induces a change of the list $\hat{G}$. The remaining point F may lie inside or outside of the conic $ABCDE$, depending on the list $L(1,\dots,7)$ (see upper and lower part of Figure 5). So there are: 21 choices for the conic $ABCDE$, two choices of G, and two possible positions of the last point F with respect to $ABCDE$. Hence in total 84 possibilities. Figure 5 gathers all of these possibilities, showing how the cubic of $\hat{G}$ with double point at C changes when 8 crosses $ABCDE$. This cubic determines the whole list $\hat{G}$. The point 8 (not represented) may be placed in six different ways on each cubic, according to the cyclic ordering of the set of points $\{8, A, B, C, D, E, F\}$. Assume for example that 8 is situated between A and B in the cyclic ordering. When 8 enters $ABCDE$, one has: $\hat{G} : E- \to A+$ (if F inside of $ABCDE$), and $\hat{G} : F- \to F+$ (if F outside of $ABCDE$). For each $G \in \{1,\dots,7\}$, we find two possible chains of degenerations of $\hat{G}$ while moving the point 8. The chain starting with $8\pm$ will be denoted by $\hat{G}\pm(8)$. For each $\hat{8}\in\{1\pm,\dots 7\pm\}$ and each $G\in\{1,\dots,7\}$ one watches which of the two possible chains is realized, see Figure 6. For each list $\hat{8}\in\{1\pm,2\pm,3\pm,4+\}$, we draw a diagram whose rows are the chains of degenerations of $\hat{1},\dots\hat{7}$. (We drop the other cases $\hat{8}\in\{7\pm,6\pm,5\pm,4-\}$, that can be deduced from the first ones by the action of D_8). Let C_2 and C_2' be two adjacent conics in a column, we add a vertical arrow from C_2 to C_2' if the following holds: 8 *outside of* C_2 *implies* 8 *outside of* C_2'. See Figures 32-38.

Chasing in the diagrams, we may find all of the admissible orbits, for the action of D_8, realizable by the lists $L(1,\dots 8)$. The explicit lists $L(1,\dots 8)$ obtained in this procedure are gathered in Figures 8-13. We denote these lists by $L_1,\dots L_{95}$, according to their appearance order. Note that we cannot completely rule out redundancies: we get sometimes several representants of the same orbit. So, we choose for each orbit one representant that we write with normal fonts, the equivalent lists are written in bold. The lists written in normal fonts will be called for convenience *principal lists*, even if their choice is not canonical.

Figures 8-9 show the 64 admissible lists with $\hat{8}=1+$. The first and the last are deduced one from the other by ± 1. The 15 lists with $\hat{5}=6+$ are mapped onto the 15 lists with $\hat{3}=4+$ by $+3$. The group of 6 lists with $\hat{2}=1-$ splits into two subgroups that are mapped one onto the other by the symmetry 15. The group of 20 lists with $\hat{4}=3-$ splits into two subgroups that are mapped one onto the other by 26. The group of 6 lists with $\hat{6}=5-$ splits into two subgroups that are mapped one onto the other by 37. Up to the action of D_8, there are 32 admissible lists with $\hat{8}=1+$. Set now $\hat{8}=1-$. First thing we rule out the orbit of $\hat{8}=1+$, that is to say any list which is mapped by some element of D_8 onto a list with $\hat{8}=1+$. In other words, we set: $\hat{1}\neq 2+,8-$, $\hat{2}\neq 3+,1-$, $\hat{3}\neq 4+,2-$, $\hat{4}\neq 5+,3-$, $\hat{5}\neq 6+,4-$, $\hat{6}\neq 7+,5-$ and $\hat{7}\neq 8+,6-$. Figure 10 shows the 6 new admissible lists obtained. They split into 2 groups that are mapped one onto the other by 15. For $\hat{8}=2+$ we rule out the orbits of $\hat{8}=1\pm$, that is we set: $\hat{1}\neq 2\pm,8\pm$, $\hat{2}\neq 3\pm,1\pm$, $\hat{3}\neq 4\pm,2\pm$, $\hat{4}\neq 5\pm,3\pm$, $\hat{5}\neq 6\pm,4\pm$, $\hat{6}\neq 7\pm,5\pm$ and $\hat{7}\neq 8\pm,6\pm$. Figure 11 shows the 4 new admissible lists with $\hat{8}=2+$. They split into 2 groups that are mapped one onto the other by 15. Set now $\hat{8}=2-$. Once we have ruled out the orbits of $\hat{8}=1\pm,2+$, we get the 13 new admissible lists shown in Figure 12. Some of them are deduced from each other by $\pm 2, 15$ or 26 so that there are only 8 new orbits. Let $\hat{8}=3+$, once we have ruled out the orbits of $\hat{8}=1\pm,2\pm$, we find no new admissible lists. For $\hat{8}=3-$, after excluding the orbits of $\hat{8}=1\pm,2\pm,3+$, we get the 8 new admissible lists shown in Figure 13. They split into 2 groups that are mapped one onto the other by ± 2. At last, let $\hat{8}=4+$, once we have excluded the orbits of $\hat{8}=1\pm,2\pm,3\pm$, we find no new admissible lists.

The total number of admissible orbits is 49. The 16 lists in each orbit are all distinct, hence there are in total 784 admisssible lists.

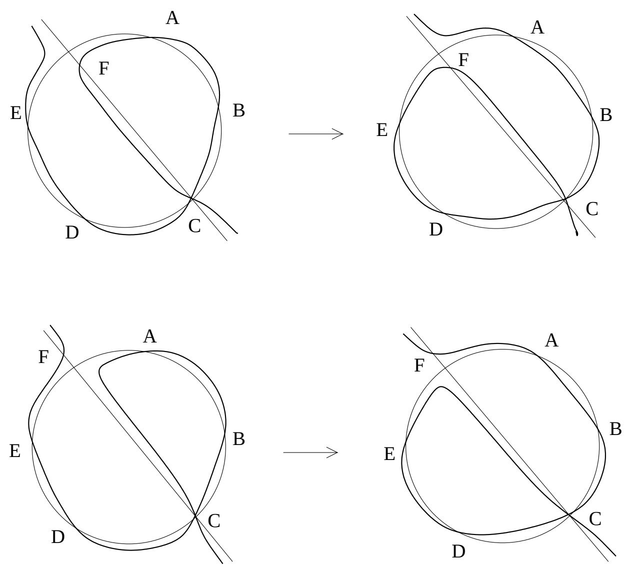

Figure 5: Degenerations of the list $\hat{G}$

	$\hat{8}$
$\hat{1}+$	$1+, 2+, 2-, 4+, 4-, 6+, 6-$
$\hat{1}-$	$1-, 3+, 3-, 5+, 5-, 7+, 7-$
$\hat{2}+$	$1+, 1-, 2-, 4+, 4-, 6+, 6-$
$\hat{2}-$	$2+, 3+, 3-, 5+, 5-, 7+, 7-$
$\hat{3}+$	$1+, 1-, 3+, 4+, 4-, 6+, 6-$
$\hat{3}-$	$2+, 2-, 3-, 5+, 5-, 7+, 7-$
$\hat{4}+$	$1+, 1-, 3+, 3-, 4-, 6+, 6-$
$\hat{4}-$	$2+, 2-, 4+, 5+, 5-, 7+, 7-$
$\hat{5}+$	$1+, 1-, 3+, 3-, 5+, 6+, 6-$
$\hat{5}-$	$2+, 2-, 4+, 4-, 5-, 7+, 7-$
$\hat{6}+$	$1+, 1-, 3+, 3-, 5+, 5-, 6-$
$\hat{6}-$	$2+, 2-, 4+, 4-, 6+, 7+, 7-$
$\hat{7}+$	$1+, 1-, 3+, 3-, 5+, 5-, 7+$
$\hat{7}-$	$2+, 2-, 4+, 4-, 6+, 6-, 7-$

Figure 6: Possible values of $\hat{8}$ for the chains $\hat{n}\pm = \hat{n} \pm (8), n = 1, \ldots, 7$

2.3 Extremal lists

Consider a configuration of eight points $1,\dots 8$ lying in convex position in the plane. Any piece of information $\hat{P} = n\pm$, with $P \in \{1,\dots 8\}$ is equivalent to a statement of the form $F < C_2$ *and* $G > C_2$, where F, G are two points among $1,\dots,8$, different from P. For example, $\hat{8} = 1+$ if and only if $7 < 23456$ and $1 > 23456$. The correspondences for $P = 8$ are signalled in bold in the tabulars of Figures 30-31, the other cases are easily deduced from these by the action of D_8. We say that a list is *maximal* (or *minimal*) for some $F \in 1,\dots,8$ if F lies outside (or inside) of all the conics determined by five of the other seven points. Say $F = 8$, then for each possible $\hat{8} = n\pm$, there is one maximal and one minimal list. The maximal list $\max(\hat{8} = n\pm)$ is obtained from the given diagram $\hat{8} = n\pm$ taking for all $\hat{m}$ with $m = 1,\dots 7$ the labels above the first column of horizontal arrows. The minimal list $\min(\hat{8} = n\pm)$ is obtained from the given diagram $\hat{8} = n\pm$ taking for all $\hat{m}$ with $m = 1,\dots 7$ the labels above the last column of horizontal arrows. These notations are consistant with D_8. The extremal lists are easily realizable as follows: start from a configuration of eight points lying on the same conic C_2. Move first 8, either to the interior or to the exterior of C_2. Then, move two further points F and G, one to the interior and the other to the exterior of C_2, so as to obtain the desired list $\hat{8}$.

The maximal lists are distributed in 7 orbits. Note that $\max(\hat{1} = n\pm)$ and $\max(\hat{1} = (10-n)\mp)$ are deduced one from the other by 15. We denote the orbits of these two lists shortly by $(n\pm, (10-n)\mp)$. It turns out that the maximal orbits $(2+, 8-)$ and $(3+, 7-)$ are also minimal: $\max(\hat{1} = 8-) = \min(\hat{2} = 1-)$ and $\max(\hat{1} = 3+) = \min(\hat{2} = 1+)$. The principal extremal lists are: $L_{64} = \max(\hat{1} = 2+) = \min(\hat{8} = 1+)$, $L_{65} = \max(\hat{1} = 7-) = \min(\hat{8} = 1-)$, $L_2 = \min(\hat{7} = 5-)$, $L_3 = \min(\hat{7} = 5+)$, $L_5 = \min(\hat{7} = 3-)$, $L_{71} = \min(\hat{1} = 4+)$, and $L_{72} = \min(\hat{1} = 4-)$.

2.4 Another encoding for the lists and the orbits

For the lists $L(1,\dots,8)$, we define the *distance* between consecutive points A, B in the cyclic ordering. Let $(A,\dots H) = (1,\dots 8)$ up to some cyclic permutation. The distance $A \to B$ is the number of conics passing through five points among C, D, E, F, G, H that separate A from B, multiplied by -1 if A is the outermost of the two points. If $\hat{A} = B\epsilon$, $\hat{B} = A\epsilon$ or $\hat{A} = \hat{B} = N\epsilon$, with $N \in \{C,\dots H\}$, then $A \to B = 0$. Let now $\hat{A} \neq \hat{B}$. We move A towards B, leaving the other points fixed and preserving the convex position. When A crosses a conic through five points among C, D, E, F, G, H, a pair of lists $\hat{B}, \hat{N}$, $N \in \{C, D, E, F, G, H\}$ changes. During the motion $A \to B$, the list $\hat{B}$ percourses (in one direction or the other) a piece of one of the complete chains $\hat{B} \pm (A)$, see Figure 7.

The upper and the lower chain correspond respectively to the case $H < CDEFG$ and $H > CDEFG$. To determine the invariant $A \to B$, we proceed as in the following example: let $\hat{A} = G+$ and $\hat{B} = A+$. One has $H < CDEFG$, $CDEFH > B > CDEGH$ and $A > CDEFG$. Thus, $A \to B = -2$. The tabulars of Figures 14-15 give the values of these invariants for all of the 49

$$A+\xrightarrow{CDEFG}G-\xrightarrow{CDEFH}G+\xrightarrow{CDEGH}E-\xrightarrow{CDFGH}E+\xrightarrow{CEFGH}C-\xrightarrow{DEFGH}C+$$

$$A-\xrightarrow{DEFGH}D+\xrightarrow{CEFGH}D-\xrightarrow{CDFGH}F+\xrightarrow{CDEGH}F-\xrightarrow{CDEFH}H+\xrightarrow{CDEFG}H-$$

Figure 7: Degenerations $\hat{B}\pm(A)$ of $\hat{B}$ while moving A towards B

successive principal lists L_n, and their sum σ. Note that the 49 octuples $(1 \to 2, \dots, 8 \to 1)$ listed in Figure 14-15 are all different. Furthermore, their orbits under the action of D_8 are also all distinct. Observing the tabulars allows to derive the following:

Proposition 3 *Any list* $L(1, \dots, 8)$ *is determined by the octuple* $(1 \to 2, 2 \to 3, \dots, 8 \to 1)$*. Each orbit is determined by an octuple of integer numbers (ranging between* -6 *and* $+6$*), defined up to cyclic permutation, and reversion with change of all signs. The absolute value* $|\sigma|$ *is an invariant of the orbits, that takes all of the even values between* 0 *and* 8*. The numbers of orbits realizing* $|\sigma| = 0, 2, 4, 6, 8$ *are respectively:* 16*,* 12*,* 16*,* 4 *and* 1*.*

We have thus a new encoding for the lists and the orbits.

Denote by $[XY]$ and $[XY]'$ the segments XY, the former belonging to the convex hull of the eight points. Note that in the motion $A \to B$, the point A may also cross some conic C_2 determined by B and four other points among C, D, E, F, G, H. So, one does not control the changes of the whole list $L(1, \dots, 8)$. In particular, let C_2 be one of the conics $BDEFG$, $BDEFH$, $BDEGH$, $BDFGH$ or $BEFGH$. If A, C lie both outside of C_2, and the second intersection of C_2 with the line BC is a point of $[BC]'$, then any path $A \to B$ must cross C_2.

3 Link between lists and pencils

3.1 Isotopies of octuples of points

The octuples of points $(1, \dots 8)$ form a stratified space Σ whose walls are formed by the configurations of points such that six of them lie on a conic.

