# A note on the Frobenius morphism on toric varieties

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**Abstract**

We give a new, shorter computation of Frobenius push-forwards of line bundles on toric varieties.

Frobenius push-forwards of line bundles on smooth toric varieties were calculated by Thomsen [1] – it was observed that they split into direct sums of line bundles. A more intrinsic approach has been developed in [2]. The algorithm of Thomsen has been adapted to the case of the toric Frobenius morphism in characteristic zero in [3] and [4]. In this note, we present a new proof of these results. This gives the shortest known computation even for the projective spaces (in which case one uses the Horrocks splitting criterion and the projection formula to prove that the direct image splits). The key point of our approach is to consider Frobenius push-forwards of all line bundles at once.

Let $X$ be a toric variety over an algebraically closed field $k$. We denote the torus acting on $X$ by $T$ and the number of rays of the fan defining $X$ by $r$. If char $k = p > 0$, we have the *honest* (absolute) Frobenius morphism $F_a : X \to X$. In fact, $X$ can be identified with its Frobenius twist $X^{(1)}$ and then $F_a$ can be seen as the quotient by the Frobenius kernel $K_a$ of $T$. In any case, for any integer $\ell > 0$ we have the *fake* (toric) Frobenius morphism $F_\ell : X \to X$ which corresponds to taking the quotient by the kernel $K_\ell$ of the $\ell$-th power map on $T$. Let $F$ be the honest or a fake Frobenius (in the honest case we define $\ell = p$) in the following theorem:

**Theorem.** *Let $D \in \operatorname{Cl} X$. Then*

$$F_* \mathscr{O}_X(D) = \bigoplus_{E \in \operatorname{Cl} X} \mathscr{O}_X(E)^{\oplus m(E,D)}, \tag{1}$$

*where the multiplicity $m(E, D)$ equals the number of points in the cube $\{0, 1, \ldots, \ell - 1\}^r \subseteq \mathbb{Z}^r = \operatorname{Div}_T X$ representing the class $D - \ell E \in \operatorname{Cl} X$ (that is, the number of $T$-divisors in $|D - \ell E|$ with coefficients less than $\ell$).*

*Proof.* Let us first prove the theorem in the case when $X$ is smooth and complete, and reduce to this case afterwards.

First of all, we remark that the push-forward $F_* \mathscr{O}_X(D)$ of a $T$-equivariant line bundle $\mathscr{O}_X(D)$ is a direct sum of line bundles. Indeed, the kernel $K$ of $F$ on $T$ (equal to $K_a$ or $K_\ell$) is

a finite diagonalizable commutative group scheme acting on $F_*\mathscr{O}_X(D)$ and the eigensheaves are line bundles. Since every line bundle on $X$ is equivariant, we get a decomposition as in (1) and we only want to compute the multiplicities.

Observe that $m(E, D)$ depends only on $D - \ell E$: by the projection formula we have $(F_*\mathscr{O}_X(D)) \otimes \mathscr{O}_X(-E) = F_*(\mathscr{O}_X(D - \ell E))$, so $m(E, D) = m(0, D - \ell E)$. Denote $m(0, D)$ simply by $m(D)$ and apply $h^0(-)$ to both sides of (1):

$$h^0(D) = h^0(F_*\mathscr{O}_X(D)) = \sum_{E \in \operatorname{Pic} X} m(E, D) \cdot h^0(E) = \sum_{E \in \operatorname{Pic} X} m(D - \ell E) \cdot h^0(E). \qquad (2)$$

We want to use some generating functions, so we fix a basis $D_1, \ldots, D_\rho$ of $\operatorname{Pic} X$ such that the effective cone lies in the positive orthant and define

$$S(x_1, \ldots, x_\rho) = \sum_{a \in \mathbb{Z}^\rho} h^0\left(\sum a_i D_i\right) x^a \quad \text{and} \quad M(x_1, \ldots, x_n) = \sum_{a \in \mathbb{Z}^\rho} m\left(\sum a_i D_i\right) x^a.$$

Then (2) just states that $S(x_1, \ldots, x_\rho) = M(x_1, \ldots, x_\rho) \cdot S(x_1^\ell, \ldots, x_\rho^\ell)$.