Proposition 4 *The octuples of points realizing a given list* $L(1, \dots 8)$ *lie all in the same chamber of* Σ*.*

Proof: Denote by h_t a homothety centered at any point P different from $1, \dots 8$, with rate t. The homotheties $h_t, t \in [1, \infty]$ give rise to an isotopy of $1, \dots 8$. During this isotopy, the list $L(1, \dots 8)$ is preserved, except for the end-point when $1, \dots 8$ are all on a line. They are also on some reducible conic. The configuration is thus on the deep stratum formed by

$\hat{1}$	**8**+	8+	8+	8+	8+	8+	8+	8+
$\hat{2}$	**8**+	8+	8+	8+	8+	8+	8+	8+
$\hat{3}$	**8**+	8+	8+	8+	8+	8+	8+	8+
$\hat{4}$	**8**+	8+	8+	8+	6−	6−	6+	3−
$\hat{5}$	**8**+	8+	6−	6+	6−	6+	6+	3−
$\hat{6}$	**8**+	5−	5−	5+	5−	5+	3−	3−
$\hat{7}$	**8**+	5−	5+	5+	3−	3−	3−	3−

$\hat{1}$	**8**+	8+	8+	8+	8+	8+	8+	**8**+
$\hat{2}$	**8**+	8+	8+	8+	8+	8+	8+	**8**+
$\hat{3}$	**6**−	6−	6−	6−	6+	6+	4−	**4**+
$\hat{4}$	**6**−	6−	6+	3−	6+	3−	3−	**3**+
$\hat{5}$	**6**−	6+	6+	3−	6+	3−	3+	**3**+
$\hat{6}$	**5**−	5+	3−	3−	3+	3+	3+	**3**+
$\hat{7}$	**3**+	3+	3+	3+	3+	3+	3+	**3**+

$\hat{1}$	**8**+	8+	8+	8+	8+	8+	8+	**8**+
$\hat{2}$	**6**−	6−	6−	6−	6−	6−	6−	**6**−
$\hat{3}$	**6**−	6−	6−	6−	6+	6+	4−	**4**+
$\hat{4}$	**6**−	6−	6+	3−	6+	3−	3−	**3**+
$\hat{5}$	**6**−	6+	6+	3−	6+	3−	3+	**3**+
$\hat{6}$	**5**−	5+	3−	3−	3+	3+	3+	**3**+
$\hat{7}$	**1**−	1−	1−	1−	1−	1−	1−	**1**−

$\hat{1}$	8+	8+	**8**+	**8**+	**8**+	**8**+	**8**+	8+
$\hat{2}$	6+	6+	**6**+	**6**+	**4**−	**4**−	**4**+	1−
$\hat{3}$	6+	6+	**4**−	**4**+	**4**−	**4**+	**4**+	1−
$\hat{4}$	6+	3−	**3**−	**3**+	**3**−	**3**+	**1**−	1−
$\hat{5}$	6+	3−	**3**+	**3**+	**1**−	**1**−	**1**−	1−
$\hat{6}$	1−	1−	**1**−	**1**−	**1**−	**1**−	**1**−	1−
$\hat{7}$	1−	1−	**1**−	**1**−	**1**−	**1**−	**1**−	1−

Figure 8: $\hat{8} = 1+$, lists $L_1, \ldots, L_{32}$

$\hat{1}$	**6**−	6−	6−	6−	6−	6−	**6**−	**6**−
$\hat{2}$	**6**−	6−	6−	6−	6−	6−	**6**−	**6**−
$\hat{3}$	**6**−	6−	6−	6−	6+	6+	**4**−	**4**+
$\hat{4}$	**6**−	6−	6+	3−	6+	3−	**3**−	**3**+
$\hat{5}$	**6**−	6+	6+	3−	6+	3−	**3**+	**3**+
$\hat{6}$	**5**−	5+	3−	3−	3+	3+	**3**+	**3**+
$\hat{7}$	**1**+	1+	1+	1+	1+	1+	**1**+	**1**+

$\hat{1}$	6−	**6**−	**6**−	**6**−	**6**−	**6**−	**6**−	6−
$\hat{2}$	6+	**6**+	**6**+	**6**+	**4**−	**4**−	**4**+	1−
$\hat{3}$	6+	**6**+	**4**−	**4**+	**4**−	**4**+	**4**+	1−
$\hat{4}$	6+	**3**−	**3**−	**3**+	**3**−	**3**+	**1**−	1−
$\hat{5}$	6+	**3**−	**3**+	**3**+	**1**−	**1**−	**1**−	1−
$\hat{6}$	1−	**1**−	**1**−	**1**−	**1**−	**1**−	**1**−	1−
$\hat{7}$	1+	**1**+	**1**+	**1**+	**1**+	**1**+	**1**+	1+

$\hat{1}$	6+	**6**+	**6**+	**6**+	**6**+	**6**+	**6**+	6+
$\hat{2}$	6+	**6**+	**6**+	**6**+	**4**−	**4**−	**4**+	1−
$\hat{3}$	6+	**6**+	**4**−	**4**+	**4**−	**4**+	**4**+	1−
$\hat{4}$	6+	**3**−	**3**−	**3**+	**3**−	**3**+	**1**−	1−
$\hat{5}$	6+	**3**−	**3**+	**3**+	**1**−	**1**−	**1**−	1−
$\hat{6}$	1+	**1**+	**1**+	**1**+	**1**+	**1**+	**1**+	1+
$\hat{7}$	1+	**1**+	**1**+	**1**+	**1**+	**1**+	**1**+	1+

$\hat{1}$	**4**−	**4**−	**4**−	**4**−	**4**+	**4**+	**2**−	2+
$\hat{2}$	**4**−	**4**−	**4**+	**1**−	**4**+	**1**−	**1**−	1+
$\hat{3}$	**4**−	**4**+	**4**+	**1**−	**4**+	**1**−	**1**+	1+
$\hat{4}$	**3**−	**3**+	**1**−	**1**−	**1**+	**1**+	**1**+	1+
$\hat{5}$	**1**+	**1**+	**1**+	**1**+	**1**+	**1**+	**1**+	1+
$\hat{6}$	**1**+	**1**+	**1**+	**1**+	**1**+	**1**+	**1**+	1+
$\hat{7}$	**1**+	**1**+	**1**+	**1**+	**1**+	**1**+	**1**+	1+

Figure 9: $\hat{8} = 1+$, lists $L_{33}, \dots L_{64}$

$\hat{1}$	7−	7+	5−	**5**+	**3**−	**3**+
$\hat{2}$	1+	1+	1+	**1**+	**1**+	**1**+
$\hat{3}$	1+	1+	1+	**1**+	**1**+	**1**−
$\hat{4}$	1+	1+	1+	**1**+	**1**−	**1**−
$\hat{5}$	1+	1+	1+	**1**−	**1**−	**1**−
$\hat{6}$	1+	1+	1−	**1**−	**1**−	**1**−
$\hat{7}$	1+	1−	1−	**1**−	**1**−	**1**−

Figure 10: $\hat{8} = 1-$, lists $L_{65}, \dots, L_{70}$

$\hat{1}$	4+	4−	**6+**	**6−**
$\hat{2}$	8−	8−	**8−**	**8−**
$\hat{3}$	8−	8−	**8−**	**8−**
$\hat{4}$	2+	8−	**8−**	**8−**
$\hat{5}$	2+	2+	**8−**	**8−**
$\hat{6}$	2+	2+	**2+**	**8−**
$\hat{7}$	2+	2+	**2+**	**2+**

Figure 11: $\hat{8} = 2+$, lists $L_{71}, \ldots, L_{74}$

$\hat{1}$	4+	**4+**	**6−**	4+	**4+**	4−	**6+**
$\hat{2}$	8+	**4−**	**8+**	6−	**6+**	8+	**8+**
$\hat{3}$	8−	**8−**	**8−**	8−	**8−**	8−	**8−**
$\hat{4}$	2+	**2+**	**8−**	2+	**2+**	8−	**8−**
$\hat{5}$	2+	**2−**	**8−**	2+	**2+**	2+	**8−**
$\hat{6}$	2+	**2−**	**8−**	2+	**2−**	2+	**2+**
$\hat{7}$	2+	**2−**	**2+**	2−	**2−**	2+	**2+**

$\hat{1}$	6+	6+	4−	**6−**	4−	4−
$\hat{2}$	6−	6+	4−	**6−**	6−	6+
$\hat{3}$	8−	8−	8−	**8−**	8−	8−
$\hat{4}$	8−	8−	8−	**8−**	8−	8−
$\hat{5}$	8−	8−	2−	**8−**	2+	2+
$\hat{6}$	2+	2−	2−	**8−**	2+	2−
$\hat{7}$	2−	2−	2−	**2−**	2−	2−

Figure 12: $\hat{8} = 2-$, lists $L_{75}, \ldots, L_{87}$

$\hat{1}$	5+	**5+**	5−	**5+**	5−	**5−**	5+	**5−**
$\hat{2}$	7+	**5−**	7+	**5−**	7+	**5−**	7+	**5−**
$\hat{3}$	7−	**7+**	7−	**7−**	7+	**7−**	7+	**7+**
$\hat{4}$	1+	**1+**	1+	**1+**	1+	**1+**	1+	**1+**
$\hat{5}$	1−	**1−**	1+	**1−**	1+	**1+**	1−	**1+**
$\hat{6}$	1−	**3+**	1−	**3+**	1−	**3+**	1−	**3+**
$\hat{7}$	3+	**3−**	3+	**3+**	3−	**3+**	3−	**3−**

Figure 13: $\hat{8} = 3-$, lists $L_{88}, \ldots, L_{95}$

n	$1 \to 2$	$2 \to 3$	$3 \to 4$	$4 \to 5$	$5 \to 6$	$6 \to 7$	$7 \to 8$	$8 \to 1$	σ
2	0	0	0	0	-1	0	5	0	4
3	0	0	0	1	0	-1	4	0	4
4	0	0	0	2	0	0	4	0	6
5	0	0	-1	0	0	-2	3	0	0
6	0	0	-1	1	0	-1	3	0	2
7	0	0	-2	0	1	0	3	0	2
8	0	0	-3	0	0	0	3	0	0
10	0	1	0	1	0	-2	2	0	2
11	0	1	-1	0	1	-1	2	0	2
12	0	1	-2	0	0	-1	2	0	0
13	0	2	0	0	2	0	2	0	6
14	0	2	-1	0	1	0	2	0	4
15	0	3	0	-1	0	0	2	0	4
18	-1	0	0	1	0	-3	1	0	-2
19	-1	0	-1	0	1	-2	1	0	-2
20	-1	0	-2	0	0	-2	1	0	-4
21	-1	1	0	0	2	-1	1	0	2
22	-1	1	-1	0	1	-1	1	0	0
23	-1	2	0	-1	0	-1	1	0	0
25	-2	0	0	0	3	0	1	0	2
26	-2	0	-1	0	2	0	1	0	0
32	-5	0	0	0	0	0	1	0	-4
34	0	0	0	1	0	-4	0	1	-2
35	0	0	-1	0	1	-3	0	1	-2
36	0	0	-2	0	0	-3	0	1	-4
37	0	1	0	0	2	-2	0	1	2
38	0	1	-1	0	1	-2	0	1	0
41	-1	0	0	0	3	-1	0	1	2
48	-4	0	0	0	0	-1	0	1	-4
49	0	0	0	0	4	0	0	2	6
56	-3	0	0	0	1	0	0	2	0
64	0	0	0	0	0	0	0	6	6

Figure 14: Distances $A \to B$ for the principal lists with $\hat{8} = 1+$

n	$1\to 2$	$2\to 3$	$3\to 4$	$4\to 5$	$5\to 6$	$6\to 7$	$7\to 8$	$8\to 1$	σ
65	-1	0	0	0	0	0	-1	6	4
66	-2	0	0	0	0	1	0	5	4
67	-3	0	0	0	-1	0	0	4	0
71	5	0	1	0	0	0	0	-2	4
72	4	0	0	-1	0	0	0	-3	0
75	4	-1	1	0	0	0	1	-1	4
78	3	-2	1	0	0	-1	0	-1	0
80	3	-1	0	-1	0	0	1	-2	0
82	1	-2	0	0	1	-1	0	-3	-4
83	0	-3	0	0	2	0	0	-3	-4
84	0	-4	0	-2	0	0	0	-2	-8
86	2	-2	0	-1	0	-1	0	-2	-4
87	1	-3	0	-1	1	0	0	-2	-4
88	-2	1	-1	1	0	1	-1	1	0
90	-1	1	-1	0	-1	1	-1	2	0
92	-1	0	-2	0	-1	2	0	2	0
94	-2	0	-2	1	0	2	0	1	0

Figure 15: Distances $A \to B$ for the principal lists with $\hat{8} \neq 1+$

the configurations of eight coconic points. Move in this stratum so that the conic becomes non-reducible. With a slight perturbation, we can make the isotopy generic, in other words, it is completely contained in the same chamber except for the end-point. Consider now two configurations realizing the same list. Perform for each of them a generic isotopy till all of the eight points lie on the same conic. Up to an affine transformation mapping one conic onto the other and an isotopy along this conic, we may assume that both isotopies have the same end-point. So we have a path connecting the two configurations and having one single non-generic point. Perturb the path in a neighbourhood of this point. In this neighbourhood, all of the walls intersect pairwise transversally. There are two possible perturbations, one crosses all of the walls twice, the other crosses no wall at all. □

3.2 Nodal lists

All along this section, $1, \ldots, 8$ are eight generic points lying in strictly convex position in $\mathbb{R}P^2$, and $\mathcal{P} = \mathcal{P}(1, \ldots, 8)$ is the pencil of cubics determined by $1, \ldots, 8$. Given two points X, Y among $1, \ldots, 8$, we denote by $(\hat{X}, Y)$ the cubic of the list $\hat{X}$ with double point at Y. Move the eight points, keeping them distinct and strictly convex. The pencil $\mathcal{P}$ degenerates into a generic singular pencil $\mathcal{P}_{sing}$ if and only if:

1. the point 9 comes together with one of the points $1, \ldots, 8$, or

2. six of the points $1, \ldots, 8$ come onto a conic.

Note that in case 1, $\mathcal{P}_{sing}$ contains a nodal cubic with node at one of the points $1, \ldots, 8$, this point being also the ninth base point of the pencil.

Definition 1 *The list* $L(1, \ldots, 8) = (\hat{1}, \hat{2}, \ldots, \hat{8})$ *is* nodal *if it is realizable by* $1, \ldots, 8$ *on a nodal cubic with double point at one of these points.*

Let C_3 be a cubic through $1, \ldots 8$ with node at one of these points. Up to the action of D_8, there are eight possible combinatorial cubics C_3, see Figure 16 where the successive cubics are denoted by $(1\pm, 1)_{nod}$, $(1-, k)_{nod}$ for $k = 8, \ldots 2$. Note that the encoding is consistant with the action of D_8.

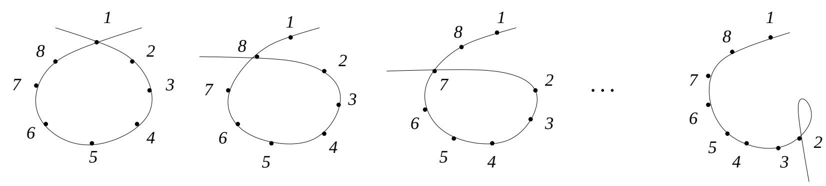

Figure 16: The cubics $(1\pm, 1)_{nod}$, $(1-, 8)_{nod}, \ldots (1-, 2)_{nod}$

Proposition 5 *Up to the action of the group* D_8, $\natural L(1, \ldots, 8)_{nodal} = 4$. *The nodal orbits are the maximal orbits* $(8-, 2+)$, $(8+, 2-)$, $(6-, 4+)$, $(6+, 4-)$. *Here are representants of each orbit along with the corresponding nodal cubics.*
$L(1, \ldots, 8) = (\hat{1}, \hat{2}, \ldots, \hat{8}) =$
$\max(\hat{1} = 8-)$, *realizable with* $(1\pm, 1)_{nod}$, $(1-, k)_{nod}, k = 2, \ldots 8$,
$\max(\hat{1} = 8+)$, *realizable with* $(1\pm, 1)_{nod}$ *and* $(1-, 8)_{nod}$,
$\max(\hat{1} = 6-)$ *and* $\max(\hat{1} = 6+)$, *both realizable with* $(1\pm, 1)_{nod}$.

Proof: Let $1, \ldots, 8$ lie in convex position on some nodal cubic C_3, one of these points being the node. Up to the action of D_8, we may assume that C_3 is one of the cubics $(1\pm, 1)_{nod}$, $(1-, k)_{nod}, k = 2 \ldots 8$. If $C_3 = (1\pm, 1)_{nod}$, one has $\hat{i} \in \{1\pm\}$ for all $i \in \{2, \ldots, 8\}$. Using the two diagrams $\hat{8} = 1\pm$ of Figures 32-33, one finds 14 possibilities for the list $L(1, \ldots, 8)$, namely the maximal lists $\max(\hat{1} = n\pm)$, $n = 2, \ldots, 8$. One has $\hat{1} = 8+ \iff 7 < 23456$ and $8 > 23456$. One can choose the points $2, \ldots, 8$ on the loop of C_3 so that this condition is achieved. One has $\hat{1} = 7- \iff 8 < 23456$ and $7 > 23456$. By Bezout's theorem between 23456 and $(1\pm, 1)_{nod}$, one cannot choose the points $1, \ldots, 8$ on the loop of this cubic verifying this condition. Finishing this argument with the other possible values of $\hat{1}$, one finds that $1, \ldots, 8$ may be chosen on $(1\pm, 1)_{nod}$ so as to realize the eight lists $\max(\hat{1} = n\pm)$, with $n = 2, 4, 6, 8$.

Similarly, one proves that $1, \ldots, 8$ on the cubic $C_3 = (1-, 8)_{nod}$ can realize exactly the first two lists $\max(\hat{1} = 8\pm)$; and points $1, \ldots, 8$ on any cubic $C_3 = (1-, k)_{nod}, k \in \{2, \ldots, 7\}$ must realize the first list $\max(\hat{1} = 8-)$. $\Box$

Proposition 6 *The three conditions hereafter are equivalent:*

1. *The list* $L(1,\dots,8)$ *determines the (combinatorial) pencil* $\mathcal{P}(1,\dots,8)$*,*
2. *The list* $L(1,\dots,8)$ *is not nodal,*
3. $\forall G \in \{1,\dots,8\}, \forall C_3$ *cubic of* $\hat{G}$*, the position of* G *with respect to* C_3 *is determined by* $L(1,\dots,8)$*.*

Proof: $\neg 2 \Rightarrow \neg 1$: let $L(1,\dots,8)$ be a nodal list. Up to the action of D_8, we may assume it is one of the four lists in Proposition 4. Any one of these lists is realizable with the eight points on a nodal cubic $C_3 = (1\pm,1)_{nod}$. The points $1,\dots,8$ give rise to a singular pencil $\mathcal{P}_{sing}$ with $1 = 9$. Perturb the pencil moving 1 away from the node onto the odd component or onto the loop of C_3 (leaving the other seven points fixed). The generic pencil obtained is deduced from $\mathcal{P}_{sing}$ replacing $(C_3, 1, \dots 9)$ by a pair C_3^1, C_3^2. One gets two pairs of distinguished cubics, see Figure 17.

$2 \Rightarrow 3$: according to Proposition 4, any two configurations of points realizing the same list may be connected by a path inside of their common chamber of Σ. As $L(1,\dots,8)$ is not nodal, none of the points G may cross any cubic of the list $\hat{G}$.

$3 \Rightarrow 1$: consider a configuration of points realizing a list $L(1,\dots 8)$ and giving rise to some pencil $\mathcal{P}$. Move the configuration preserving the list, the combinatorial pencil degenerates only if some base point A among $1,\dots 8$ comes together with 9. But this amounts to say that each point $G \neq A \in \{1,\dots,8\}$ comes onto the cubic $(\hat{G}, A)$. $\Box$

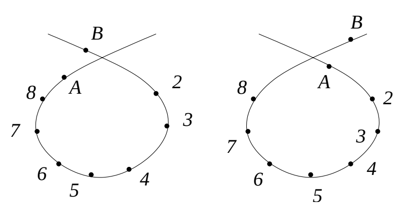

Figure 17: Pairs of distinguished cubics, $(A,B) = (1,9)$ or $(9,1)$

3.3 Pairs of distinguished cubics

Let now $1,\dots,8$ realize any list $L(1,\dots,8)$. Up to some isotopy replacing the initial configuration $1,\dots,8$ by another one in the same chamber and closer to the deep stratum (configurations of eight coconic points) , one may achieve the following condition: for any pair of consecutive points A, B, any one of the 15 conics passing through B but not A, has its second intersection point with the line AB on the segment $[AB]$. As an immediate consequence, we get the following

Proposition 7 *Let $1, \dots, 8$ lie in convex position. Let A, B be two of these eight points, consecutive for the cyclic ordering.*

1. *If $A \to B = 0$, one may move A towards B till $A = B$, without degeneration of the list $L(1, \dots 8)$ inbetween.*
2. *If $A \to B = \pm n$, with $n > 0$, the points A and B are separated from each other by n of the 6 conics determined by C, D, E, F, G, H. One may move A towards B, till $A = B$, in such a way that the list $L(1, \dots, 8)$ undergoes exactly n degenerations, corresponding to the n conics.*

Furthermore, if the initial list $L(1, \dots, 8)$ is non-nodal, then the combinatorial pencil does not degenerate till $A = B$ (case 1) or till A reaches the first of the n conics (case 2).

In what follows, we call cubic C_3 a combinatorial cubic $(C_3, 1, \dots 8)$ (the position of 9 is not yet specified). We define an encoding for distinguished cubics, that is consistant with the action of D_8. Denote by $(81, L)$ and $(81, C)$ the first and the second cubic in Figure 18. The letters L and C stand respectively for *loop* and *odd component*.

Let $(1-, N), N = 3, \dots 8$ be the cubic obtained from $(1-, N)_{nod}$, shifting N away from the node onto the loop, and let $(N, 1-), N = 3, \dots 7$ be the cubic obtained from $(1-, N)_{nod}$, shifting N away from the node onto the odd component. For $(P, Q) = (3, 4), (4, 5), \dots, (8, 1)$, denote by $(1-, PQ)$, the cubic obtained from $(1-, P)_{nod}$ (or $(1-, Q)_{nod}$) shifting the node away from P (or Q) into the interior of the arc PQ. Let $(1-, E)$ (E stands for *end*) be the cubic that could be defined as $(1-, N)$, with $N = 2$ or as $(1-, PQ)$, with $(P, Q) = (2, 3)$. We chose the specific notation to avoid double notation for one single cubic type. The cubics encoded hereabove are called *cubics of the family* $1-$. In section 5 ahead where we classify the pencils, we will consider combinatorial cubics $(C_3, 1, \dots 9)$. To encode them, we use the notation defined here for $(C_3, 1, \dots, 8)$, enhanced with the position of 9. The cubic $C_3 = (C_3, 1, \dots, 8)$ is divided into 10 successive arcs by the points $1, \dots 8, X$ (X is the node). If $C_3 = (81, L)$ or $(81, C)$, then 9 lies on the arc XX. If C_3 is of the family $N\epsilon$, percourse C_3 starting from N in the direction ϵ, and write which oriented arc contains 9 (see Figures 27-29).

Denote by $\mathcal{P}$ the combinatorial pencil of cubics determined by $1, \dots, 8$. Move 8 towards 1 till the pencil degenerates, see Proposition 7. The only other base point of the pencil that moves is 9. Let $\mathcal{P}_{sing}$ be the singular pencil obtained, let C_3, C_3^1 and C_3^2 be the three singular cubics involved (see section 1.2). If the list $L(1, \dots, 8)$ is non-nodal, we may recover pairs of distinguished cubics of $\mathcal{P}$ from the list as explained hereafter.

1. If $8 \to 1 = 0$, the singular cubic C_3 (with double point at $8 = 1$) is identical to both $(\hat{8}, 1)$ and $(\hat{1}, 8)$. If $\hat{8} = \hat{1} = 2+$ or $7-$, one does not know a priori whether the double points of these two auxiliary cubics are isolated or not. Therefore, the cubics C_3^1, C_3^2 may be non-real. If $\hat{8} = \hat{1} \neq 2+$ or

7−, the cubics $(\hat{8}, 1)$ and $(\hat{1}, 8)$ have each a non-isolated double point. All along the motion $8 \to 1$,the position of 1 with respect to $(\hat{1}, 8)$ and the position of 8 with respect to $(\hat{8}, 1)$ are preserved. Using Bezout's theorem with these auxiliary cubics, one finds out the correponding pair C_3^1, C_3^2. In all of the cases, any one of the four data: cubic $(\hat{1}, 8)$ enhanced with the position of 1, cubic $(\hat{8}, 1)$ enhanced with the position of 8, cubic C_3^1, and cubic C_3^2 determines the other three. Either case $\hat{8} = 1+, \hat{1} = 8+$ and $\hat{8} = 1-, \hat{1} = 8-$ gives rise to three pairs C_3^1, C_3^2 (these three pairs, along with the corresponding auxiliary cubics, are shown in the upper part of Figure 18).

$$(81, L), (81, C)$$
$$(8+, 1), (8-, 78)$$
$$(1+, 12), (1-, 8)$$

Each of the other cases gives rise to two pairs C_3^1, C_3^2, see the tabular hereafter, where N ranges from 3 to 6. Note that all of these cubics C_3^1, C_3^2 are distinguished. The case $\hat{8} = \hat{1} = 7+$ is shown in the lower part of Figure 18.

$\hat{8} = \hat{1} = 7+$	$\hat{8} = \hat{1} = 2-$	$\hat{8} = \hat{1} = N+$	$\hat{8} = \hat{1} = N-$
$(7+, 8), (1, 7+)$	$(2-, 8), (12, C)$	$(N+, 1), (8, N+)$	$(N-, 8), (1, N-)$
$(78, C), (7+, 1)$	$(2-, 1), (8, 2-)$	$(N+, 8), (1, N+)$	$(N-, 1), (8, N-)$

2. If $8 \to 1 \neq 0$, one has $C_3 = C_2 \cup (1P), P \in \{2, \dots, 7\}$. If: $P = 2$ and both 1 and 2 are outside of $C_2 = 34567$, then one does not know a priori whether the double points of the reducible cubic C_3 are real or complex conjugated. Therefore, the cubics C_3^1, C_3^2 may be non-real. The points $1, 2$ lie both outside of 34567 if and only if $\hat{8} \in \{1+, 2-, 4\pm, 6\pm\}$. The supplementary condition $8 < 34567$ is achieved if and only if $\hat{1} = 2+$. So we rule out the case $\hat{8} \in \{1+, 2-, 4\pm, 6\pm\}$ and $\hat{1} = 2+$. For the other cases, find out C_2 looking up the sequence $\hat{1} \pm (8)$. Either cubic C_3^1, C_3^2 may be obtained from the reducible cubic C_3 by perturbing one of the double points of C_3. All of the pairs obtained are distinguished cubics. Indeed, let s be the intersection of the line $(1P)$ with the interior of C_2, and let s_1, s_2 be the two arcs of C_2 on either side of $(1P)$. The loop of each cubic C_3^1, C_3^2 is obtained perturbing one of the $s_i \cup s, i \in \{1, 2\}$, and both $s_i \cup s$ contain some points among $1, \dots, 8$. We give an example in Figure 19, with $\hat{1} = 5+, \hat{5} = 8-$, the corresponding pair of distinguished cubics is $(5, 8-), (5+, 1)$. Note that $\hat{1} = 5+$ is part of the chain $\hat{1}-$ and $\hat{5} = 8-$ is part of the chain $\hat{5}-$. Using Figure 6, we deduce that $\hat{8} \in \{5-, 7\pm\}$.