Let us compute the series $S$. Consider the map $L : \mathbb{Z}^r = \operatorname{Div}_T X \to \operatorname{Pic} X = \mathbb{Z}^\rho$ (the first identification being given by the basis of „ray" divisors, the second by $D_1, \ldots, D_\rho$) taking a $T$-divisor to its class. Because $h^0(D)$ equals the number of effective $T$-divisors linearly equivalent to $D$, we get

$$S(x_1, \ldots, x_\rho) = \sum_{b \in \mathbb{Z}^r_{\geq 0}} x^{L(b)} = \prod_{i=1}^r \frac{1}{1 - x^{L(e_i)}},$$

$e_1, \ldots, e_r$ being the basis in $\mathbb{Z}^r$. Therefore

$$M(x_1, \ldots, x_\rho) = \frac{S(x_1, \ldots, x_\rho)}{S(x_1^\ell, \ldots, x_\rho^\ell)} = \prod_{i=1}^r \frac{1 - x^{\ell L(e_i)}}{1 - x^{L(e_i)}} = \prod_{i=1}^r (1 + x^{L(e_i)} + x^{2L(e_i)} + \ldots + x^{(\ell-1)L(e_i)}),$$

hence $m(D)$ is the number of points $p = \sum a_i e_i$ with $0 \leq a_i < \ell$ and $L(p) = D$.

We turn to the case $X$ not necessarily smooth nor complete. First, by taking an appropriate subdivision of the fan we get a toric resolution of singularities $\pi : \widetilde{X} \to X$. Next, by adding extra cones, we embed $i : \widetilde{X} \to \bar{X}$ into a smooth complete toric variety $\bar{X}$. Using the diagrams

$$
\begin{array}{ccc}
\mathbb{Z}^{\bar{r}-\tilde{r}} & \overset{\simeq}{\longrightarrow} & \mathbb{Z}^{\bar{r}-\tilde{r}} \\
\downarrow & & \downarrow \\
M \hookrightarrow \operatorname{Div}_T \bar{X} \longrightarrow\!\!\!\!\to \operatorname{Pic} \bar{X} \\
\downarrow{\simeq} \quad\quad \downarrow \quad\quad \downarrow \\
M \hookrightarrow \operatorname{Div}_T \widetilde{X} \longrightarrow\!\!\!\!\to \operatorname{Pic} \widetilde{X}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathbb{Z}^{\tilde{r}-r} & \overset{\simeq}{\longrightarrow} & \mathbb{Z}^{\tilde{r}-r} \\
\uparrow & & \uparrow \\
M \hookrightarrow \operatorname{Div}_T \widetilde{X} \longrightarrow\!\!\!\!\to \operatorname{Pic} \widetilde{X} \\
{\simeq}\uparrow \quad\quad \uparrow \quad\quad \uparrow \\
M \hookrightarrow \operatorname{Div}_T X \longrightarrow\!\!\!\!\to \operatorname{Cl} X
\end{array}
$$

it is easy to see that if the theorem holds for $\bar{X}$, then it also holds for $\widetilde{X}$ and for $X$ (the variety $X$ being normal). $\qquad \square$

# References

[1] Thomsen, J. F. *Frobenius direct images of line bundles on toric varieties.* J. Algebra 226 (2000), no. 2, 865–874.

[2] Bøgvad, R. *Splitting of the direct image of sheaves under the Frobenius.* Proc. Amer. Math. Soc. 126 (1998), no. 12, 3447–3454.

[3] Costa, L., Miró-Roig, R. M. *Frobenius splitting and Derived category of toric varieties* `arXiv:1006.5315v1`

[4] Lasoń, M., Michałek, M. *On the full, strongly exceptional collections on toric varieties with Picard number three*, `arXiv:1003.2047v1`

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