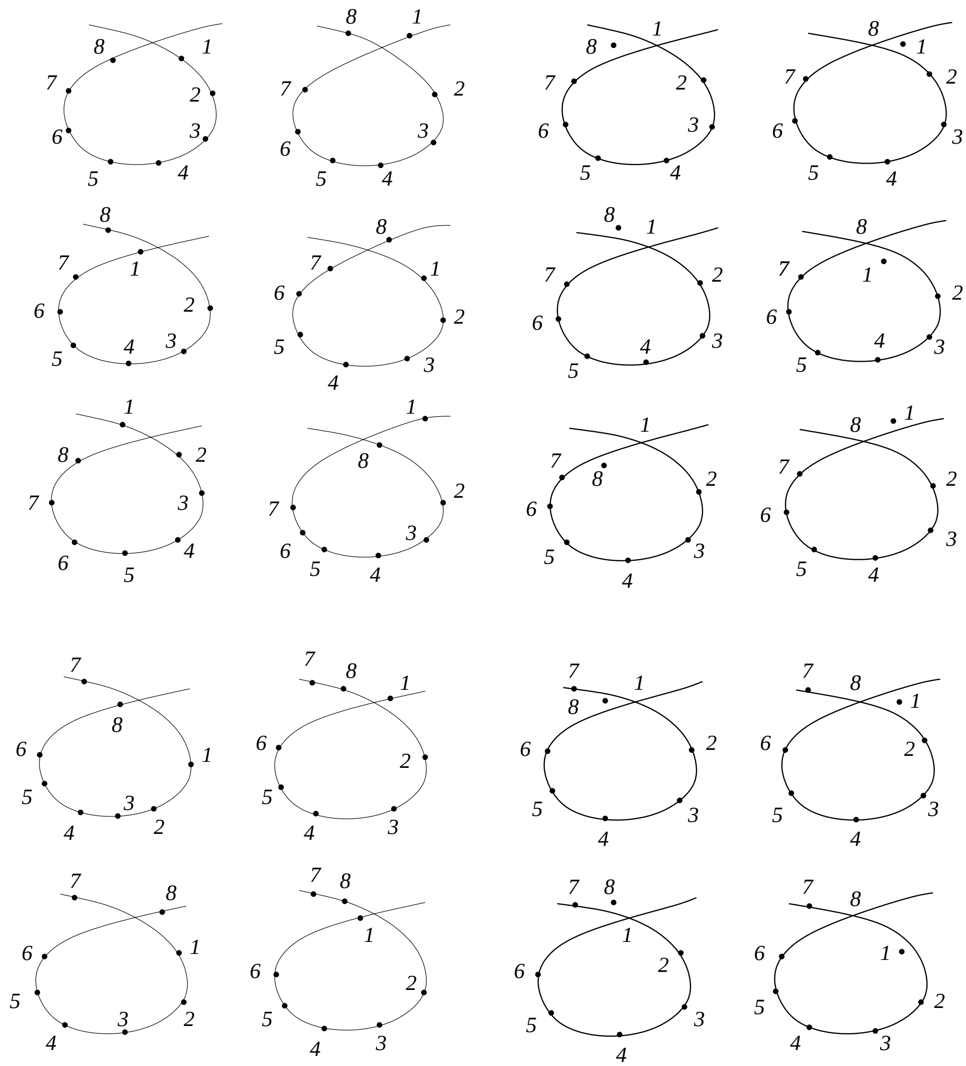

Figure 18: Two examples: $\hat{8} = 1\pm, \hat{1} = 8\pm$, and $\hat{8} = \hat{1} = 7+$

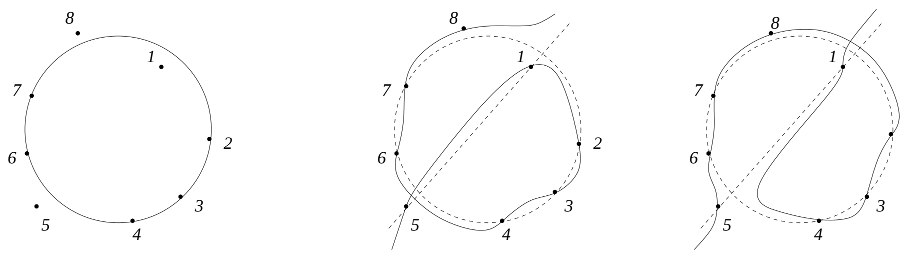

Figure 19: $\hat{1} = 5+, \hat{5} = 8-, \hat{8} \in \{5-, 7\pm\}$

4 Constructions

4.1 Elementary changes and inductive constructions

In this section, we finish the proof of Proposition 2. Let us call *elementary change* the change induced on a list letting one point among $1, \dots, 8$ cross a conic determined by five others, in some direction. Up to the action of D_8, there are 19 possible such changes, see Figures 20-21. The first one occurs when 8 goes to the outside of the conic 34567, and $1, 2$ lie both outside of 34567. If the conic 34567 has no real intersection points with the line 12, the combinatorial pencil does not degenerates. Otherwise, one recovers the same set of eight distinguished cubics after the degeneration. The changes of pairs of distinguished cubics induced by the other elementary changes are shown in Figure 22. Recall that a statement of the form 6 *crosses* 12345 *from the inside to the outside* is equivalent to 5 *crosses* 12346 *from the outside to the inside*. All of the 19 elementary pairs but one may be thus interpreted as motions of a point X towards a consecutive point Y such that $X \to Y \neq 0$, till X reaches the first conic separating it from Y. The only exception is the third change:

$$\hat{1} : 8- \to 3+ \qquad \hat{2} : 1- \to 1+$$

By Proposition 7, if a list $L(1, \dots, 8)$ is realizable, then any list that may be obtained from $L(1, \dots, 8)$ by an elementary change (different from the particular one hereabove) is also realizable. We say that the elementary change is *always possible*. Consider now a list with $\hat{1} = 8-$, $\hat{2} = 1-$. Note that $\hat{1} = 8-$ belongs to the chain $\hat{1} - (8)$, and $\hat{2} = 1-$ belongs to the chain $\hat{2} + (8)$. Using Figure 6, we deduce that $\hat{8} = 1-$; using Figure 33, we see that $\hat{N} = 1-$ for $N = 3, \dots 7$. The list is $\max(\hat{1} = 8-)$. The point 8 lies inside of all the conics determined by 1 and four points among $2, \dots 7$. Thus one may let 8 move towards the conic 34567 and cross it without degenerations inbetween. This elementary change is also always possible. The total number of elementary changes is 224, see Figures 39-40. Start from a list that is already realized, we will say that we perform the change $(\hat{N}, \hat{M})$ without further precision, as there is no ambiguity possible.

Any principal list may be obtained from an extremal list by some sequence of elementary changes and actions of elements of D_8. See Figure 23 where n

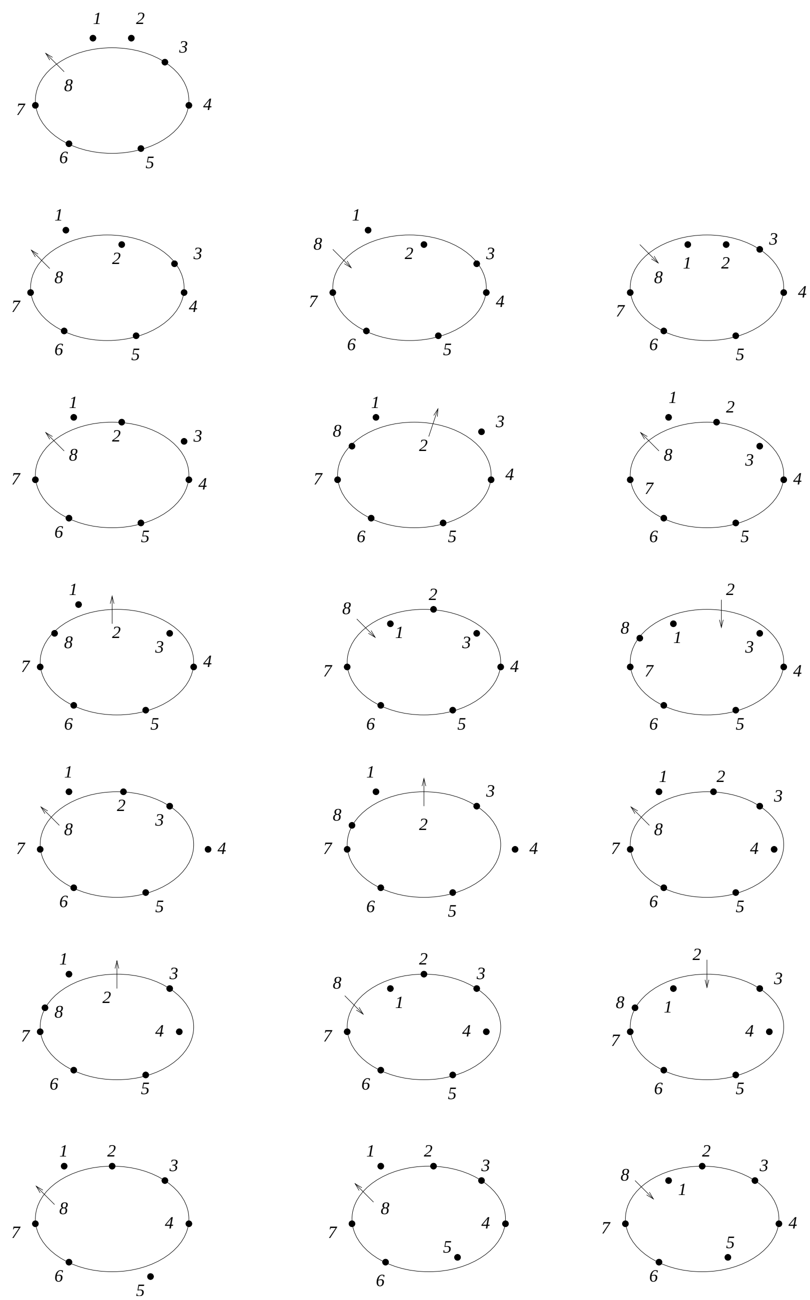

Figure 20: Elementary changes

$\hat{1} : 2+ \to 2-$		
$\hat{2} : 1+ \to 1-$		
$\hat{1} : 3+ \to 8-$	$\hat{1} : 8- \to 3+$	$\hat{1} : 8- \to 3+$
$\hat{2} : 1+ \to 1-$	$\hat{2} : 1- \to 1+$	$\hat{2} : 8- \to 3+$
$\hat{1} : 3- \to 3+$	$\hat{1} : 3+ \to 3-$	$\hat{1} : 2- \to 4+$
$\hat{3} : 1+ \to 1-$	$\hat{3} : 1- \to 1+$	$\hat{3} : 1+ \to 1-$
$\hat{1} : 4+ \to 2-$	$\hat{1} : 4+ \to 2-$	$\hat{1} : 2- \to 4+$
$\hat{3} : 1- \to 1+$	$\hat{3} : 8- \to 2+$	$\hat{3} : 2+ \to 8-$
$\hat{1} : 4+ \to 4-$	$\hat{1} : 4- \to 4+$	$\hat{1} : 5+ \to 3-$
$\hat{4} : 1+ \to 1-$	$\hat{4} : 1- \to 1+$	$\hat{4} : 1+ \to 1-$
$\hat{1} : 3- \to 5+$	$\hat{1} : 3- \to 5+$	$\hat{1} : 5+ \to 3-$
$\hat{4} : 1- \to 1+$	$\hat{4} : 8- \to 2+$	$\hat{4} : 2+ \to 8-$
$\hat{1} : 5- \to 5+$	$\hat{1} : 4- \to 6+$	$\hat{1} : 6+ \to 4-$
$\hat{5} : 1+ \to 1-$	$\hat{5} : 1+ \to 1-$	$\hat{5} : 8- \to 2+$

Figure 21: Representants of the 19 orbits of elementary changes

$(1+, 2) \to (1-, E)$	$(1-, E) \to (1+, 2)$	$(8-, 2) \to (23, L)$
$(1, 3+) \to (1+, 12)$	$(1+, 12) \to (1, 3+)$	$(81, L) \to (3+, 1)$
$(3, 1+) \to (3, 1-)$	$(3, 1-) \to (3, 1+)$	$(1+, 3) \to (1-, 3)$
$(1, 3-) \to (1, 3+)$	$(1, 3+) \to (1, 3-)$	$(12, C) \to (1, 4+)$
$(1-, 3) \to (1+, 3)$	$(8-, 3) \to (2+, 3)$	$(2+, 3) \to (8-, 3)$
$(1, 4+) \to (12, C)$	$(4+, 1) \to (2-, 1)$	$(2-, 1) \to (4+, 1)$
$(4, 1+) \to (4, 1-)$	$(4, 1-) \to (4, 1+)$	$(1+, 4) \to (1-, 4)$
$(1, 4+) \to (1, 4-)$	$(1, 4-) \to (1, 4+)$	$(1, 5+) \to (1, 3-)$
$(1-, 4) \to (1+, 4)$	$(8-, 4) \to (2+, 4)$	$(2+, 4) \to (8-, 4)$
$(1, 3-) \to (1, 5+)$	$(3-, 1) \to (5+, 1)$	$(5+, 1) \to (3-, 1)$
$(5, 1+) \to (5, 1-)$	$(1+, 5) \to (1-, 5)$	$(8-, 5) \to (2+, 5)$
$(1, 5-) \to (1, 5+)$	$(1, 4-) \to (1, 6+)$	$(6+, 1) \to (4-, 1)$

Figure 22: Changes of pairs of distinguished cubics

1	$(\hat{6},\hat{7})$	2	$(\hat{5},\hat{7})$	3	$(\hat{5},\hat{6})$	4	$(\hat{4},\hat{7})$	6	$(\hat{4},\hat{6})$	7	$(\hat{4},\hat{5})$	8
6	$(\hat{5},\hat{6})$	5	(37)	**9**								
9	$(\hat{5},\hat{6})$	10	$(\hat{4},\hat{6})$	11	$(\hat{4},\hat{5})$	12	$(\hat{3},\hat{6})$	14	$(\hat{4},\hat{5})$	13		
14	$(\hat{3},\hat{5})$	15										
3	(37)	**17**	$(\hat{5},\hat{6})$	18	$(\hat{4},\hat{6})$	19	$(\hat{4},\hat{5})$	20	$(\hat{3},\hat{6})$	22	$(\hat{3},\hat{5})$	23
22	$(\hat{4},\hat{5})$	21	$(\hat{2},\hat{6})$	25	$(\hat{4},\hat{5})$	26						
2	(37)	**33**	$(\hat{5},\hat{6})$	34	$(\hat{4},\hat{6})$	35	$(\hat{4},\hat{5})$	36	$(\hat{3},\hat{6})$	38		
35	$(\hat{3},\hat{6})$	37	$(\hat{2},\hat{6})$	41	$(\hat{1},\hat{6})$	49						
65	$(\hat{1},\hat{7})$	66	$(\hat{1},\hat{6})$	67								
71	$(\hat{1},\hat{4})$	72										
71	$(\hat{2},\hat{8})$	75	$(\hat{2},\hat{7})$	78								
75	$(\hat{1},\hat{4})$	80	$(\hat{2},\hat{7})$	86	$(\hat{2},\hat{6})$	87	$(\hat{2},\hat{5})$	84				
87	$(\hat{1},\hat{5})$	83	$(\hat{2},\hat{6})$	82								

Figure 23: Inductive construction of the principal lists

stands for a list L_n (as in Figures 14-15) and each row is a new sequence whose first list was already realized. These sequences are chosen so as to reach all of the principal lists with the least possible number of starting lists (note that some intermediate lists are also extremal).

This finishes the proof of Proposition 2. □

4.2 Lists obtained perturbing four reducible cubics

Among the principal lists, exactly twelve have a set of four distinct elementary pairs. The lists L_n with $n = 11, 12, 13, 14, 19, 20, 21, 22$, have each distinct elementary pairs $(\hat{1},\hat{8})$, $(\hat{2},\hat{7})$, $(\hat{3},\hat{6})$, $(\hat{4},\hat{5})$. Each of the four lists L_n with $n = 88, 90, 92, 94$ has distinct elementary pairs $(\hat{1},\hat{5})$, $(\hat{2},\hat{6})$, $(\hat{3},\hat{7})$, $(\hat{4},\hat{8})$. We explain hereafter how to realize these lists directly.

Consider a pair of ellipses intersecting at four points, $1, 3, 5, 7$, see Figure 24. Denote by 9 the intersection of the diagonal lines 15 and 37. Draw a vertical and a horizontal line, both passing through 9. Let $4, 8, 2, 6$ be four supplementary points chosen such as: $4, 8$ are the intersections of the vertical line with one ellipse; $2, 6$ are the intersections of the horizontal line with the other ellipse; each pair of points lying on one ellipse is in the interior of the other. By construction, the points $1, \ldots 8$ lie in convex position. Moreover, there exist two supplementary conics passing through six points: 234678 and 124568. As a matter of fact, the pencil of cubics $\mathcal{P}$ determined by $1, \ldots, 8$ has 9 as ninth base point and four distinguished cubics, all of them reducible: $498 \cup 123567$, $296 \cup 134578$, $397 \cup 124568$ and $195 \cup 234678$. Let us perturb the pencil, moving slightly the points 8 and 9. As $1, 2, 3, 5, 6, 7$ are on a conic, $8, 4$ and 9 must stay aligned. The point 8 leaves the three conics $23467, 12456, 13457$. Letting 8 cross each conic from the inside to the outside yields the following elementary moves: Conic 23467: $(\hat{1} = 5-, \hat{5} = 1+) \to (\hat{1} = 5+, \hat{5} = 1-)$. Conic 12456: $(\hat{3} = 7-, \hat{7} = 3+) \to (\hat{3} = 7+, \hat{7} = 3-)$. Conic 13457: $(\hat{2} = 7+, \hat{6} = 1-) \to$

$(\hat{2} = 5-, \hat{6} = 3+)$. Let us now move 3 away from the conic 12567. Letting 3 cross 12567 from the inside to the outside yields the following elementary move: $(\hat{4} = 7-, \hat{8} = 5+) \to (\hat{4} = 1+, \hat{8} = 3-)$. The first perturbation may be done so as to realize six different positions of 8 with respect to the set of conics 23467, 12456, 13457. Then, move 3 to the outside of 12567. We obtain the first five lists, and the last list of Figure 13, four of these lists are: L_{88}, L_{90}, L_{92} and $(+2) \cdot L_{94}$.

Consider a pair of ellipses intersecting at four points $2, 4, 5, 7$, see Figure 25. Denote by 9 the intersection of the lines 27 and 45. Draw a line Δ passing through 9 and cutting the ellipses on their arcs 57 and 24. Let 6 and 3 be the intersections of Δ with one ellipse, chosen so that these points lie outside of the second ellipse. Draw a line Δ' passing through 9 and cutting the second ellipse at two points $8, 1$ on the arc 72. One may choose Δ' in such a way that the points $1, \dots 8$ lie in convex position. By construction, there exist two supplementary conics passing through six of the points: 123678 and 134568. The pencil of cubics $\mathcal{P}$ determined by $1, \dots, 8$ has 9 as ninth base point and four distinguished cubics, all of them reducible: $189 \cup 234567$, $369 \cup 124578$, $459 \cup 123678$ and $279 \cup 134568$. Let us perturb the pencil, moving the points 8 and 9. The point 8 leaves the three conics 13456, 12457, 12367. Letting 8 cross each conic from the inside to the outside yields the following elementary moves: Conic 13456: $(\hat{2} = 6-, \hat{7} = 1-) \to (\hat{2} = 8+, \hat{7} = 3+)$. Conic 12457: $(\hat{3} = 6+, \hat{6} = 3+) \to (\hat{3} = 6-, \hat{6} = 3-)$. Conic 12367: $(\hat{4} = 3-, \hat{5} = 3-) \to (\hat{4} = 6+, \hat{5} = 6+)$. Let us now move 7 away from the conic 23456. Letting 7 cross 23456 from the inside to the outside yields the following elementary move: $(\hat{1} = 8+, \hat{8} = 1+) \to (\hat{1} = 8-, \hat{8} = 1-)$. The first perturbation may be done so as to realize six different positions of 8 with respect to the set of conics 13456, 12457, 12367. Then, move 7 to the inside of 23456. We obtain the lists $L_{11}, L_{12}, L_{14}, L_{19}, L_{21}, L_{22}$. Starting again from the pencil with 4 reducible cubics we realize the two missing lists as follows. List L_{13}: move first 7 to the outside of 12458 and the inside of 12368; then, move 8 to the outside of 13456. List L_{20}: move first 6 to the outside of 23457 and the inside of 12378; then move 8 to the outside of 12457.

Proposition 8 *Let $1, \dots 8$ be eight points lying in convex position in the plane and let k be the number of conics passing through exactly 6 of them. One has $k \leq 4$. If $k = 4$, then the points realize, up to the action of D_8, one of the two non-generic lists shown in Figures 24-25. The orbit of the first list has two elements, the orbit of the second list has 8 elements.*

Proof: Perturbing slightly the configuration $1, \dots 8$ must yield a generic list with four distinct elementary pairs, otherwise stated a list that is in the orbit of some of the 12 lists L_n considered hereabove. Up to the action of D_8, the original configuration $1, \dots 8$ must be as shown in Figure 24 or 25. □

Let us finish this section with a few remarks. The list l of Figure 24 is encoded by the data: $2, 6 < 134578$, $4, 8 < 123567$, $1, 5 > 234678$, $3, 7 > 124568$. This

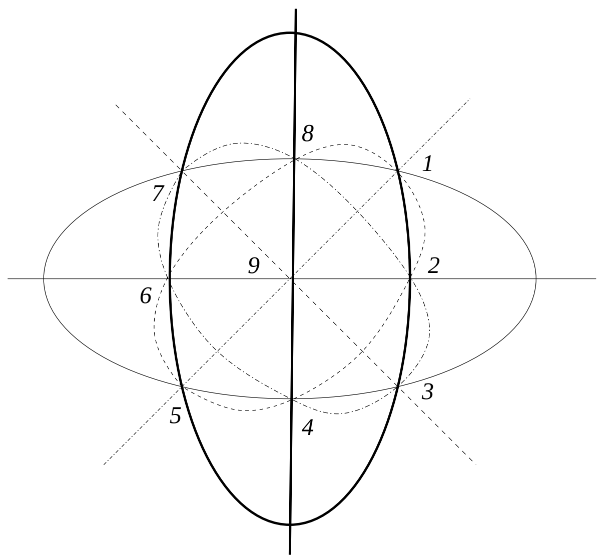

Figure 24: First configuration of points with 4 reducible cubics

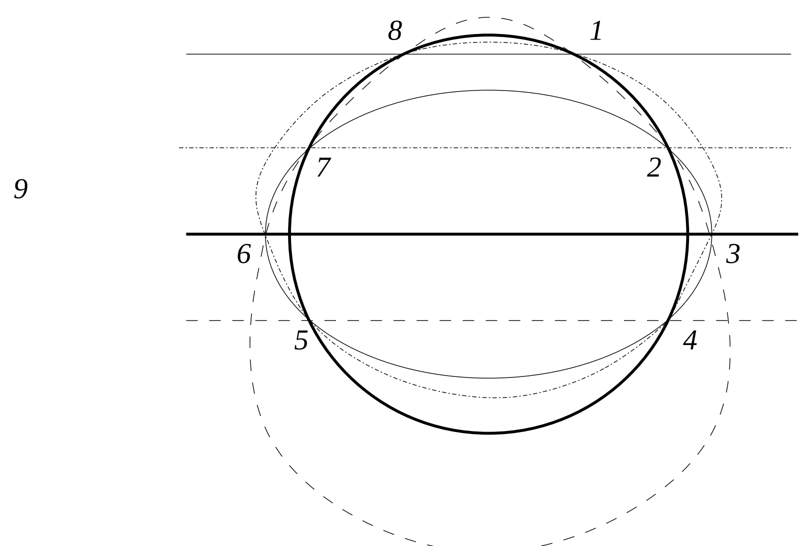

Figure 25: Second configuration of points with 4 reducible cubics

list is invariant by the action of id, 15, 26, 37, 48, $+2$, -2, $+4$. The list l' of Figure 25 is encoded by the data: $1,8 > 234567$, $5,4 < 123678$, $2,7 < 134568$, $3,6 > 124578$. This list is invariant by the action of $(+1)(48)$. The element $(+1)(48)$ preserves only two lists: l' and $(+4) \cdot l' = (+1)(26) \cdot l'$, whereas the element 15 leaves many lists unchanged. The lists invariant by 15 must contain a conic 234678. Let us say that a list is *almost generic* if it contains only one configuration of 6 coconic points. There are in total 24 almost generic lists invariant by 15, distributed in two groups that are deduced from one another by $+4$ (or 37).

A pencil of cubics with eight base points lying in convex position in the real plane (no 7 of them being coconic) has at most four reducible cubics. If we drop the condition of convexity, we can find easily configurations of base points realizing the maximum number of six reducible cubics. Let us say that a cubic is *completely reducible* if it is the product of three lines. A complex pencil contains at most four completely reducible cubics [5], [6], this upper bound is realized by the *Hessian pencils*. Recall that the nine inflection points of a complex cubic C_3 are disposed on a configuration of twelve lines such that: each point lies on four lines and each line passes through three points. This set of twelve lines splits into a union of four completely reducible cubics, belonging to the pencil of cubics generated by C_3 and its Hessian (this pencil has the nine inflection points of C_3 as base points).

5 Classification of the pencils of cubics

Denote by $\natural\mathcal{P}(1,\dots,8)$ the number of generic combinatorial pencils of cubics with eight base points in convex position in the plane.

Proposition 9 *Up to the action of D_8, $\natural\mathcal{P}(1,\dots,8) = 45$*

Proof: Let us call *nodal pencil* a pencil corresponding to a nodal list. Up to the action of D_8, there are four nodal lists, see Proposition 5. It follows from this proposition that the first list gives rise to nine pencils, that are deduced from each others switching 9 with the other base points. The second list gives rise to three pencils obtained from each others by the swaps $9 \leftrightarrow 1$, $9 \leftrightarrow 8$; each of the last two lists gives rise to two pencils obtained one from another by the swap $9 \leftrightarrow 1$.

Let T be the triangle $(12), (67), (17)$ containing 8. The condition that $1, \dots, 8$ realizes the list $\max(\hat{1} = 8-)$ splits into eight disjoint subconditions. There is an ordering $14567 > (\hat{8}, 1) > (\hat{8}, 7) > (\hat{8}, 6) > \dots > (\hat{8}, 2) > 34567$ in T: the point 8 lies between two consecutive of these curves. When 8 lies on a cubic $(\hat{8}, N)$, otherwise stated, when the eight chosen base points are on a cubic $(1-, N)_{nod}$, the pencil is singular, with $9 = N$. By Bezout's theorem with the cubics $(\hat{8}, N)$, the degeneration $9 = 8$ may occur only if 8 lies between $(\hat{8}, 1)$ and $(\hat{8}, 7)$. Using the method exposed in section 3.3, one gets thus the nine pencils of Figure 26 with

$(G, H, I, A, B, C, D, E, F) = (9, 2, 3, 4, 5, 6, 7, 8, 1)$, $(1, 2, 3, 4, 5, 6, 7, 8, 9)$, $(1, 2, 3, 4, 5, 6, 7, 9, 8)$, $(1, 2, 3, 4, 5, 6, 9, 7, 8)$, $(1, 2, 3, 4, 5, 9, 6, 7, 8)$, $(1, 2, 3, 4, 9, 5, 6, 7, 8)$, $(1, 2, 3, 9, 4, 5, 6, 7, 8)$, $(1, 2, 9, 3, 4, 5, 6, 7, 8)$, $(1, 9, 2, 3, 4, 5, 6, 7, 8)$.

If 8 lies outside of the loop of $(\hat{8}, 1)$, then one gets the pencil with $G = 9$. If 8 lies between $(\hat{8}, 1)$ and $(\hat{8}, 7)$, one gets the next two pencils with $(E, F) = (8, 9)$ and $(9, 8)$. The other positions of 8 give rise to the other pencils, switching successively 9 with $7, 6, \dots, 2$.

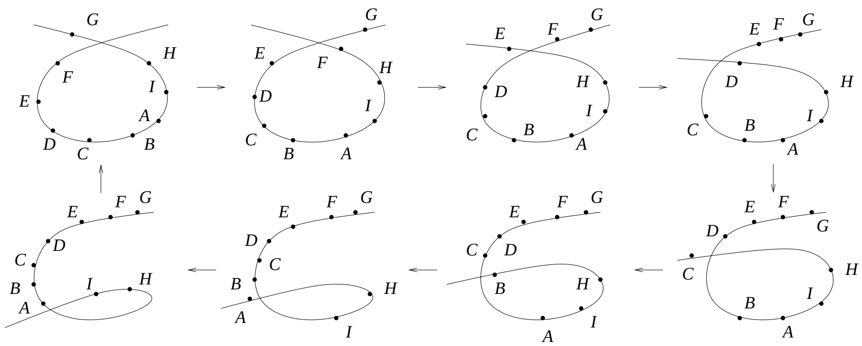

Figure 26: Nodal pencils obtained from the list $\max(\hat{1} = 8-)$

We say that an elementary change is possible for a *pencil* if one may move the base points till the change occurs, without degeneration of the pencil inbetween. The elementary change $\hat{1} : 8- \to 8+$, $\hat{8} : 1- \to 1+$ (7 enters the conic 23456, $1, 8$ outside of 23456) is possible only for the first three nodal pencils corresponding to $\max(\hat{1} = 8-)$, by Proposition 5. This elementary change leaves the sequence of distinguished cubics unchanged. The nodal list with $\hat{1} = 8+$ is thus realizable by the three pencils of Figure 26 with

$(G, H, I, A, B, C, D, E, F) = (9, 2, 3, 4, 5, 6, 7, 8, 1)$, $(1, 2, 3, 4, 5, 6, 7, 8, 9)$, $(1, 2, 3, 4, 5, 6, 7, 9, 8)$

The elementary change $\hat{1} : 8+ \to 6-$, $\hat{7} : 1- \to 1+$ (8 enters the conic 23456, 7 inside and 1 outside of 23456) is possible only for the first two pencils. For both, perform the change of pairs: $(1-, 7)$, $(81, C) \to (1+, 7), (1, 6-)$, one gets the two pencils corresponding to the nodal list $\max(\hat{1} = 6-)$. The elementary change $\hat{1} : 6- \to 6+$, $\hat{6} : 1- \to 1+$ (8 enters the conic 23457, $1, 6$ outside of 23457) may be performed on both previous pencils replacing the pair $(6, 1-), (1, 6-)$ by the pair $(6, 1+), (1, 6+)$, the new pencils obtained realize the nodal list $\max(\hat{1} = 6+)$. In Figures 27-29, we classify the pencils of cubics with eight base points lying in convex position. Each pencil is described as a sequence of eight successive combinatorial distinguished cubics $C_3(1, \dots, 9)$, encoded as explained in section 3.3. The upper nine pencils displayed in Figure 27 are the nodal pencils obtained from the list $\max(\hat{1} = 8-) = (15) \cdot L_{64}$. The first three of them correspond also to the list $\max(\hat{1} = 8+) = L_{32}$. The next two pencils after the blank line correspond to the list $\max(\hat{1} = 6-) = L_{48}$. The last two pencils after the second blank line correspond to the list $\max(\hat{1} = 6+) = L_{56}$.

Our next concern is to count the pencils of cubics up to the action of D_8 and construct representants of each equivalence class. Let us call *non-essential changes* the elementary changes that leave the pencils unchanged (the orbit of the first change in Figures 20-21). We have already established that the nodal lists give rise to 13 orbits of pencils, see Figure 27. Two lists obtained from one another by a non-essential change must be both nodal or both non-nodal. For any non-nodal list L_n, denote by $\mathcal{P}_n$ the corresponding pencil of cubics. Hereafter, $(\hat{5}, \hat{6})$ stand always for the same elementary change $(\hat{5} : 6- \to 6+, \hat{6} : 5- \to 5+)$; and $(\hat{1}, \hat{8})$ stands for $(\hat{1} : 8+ \to 8-, \hat{8} : 1+ \to 1-)$. One has:

$(\hat{5}, \hat{6}) \cdot L_3 = L_4$,
$26 \circ (+1)(48) \circ (\hat{1}, \hat{8}) \cdot L_4 = (+3) \circ (\hat{1}, \hat{8}) \cdot L_4 = L_{15}$,
$(\hat{5}, \hat{6}) \circ 37 \cdot L_3 = (\hat{5}, \hat{6}) \cdot L_{17} = L_{18}$.

Thus $\mathcal{P}_4 = \mathcal{P}_3$; $\mathcal{P}_{15} = (+3) \cdot \mathcal{P}_4$; $\mathcal{P}_{18} = 37 \cdot \mathcal{P}_3$. One has:

$(\hat{5}, \hat{6}) \cdot L_5 = L_6$,
$(\hat{5}, \hat{6}) \circ 37 \cdot L_5 = (\hat{5}, \hat{6}) \cdot L_9 = L_{10}$,
$26 \circ (+1)(48) \circ (\hat{1}, \hat{8}) \cdot L_6 = (+3) \circ (\hat{1}, \hat{8}) \cdot L_6 = L_{23}$.

Thus, $\mathcal{P}_6 = \mathcal{P}_5$; $\mathcal{P}_{10} = 37 \cdot \mathcal{P}_5$; $\mathcal{P}_{23} = (+3) \cdot \mathcal{P}_6$. One has $(\hat{5}, \hat{6}) \circ 37 \cdot L_2 = L_{34}$. Thus, $\mathcal{P}_{34} = 37 \cdot \mathcal{P}_2$. Finally, $(+1)(48) \circ (\hat{1}, \hat{8}) \cdot L_n = L_m$ for $(n, m) = (7, 26)$, $(8, 25)$, $(11, 22)$, $(12, 21)$, $(13, 20)$, $(14, 19)$. Thus $\mathcal{P}_m = (+1)(48) \cdot \mathcal{P}_n$. The non-nodal principal lists split into two subsets: 28 lists with $\hat{8} = 1+$ and 17 lists with $\hat{8} \neq 1+$. The first set gives rise to 15 orbits of pencils. In the second set, there is a one-to-one correspondence between the lists and the equivalence classes of pencils, see Figures 28-29. There are in total $13 + 15 + 17 = 45$ equivalence classes of pencils.

To construct the non-nodal pencils in the easiest way, we may follow the sequences of elementary changes from Figure 23, with four starting lists: L_2, L_{65}, L_{71}, L_{88}. Construct directly the starting pencils L_{65}, L_{71}, L_{88} using the method exposed in section 3.3. To get the starting pencil L_2 in the shortest way, observe that the list L_2 is obtained from the (non-principal and nodal) list L_1 by an elementary move $(\hat{6}, \hat{7})$. The list $L_1 = \max(\hat{8} = 1+) = (+1)(48) \max(\hat{1} = 8-)$ is realizable by nine pencils. However, the elementary change $(\hat{6}, \hat{7})$ is possible only for the first of them, shown in the first row of Figure 28. □

$(12,L)$ XX	$(81,L)$ XX	$(81,C)$ 18	$(1-,7)$ $X1$	$(6,1-)$ 61	$(1-,5)$ $X1$	$(4,1-)$ 41	$(1-,3)$ $X1$
$(1+,12)$ $8X$	$(1-,81)$ $2X$	$(81,C)$ $X1$	$(1-,7)$ 18	$(6,1-)$ 18	$(1-,5)$ 18	$(4,1-)$ 18	$(1-,3)$ 18
$(1+,12)$ 78	$(1-,8)$ $X7$	$(1-,78)$ $X1$	$(1-,7)$ $8X$	$(6,1-)$ 87	$(1-,5)$ 87	$(4,1-)$ 87	$(1-,3)$ 87
$(1+,12)$ 67	$(1-,8)$ 76	$(7,1-)$ $X6$	$(1-,67)$ $2X$	$(6,1-)$ $7X$	$(1-,5)$ 76	$(4,1-)$ 76	$(1-,3)$ 76
$(1+,12)$ 56	$(1-,8)$ 65	$(7,1-)$ 65	$(1-,6)$ $X5$	$(1-,56)$ $X1$	$(1-,5)$ $6X$	$(4,1-)$ 65	$(1-,3)$ 65
$(1+,12)$ 45	$(1-,8)$ 54	$(7,1-)$ 54	$(1-,6)$ 54	$(5,1-)$ $X4$	$(1-,45)$ $2X$	$(4,1-)$ $5X$	$(1-,3)$ 54
$(1+,12)$ 34	$(1-,8)$ 43	$(7,1-)$ 43	$(1-,6)$ 43	$(5,1-)$ 43	$(1-,4)$ $X3$	$(1-,34)$ $X1$	$(1-,3)$ $4X$
$(1+,12)$ 23	$(1-,8)$ 32	$(7,1-)$ 32	$(1-,6)$ 32	$(5,1-)$ 32	$(1-,4)$ 32	$(3,1-)$ $X2$	$(1-,E)$ $2X$
$(1+,12)$ $X2$	$(1-,8)$ 28	$(7,1-)$ $2X$	$(1-,6)$ 26	$(5,1-)$ $2X$	$(1-,4)$ 24	$(3,1-)$ $2X$	$(1-,E)$ $X2$

$(12,L)$ XX	$(81,L)$ XX	$(1+,7)$ $X1$	$(1,6-)$ 16	$(6,1-)$ 61	$(1-,5)$ $X1$	$(4,1-)$ 41	$(1-,3)$ $X1$
$(1+,12)$ $8X$	$(1-,18)$ $2X$	$(1+,7)$ 12	$(1,6-)$ $X1$	$(6,1-)$ 18	$(1-,5)$ 18	$(4,1-)$ 18	$(1-,3)$ 18

$(12,L)$ XX	$(81,L)$ XX	$(1+,7)$ $X1$	$(6,1+)$ 61	$(1,6+)$ 16	$(1-,5)$ $X1$	$(4,1-)$ 41	$(1-,3)$ $X1$
$(1+,12)$ $8X$	$(1-,18)$ $2X$	$(1+,7)$ 12	$(6,1+)$ 12	$(1,6+)$ $X1$	$(1-,5)$ 18	$(4,1-)$ 18	$(1-,3)$ 18

Figure 27: Pencils $\max(\hat{1} = 8-)$, $\max(\hat{1} = 8+)$, $\max(\hat{1} = 6-)$, $\max(\hat{1} = 6+)$

$(78, L)$ XX	$(81, L)$ XX	$(81, C)$ 18	$(8+, 2)$ $X8$	$(3, 8+)$ 38	$(8+, 4)$ $X8$	$(5, 8+)$ 58	$(8+, 6)$ $X8$	**1**
$(56, L)$ XX	$(81, L)$ XX	$(81, C)$ 18	$(8+, 2)$ $X8$	$(3, 8+)$ 38	$(8+, 4)$ $X8$	$(5, 8+)$ 58	$(5-, 7)$ $X5$	2
$(56, L)$ XX	$(81, L)$ XX	$(81, C)$ 18	$(8+, 2)$ $X8$	$(3, 8+)$ 38	$(8+, 4)$ $X8$	$(5+, 7)$ $X5$	$(56, C)$ 65	$3, 4$
$(56, L)$ XX	$(81, L)$ XX	$(81, C)$ 18	$(8+, 2)$ $X8$	$(3, 8+)$ 38	$(3-, 7)$ $X3$	$(6-, 4)$ $X6$	$(56, C)$ 65	$6, 5$
$(56, L)$ XX	$(81, L)$ XX	$(81, C)$ 18	$(8+, 2)$ $X8$	$(3, 8+)$ 38	$(3-, 7)$ $X3$	$(6, 3-)$ 63	$(6+, 4)$ $X6$	7
$(34, L)$ XX	$(81, L)$ XX	$(81, C)$ 18	$(8+, 2)$ $X8$	$(3, 8+)$ 38	$(3-, 7)$ $X3$	$(6, 3-)$ 63	$(3-, 5)$ $X3$	8
$(18, L)$ XX	$(65, L)$ XX	$(6+, 4)$ $X6$	$(6, 3-)$ 63	$(3, 6-)$ 36	$(3+, 7)$ $X3$	$(8+, 2)$ $X8$	$(18, C)$ 81	11
$(18, L)$ XX	$(34, L)$ XX	$(3-, 5)$ $X3$	$(6, 3-)$ 63	$(3, 6-)$ 36	$(3+, 7)$ $X3$	$(8+, 2)$ $X8$	$(18, C)$ 81	12
$(18, L)$ XX	$(34, L)$ XX	$(3-, 5)$ $X3$	$(3, 6+)$ 36	$(6, 3+)$ 63	$(3+, 7)$ $X3$	$(8+, 2)$ $X8$	$(18, C)$ 81	14
$(18, L)$ XX	$(65, L)$ XX	$(6+, 4)$ $X6$	$(3, 6+)$ 36	$(6, 3+)$ 63	$(3+, 7)$ $X3$	$(8+, 2)$ $X8$	$(18, C)$ 81	13
$(18, L)$ XX	$(65, L)$ XX	$(6+, 4)$ $X6$	$(6, 3-)$ 63	$(3, 6-)$ 36	$(6-, 2)$ $X6$	$(1, 6-)$ 16	$(1+, 7)$ $X1$	35
$(18, L)$ XX	$(34, L)$ XX	$(3-, 5)$ $X3$	$(6, 3-)$ 63	$(3, 6-)$ 36	$(6-, 2)$ $X6$	$(1, 6-)$ 16	$(1+, 7)$ $X1$	36
$(18, L)$ XX	$(34, L)$ XX	$(3-, 5)$ $X3$	$(3, 6+)$ 36	$(6, 3+)$ 63	$(6-, 2)$ $X6$	$(1, 6-)$ 16	$(1+, 7)$ $X1$	38
$(18, L)$ XX	$(65, L)$ XX	$(6+, 4)$ $X6$	$(3, 6+)$ 36	$(6, 3+)$ 63	$(6-, 2)$ $X6$	$(1, 6-)$ 16	$(1+, 7)$ $X1$	37
$(18, L)$ XX	$(65, L)$ XX	$(6+, 4)$ $X6$	$(3, 6+)$ 36	$(6+, 2)$ $X6$	$(6, 1-)$ 61	$(1, 6-)$ 16	$(1+, 7)$ $X1$	41
$(18, L)$ XX	$(65, L)$ XX	$(6+, 4)$ $X6$	$(3, 6+)$ 36	$(6+, 2)$ $X6$	$(1, 6+)$ 16	$(6, 1+)$ 61	$(1+, 7)$ $X1$	49

Figure 28: Pencils with $\hat{8} = 1+$

$(1+,2)$ 82	$(3,1+)$ $8X$	$(1+,4)$ 84	$(5,1+)$ $8X$	$(1+,6)$ 86	$(7,1+)$ $8X$	$(1,7-)$ $8X$	$(1-,8)$ 28	65
$(1+,2)$ 82	$(3,1+)$ $8X$	$(1+,4)$ 84	$(5,1+)$ $8X$	$(1+,6)$ 86	$(1,7+)$ $6X$	$(7,1-)$ $2X$	$(1-,8)$ 28	66
$(1+,2)$ 82	$(3,1+)$ $8X$	$(1+,4)$ 84	$(5,1+)$ $8X$	$(1,5-)$ $6X$	$(1-,6)$ 26	$(7,1-)$ $2X$	$(1-,8)$ 28	67
$(4,2+)$ $1X$	$(2+,5)$ 15	$(6,2+)$ $1X$	$(2+,7)$ 17	$(8,2+)$ $1X$	$(2,8-)$ $1X$	$(8-,3)$ 13	$(4+,1)$ 31	71
$(4-,1)$ 51	$(2+,5)$ 15	$(6,2+)$ $1X$	$(2+,7)$ 17	$(8,2+)$ $1X$	$(2,8-)$ $1X$	$(8-,3)$ 13	$(4,8-)$ $1X$	72
$(4,2+)$ $1X$	$(2+,5)$ 15	$(6,2+)$ $1X$	$(2+,7)$ 17	$(2,8+)$ $7X$	$(8,2-)$ $3X$	$(8-,3)$ 13	$(4+,1)$ 31	75
$(4,2+)$ $1X$	$(2+,5)$ 15	$(6,2+)$ $1X$	$(2,6-)$ $7X$	$(2-,7)$ 37	$(8,2-)$ $3X$	$(8-,3)$ 13	$(4+,1)$ 31	78
$(4-,1)$ 51	$(2+,5)$ 15	$(6,2+)$ $1X$	$(2+,7)$ 17	$(2,8+)$ $7X$	$(8,2-)$ $3X$	$(8-,3)$ 13	$(4,8-)$ $1X$	80
$(4-,1)$ 51	$(2+,5)$ 15	$(6,2+)$ $1X$	$(2,6-)$ $7X$	$(2-,7)$ 37	$(8,2-)$ $3X$	$(8-,3)$ 13	$(4,8-)$ $1X$	86
$(4-,1)$ 51	$(2+,5)$ 15	$(2,6+)$ $5X$	$(6,2-)$ $3X$	$(2-,7)$ 37	$(8,2-)$ $3X$	$(8-,3)$ 13	$(4,8-)$ $1X$	87
$(4-,1)$ 51	$(2,4-)$ $5X$	$(2-,5)$ 35	$(6,2-)$ $3X$	$(2-,7)$ 37	$(8,2-)$ $3X$	$(8-,3)$ 13	$(4,8-)$ $1X$	84
$(8-,5)$ 15	$(6+,1)$ 51	$(2,6+)$ $5X$	$(6,2-)$ $3X$	$(2-,7)$ 37	$(8,2-)$ $3X$	$(8-,3)$ 13	$(4,8-)$ $1X$	83
$(8-,5)$ 15	$(6+,1)$ 51	$(6,2+)$ $1X$	$(2,6-)$ $7X$	$(2-,7)$ 37	$(8,2-)$ $3X$	$(8-,3)$ 13	$(4,8-)$ $1X$	82
$(5,1-)$ $2X$	$(1,5+)$ $4X$	$(1+,4)$ 84	$(3-,8)$ 48	$(3,7-)$ $8X$	$(7,3+)$ $2X$	$(7+,2)$ 62	$(1-,6)$ 26	88
$(1,5-)$ $6X$	$(5,1+)$ $8X$	$(1+,4)$ 84	$(3-,8)$ 48	$(3,7-)$ $8X$	$(7,3+)$ $2X$	$(7+,2)$ 62	$(1-,6)$ 26	90
$(1,5-)$ $6X$	$(5,1+)$ $8X$	$(1+,4)$ 84	$(3-,8)$ 48	$(7,3-)$ $4X$	$(3,7+)$ $6X$	$(7+,2)$ 62	$(1-,6)$ 26	92
$(5,1-)$ $2X$	$(1,5+)$ $4X$	$(1+,4)$ 84	$(3-,8)$ 48	$(7,3-)$ $4X$	$(3,7+)$ $6X$	$(7+,2)$ 62	$(1-,6)$ 26	94

Figure 29: Pencils with $\hat{8} \neq 1+$

$\hat{8}$	1+		1−		2+		2−		3+		3−		4+	
C_2	in	out	in	out	in	out	in	out	in	out	in	out	in	out
23456	**7**	**1**		1, 7	1, 7		1, 7			1, 7		1, 7	1, 7	
23457		1, 6	6	1	1	6	1	6	6	1	6	1	1	6
23467	5	1		1, 5	1, 5		1, 5			1, 5		1, 5	1, 5	
23567		1, 4	4	1	1	4	1	4	4	1	4	1	1	4
24567	3	1		1, 3	1, 3		1, 3			1, 3	1	3	3	1
34567		1, 2	**2**	**1**	**1**	**2**		1, 2	1, 2		1, 2			1, 2
13456	2, 7		2, 7			2, 7	7	2	2	7	2	7	7	2
13457	2	6	2	6	6	2		2, 6	2, 6		2, 6			2, 6
13467	2, 5		2, 5			2, 5	5	2	2	5	2	5	5	2
13567	2	4	2	4	4	2		2, 4	2, 4		2, 4			2, 4
14567	2, 3		2, 3			2, 3	**3**	**2**	**2**	**3**		2, 3	2, 3	
12456	7	3	7	3	3	7	3	7	7	3		3, 7	3, 7	
12457		3, 6		3, 6	3, 6		3, 6			3, 6	6	3	3	6
12467	5	3	5	3	3	5	3	5	5	3		3, 5	3, 5	
12567		3, 4		3, 4	3, 4		3, 4			3, 4	**4**	**3**	**3**	**4**
12356	4, 7		4, 7			4, 7		4, 7	4, 7		4, 7			4, 7
12357	4	6	4	6	6	4	6	4	4	6	4	6	6	4
12367	4, 5		4, 5			4, 5		4, 5	4, 5		4, 5			4, 5
12346	7	5	7	5	5	7	5	7	7	5	7	5	5	7
12347		5, 6		5, 6	5, 6		5, 6			5, 6		5, 6	5, 6	
12345	6, 7		6, 7			6, 7		6, 7	6, 7		6, 7			6, 7

Figure 30: The lists $\hat{8} = L(1, \dots 7)$

6 Tabulars

$\hat{8}$	4−		5+		5−		6+		6−		7+		7−	
C_2	in	out	in	out	in	out	in	out	in	out	in	out	in	out
23456	1, 7			1, 7		1, 7	1, 7		1, 7			1, 7	**1**	**7**
23457	1	6	6	1	6	1	1	6		1, 6	1, 6		1, 6	
23467	1, 5			1, 5	1	5	5	1	5	1	1	5	1	5
23567		1, 4	1, 4		1, 4			1, 4		1, 4	1, 4		1, 4	
24567	3	1	1	3	1	3	3	1	3	1	1	3	1	3
34567		1, 2	1, 2		1, 2			1, 2		1, 2	1, 2		1, 2	
13456	7	2	2	7	2	7	7	2	7	2	2	7		2, 7
13457		2, 6	2, 6		2, 6			2, 6	2	6	6	2	6	2
13467	5	2	2	5		2, 5	2, 5		2, 5			2, 5		2, 5
13567	2	4	4	2	4	2	2	4	2	4	4	2	4	2
14567	2, 3			2, 3		2, 3	2, 3		2, 3			2, 3		2, 3
12456	3, 7			3, 7		3, 7	3, 7		3, 7			3, 7	3	7
12457	3	6	6	3	6	3	3	6		3, 6	3, 6		3, 6	
12467	3, 5			3, 5	3	5	5	3	5	3	3	5	3	5
12567		3, 4	3, 4		3, 4			3, 4		3, 4	3, 4		3, 4	
12356	7	4	4	7	4	7	7	4	7	4	4	7		4, 7
12357		4, 6	4, 6		4, 6			4, 6	4	6	6	4	6	4
12367	**5**	**4**	**4**	**5**		4, 5	4, 5		4, 5			4, 5		4, 5
12346	5	7	7	5		5, 7	5, 7		5, 7			5, 7	5	7
12347	5, 6			5, 6	**6**	**5**	**5**	**6**		5, 6	5, 6		5, 6	
12345		6, 7	6, 7		6, 7			6, 7	**7**	**6**	**6**	**7**		6, 7

Figure 31: The lists $\hat{8} = L(1, \ldots 7)$, continued

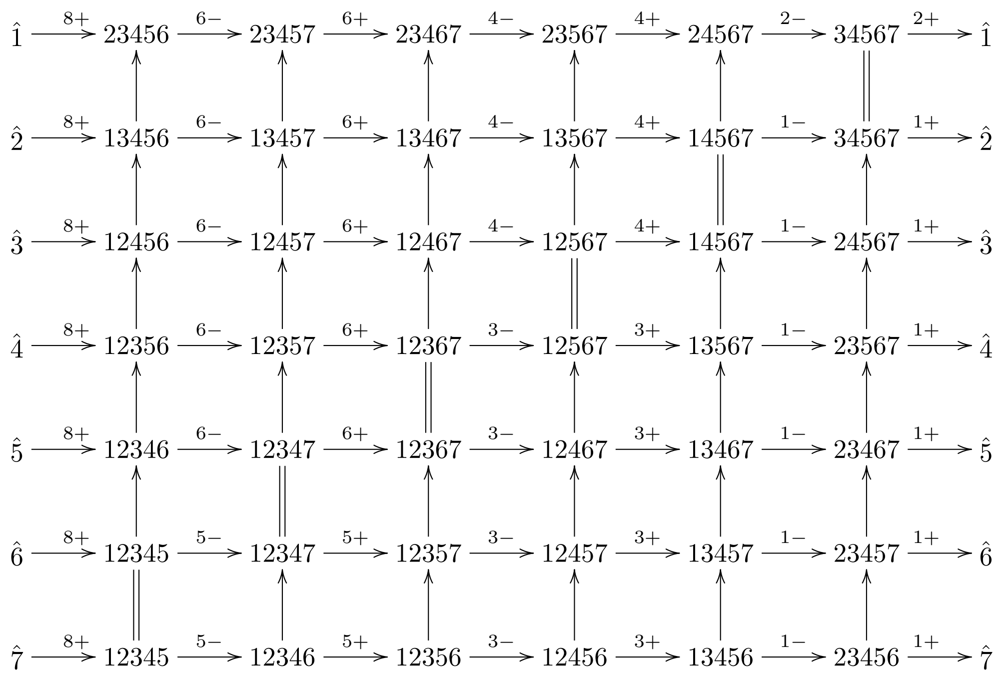

Figure 32: $\hat{8} = 1+$

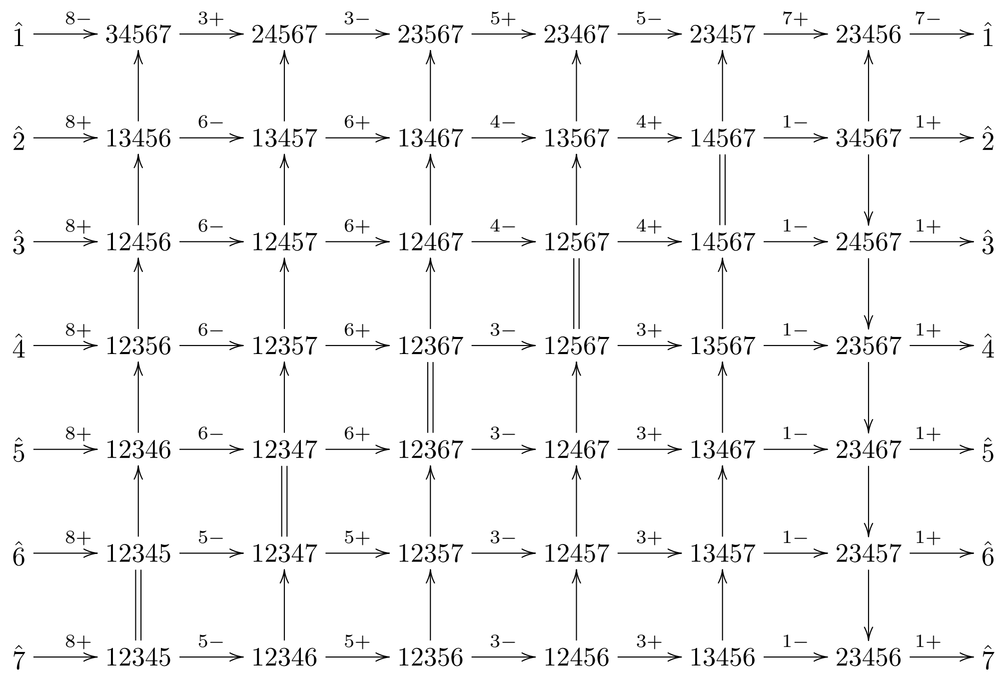

Figure 33: $\hat{8} = 1-$

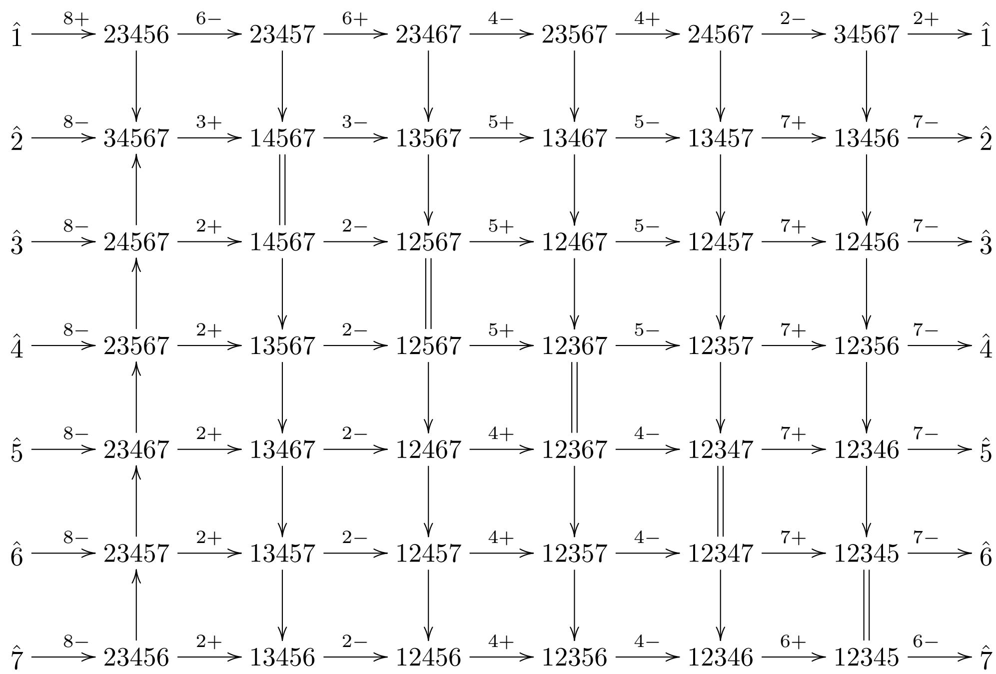

Figure 34: $\hat{8} = 2+$

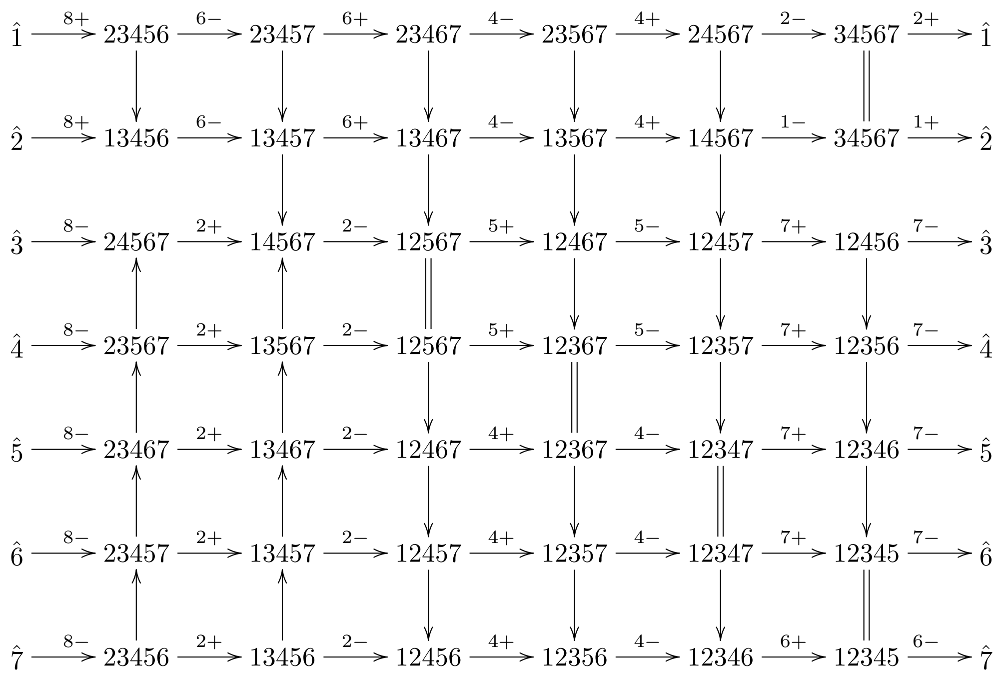

Figure 35: $\hat{8} = 2-$

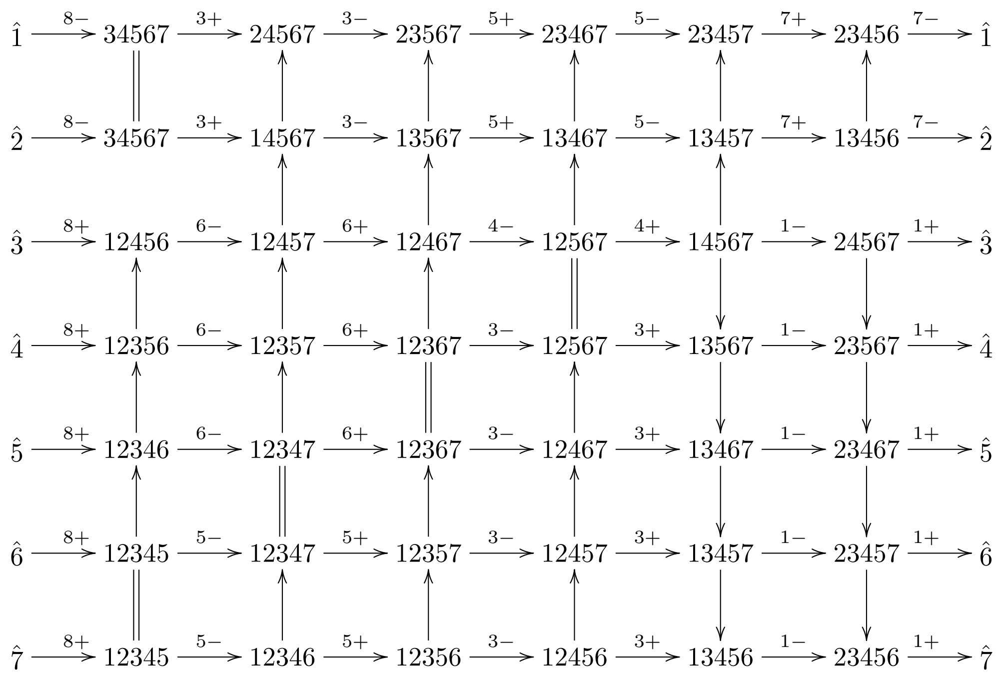

Figure 36: $\hat{8} = 3+$

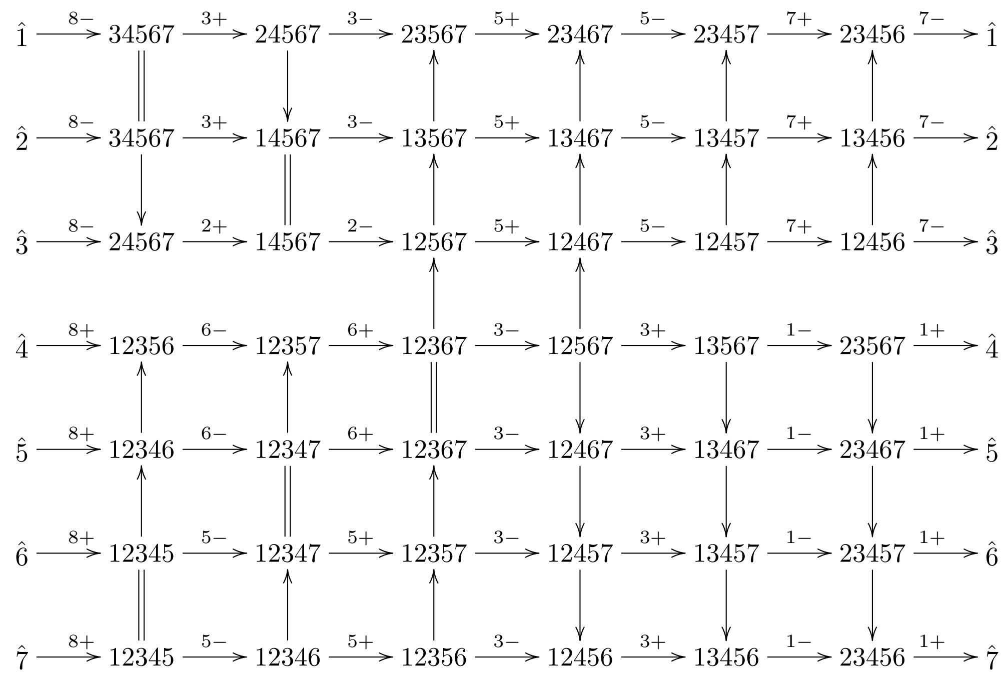

Figure 37: $\hat{8} = 3-$

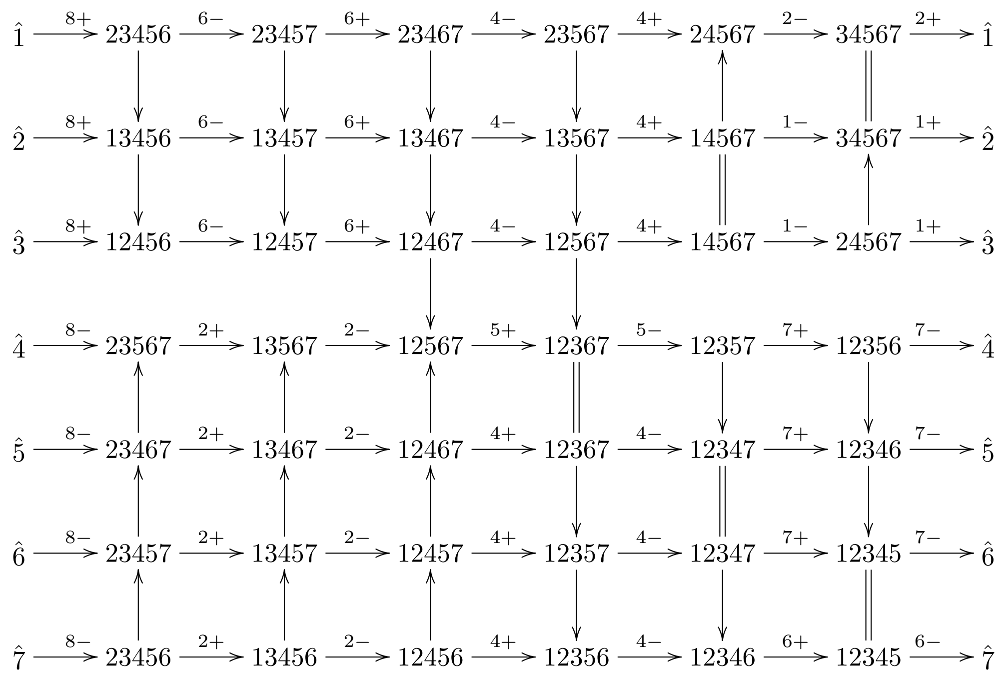

Figure 38: $\hat{8} = 4+$

$\hat{1}$	$3+ \leftrightarrow 8-$	$3+ \leftrightarrow 8-$	$2+ \leftrightarrow 2-$	$2+ \leftrightarrow 2-$
$\hat{2}$	$3+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$	$3+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$
$\hat{1}$	$3- \leftrightarrow 3+$	$3- \leftrightarrow 3+$	$2- \leftrightarrow 4+$	$2- \leftrightarrow 4+$
$\hat{3}$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$
$\hat{1}$	$5+ \leftrightarrow 3-$	$5+ \leftrightarrow 3-$	$4+ \leftrightarrow 4-$	$4+ \leftrightarrow 4-$
$\hat{4}$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$
$\hat{1}$	$5- \leftrightarrow 5+$	$5- \leftrightarrow 5+$	$4- \leftrightarrow 6+$	$4- \leftrightarrow 6+$
$\hat{5}$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$
$\hat{1}$	$7+ \leftrightarrow 5-$	$7+ \leftrightarrow 5-$	$6+ \leftrightarrow 6-$	$6+ \leftrightarrow 6-$
$\hat{6}$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$
$\hat{1}$	$7- \leftrightarrow 7+$	$7- \leftrightarrow 7+$	$6- \leftrightarrow 8+$	$6- \leftrightarrow 8+$
$\hat{7}$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$	$2+ \leftrightarrow 8-$	$1+ \leftrightarrow 1-$
$\hat{1}$	$2+ \leftrightarrow 7-$	$2+ \leftrightarrow 7-$	$8+ \leftrightarrow 8-$	$8+ \leftrightarrow 8-$
$\hat{8}$	$2+ \leftrightarrow 7-$	$1+ \leftrightarrow 1-$	$2+ \leftrightarrow 7-$	$1+ \leftrightarrow 1-$
$\hat{2}$	$3- \leftrightarrow 3+$	$3- \leftrightarrow 3+$	$1- \leftrightarrow 4+$	$1- \leftrightarrow 4+$
$\hat{3}$	$1- \leftrightarrow 4+$	$2- \leftrightarrow 2+$	$1- \leftrightarrow 4+$	$2- \leftrightarrow 2+$
$\hat{2}$	$5+ \leftrightarrow 3-$	$5+ \leftrightarrow 3-$	$4+ \leftrightarrow 4-$	$4+ \leftrightarrow 4-$
$\hat{4}$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$
$\hat{2}$	$5- \leftrightarrow 5+$	$5- \leftrightarrow 5+$	$4- \leftrightarrow 6+$	$4- \leftrightarrow 6+$
$\hat{5}$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$
$\hat{2}$	$7+ \leftrightarrow 5-$	$7+ \leftrightarrow 5-$	$6+ \leftrightarrow 6-$	$6+ \leftrightarrow 6-$
$\hat{6}$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$
$\hat{2}$	$7- \leftrightarrow 7+$	$7- \leftrightarrow 7+$	$6- \leftrightarrow 8+$	$6- \leftrightarrow 8+$
$\hat{7}$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$
$\hat{2}$	$1+ \leftrightarrow 7-$	$1+ \leftrightarrow 7-$	$8+ \leftrightarrow 8-$	$8+ \leftrightarrow 8-$
$\hat{8}$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$	$1- \leftrightarrow 3+$	$2- \leftrightarrow 2+$
$\hat{3}$	$4- \leftrightarrow 4+$	$4- \leftrightarrow 4+$	$2- \leftrightarrow 5+$	$2- \leftrightarrow 5+$
$\hat{4}$	$2- \leftrightarrow 5+$	$3- \leftrightarrow 3+$	$2- \leftrightarrow 5+$	$3- \leftrightarrow 3+$

Figure 39: Elementary changes

$\hat{3}$	$6+ \leftrightarrow 4-$	$6+ \leftrightarrow 4-$	$5+ \leftrightarrow 5-$	$5+ \leftrightarrow 5-$
$\hat{5}$	$2- \leftrightarrow 4+$	$3- \leftrightarrow 3+$	$2- \leftrightarrow 4+$	$3- \leftrightarrow 3+$
$\hat{3}$	$6- \leftrightarrow 6+$	$6- \leftrightarrow 6+$	$5- \leftrightarrow 7+$	$5- \leftrightarrow 7+$
$\hat{6}$	$2- \leftrightarrow 4+$	$3- \leftrightarrow 3+$	$2- \leftrightarrow 4+$	$3- \leftrightarrow 3+$
$\hat{3}$	$8+ \leftrightarrow 6-$	$8+ \leftrightarrow 6-$	$7+ \leftrightarrow 7-$	$7+ \leftrightarrow 7-$
$\hat{7}$	$2- \leftrightarrow 4+$	$3- \leftrightarrow 3+$	$2- \leftrightarrow 4+$	$3- \leftrightarrow 3+$
$\hat{3}$	$8- \leftrightarrow 8+$	$8- \leftrightarrow 8+$	$7- \leftrightarrow 1+$	$7- \leftrightarrow 1+$
$\hat{8}$	$2- \leftrightarrow 4+$	$3- \leftrightarrow 3+$	$2- \leftrightarrow 4+$	$3- \leftrightarrow 3+$
$\hat{4}$	$5- \leftrightarrow 5+$	$5- \leftrightarrow 5+$	$3- \leftrightarrow 6+$	$3- \leftrightarrow 6+$
$\hat{5}$	$3- \leftrightarrow 6+$	$4- \leftrightarrow 4+$	$3- \leftrightarrow 6+$	$4- \leftrightarrow 4+$
$\hat{4}$	$7+ \leftrightarrow 5-$	$7+ \leftrightarrow 5-$	$6+ \leftrightarrow 6-$	$6+ \leftrightarrow 6-$
$\hat{6}$	$3- \leftrightarrow 5+$	$4- \leftrightarrow 4+$	$3- \leftrightarrow 5+$	$4- \leftrightarrow 4+$
$\hat{4}$	$7- \leftrightarrow 7+$	$7- \leftrightarrow 7+$	$6- \leftrightarrow 8+$	$6- \leftrightarrow 8+$
$\hat{7}$	$3- \leftrightarrow 5+$	$4- \leftrightarrow 4+$	$3- \leftrightarrow 5+$	$4- \leftrightarrow 4+$
$\hat{4}$	$1+ \leftrightarrow 7-$	$1+ \leftrightarrow 7-$	$8+ \leftrightarrow 8-$	$8+ \leftrightarrow 8-$
$\hat{8}$	$3- \leftrightarrow 5+$	$4- \leftrightarrow 4+$	$3- \leftrightarrow 5+$	$4- \leftrightarrow 4+$
$\hat{5}$	$6- \leftrightarrow 6+$	$6- \leftrightarrow 6+$	$4- \leftrightarrow 7+$	$4- \leftrightarrow 7+$
$\hat{6}$	$4- \leftrightarrow 7+$	$5- \leftrightarrow 5+$	$4- \leftrightarrow 7+$	$5- \leftrightarrow 5+$
$\hat{5}$	$8+ \leftrightarrow 6-$	$8+ \leftrightarrow 6-$	$7+ \leftrightarrow 7-$	$7+ \leftrightarrow 7-$
$\hat{7}$	$4- \leftrightarrow 6+$	$5- \leftrightarrow 5+$	$4- \leftrightarrow 6+$	$5- \leftrightarrow 5+$
$\hat{5}$	$8- \leftrightarrow 8+$	$8- \leftrightarrow 8+$	$7- \leftrightarrow 1+$	$7- \leftrightarrow 1+$
$\hat{8}$	$4- \leftrightarrow 6+$	$5- \leftrightarrow 5+$	$4- \leftrightarrow 6+$	$5- \leftrightarrow 5+$
$\hat{6}$	$7- \leftrightarrow 7+$	$7- \leftrightarrow 7+$	$5- \leftrightarrow 8+$	$5- \leftrightarrow 8+$
$\hat{7}$	$5- \leftrightarrow 8+$	$6- \leftrightarrow 6+$	$5- \leftrightarrow 8+$	$6- \leftrightarrow 6+$
$\hat{6}$	$1+ \leftrightarrow 7-$	$1+ \leftrightarrow 7-$	$8+ \leftrightarrow 8-$	$8+ \leftrightarrow 8-$
$\hat{8}$	$5- \leftrightarrow 7+$	$6- \leftrightarrow 6+$	$5- \leftrightarrow 7+$	$6- \leftrightarrow 6+$
$\hat{7}$	$8- \leftrightarrow 8+$	$8- \leftrightarrow 8+$	$6- \leftrightarrow 1+$	$6- \leftrightarrow 1+$
$\hat{8}$	$6- \leftrightarrow 1+$	$7- \leftrightarrow 7+$	$6- \leftrightarrow 1+$	$7- \leftrightarrow 7+$

Figure 40: Elementary changes, continued

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