WHEN EVERY PRINCIPAL IDEAL IS FLAT

FATIMA CHENIOUR AND NAJIB MAHDOU

Abstract. This paper deals with well-known notion of PF-rings, that is, rings in which principal ideals are flat. We give a new characterization of PF-rings. Also, we provide a necessary and sufficient condition for $R \bowtie I$ (resp., R/I when R is a Dedekind domain or I is a primary ideal) to be PF-ring. The article includes a brief discussion of the scope and precision of our results.

1. Introduction

All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary.
We start by recalling some definitions.

A ring R is called a PF-ring if principal ideals of R are flat. Recall that R is a PF-ring if and only if R_Q is a domain for every prime (resp., maximal) ideal Q of R. For example, any domain and any semihereditary ring is a PF-ring (since a localization of a semihereditary ring by a prime (resp., maximal) ideal is a Prüfer domain). Note that a PF-ring is reduced by [12, Theorem 4.2.2 , p. 114]. See for instance [12, 13].

An R-module M is called P-flat if, for any $(s, x) \in R \times M$ such that $sx = 0$, then $x \in (0 : s)M$. If M is flat, then M is naturally P-flat. When R is a domain, M is P-plat if and only if it is torsion-free. When R is an arithmetical ring, then any P-flat module is flat (by [5, p. 236]). Also, every P-flat cyclic module is flat (by [5, Proposition 1(2)]). See for instance [5, 12].

The amalgamated duplication of a ring R along an ideal I is a ring that is defined as the following subring with unit element $(1, 1)$ of $R \times R$:

$$R \bowtie I = \{(r, r + i)/r \in R, i \in I\}.$$

This construction has been studied, in the general case, and from the different point of view of pullbacks, by D'Anna and Fontana [8]. Also, in [7], they have considered the case of the amalgamated duplication of a ring, in not necessarily Noetherian setting, along a multiplicative canonical ideal in the sense of [14]. In [6] D'Anna has studied some properties of $R \bowtie I$, in

2000 *Mathematics Subject Classification.* 13D05, 13D02.
Key words and phrases. PF-ring, direct product, localization, Dedekind domain, homomorphic image, amalgamated duplication of a ring along an ideal, Pullbacks.

order to construct reduced Gorenstein rings associated to Cohen-Macaulay rings and has applied this construction to curve singularities. On the other hand, Maimani and Yassemi, in [16], have studied the diameter and girth of the zero- divisor of the ring $R \bowtie I$. Some references are [7, 8, 9, 10, 16].

Let A and B be rings and let $\varphi : A \to B$ be a ring homomorphism making B an A-module. We say that A is a module retract of B if there exists a ring homomorphism $\psi : B \to A$ such that $\psi o \varphi = id_A$. ψ is called retraction of φ. See for instance [12].

Our first main result in this paper is Theorem 2.1 which gives us a new characterization of PF-rings. Also, we provide a necessary and sufficient condition for $R \bowtie I$ (resp., R/I when R is a Dedekind domain or I is a primary ideal) to be PF-ring. Our results generate new and original examples which enrich the current literature with new families of PF-rings with zero-divisors.

2. Main Results

Recall that an R-module M is called P-flat if, for any $(s, x) \in R \times M$ such that $sx = 0$, then $x \in (0 : s)M$. Now, we give a new characterization for a class of PF-rings, which is the first main result of this paper.

Theorem 2.1. *Let R be a commutative ring. Then the following conditions are equivalent:*
(1) *Every ideal of R is P-flat.*
(2) *Every principal ideal of R is P-flat.*
(3) *R is a PFring, that is every principal ideal of R is flat.*
(4) *For any elements $(s, x) \in R^2$ such that $sx = 0$, there exists $\alpha \in (0 : s)$ such that $x = \alpha x$.*

Proof. (1) $\Longrightarrow$ (2) Clear.
(2) $\Longrightarrow$ (3) Let Ra be a principal ideal of R generated by a. Our aim is to show that Ra is flat.
Let J be an ideal of R. We must show that $u : Ra \otimes J \longrightarrow Ra \otimes R$, where $u(a \otimes x) = ax$, is injective. Let $a \in R$ and $x \in J$ such that $ax = 0$. Hence, there exists $\beta \in (0 : x)$ and $\lambda \in R$ such that $a = \beta\lambda a$ (since Ra is P-flat). Therefore, $a \otimes x = \beta\lambda a \otimes x = \lambda a \otimes \beta x = 0$, as desired.
(3) $\Longrightarrow$ (4) Let (s, x) be an element of R^2 such that $sx = 0$. Our aim is to show that there exists $\beta \in (0 : s)$ such that $x = \beta x$. The principal ideal generated by x is P-flat (since it is flat), so there exists $\alpha \in (0 : s)$ and $r \in R$ such that $x = \alpha r x = \beta x$ with $\beta = \alpha r \in (0 : s)$.
(4) $\Longrightarrow$ (1) Let I be an ideal of R. Let $(s, x) \in R \times I$ such that $sx = 0$.

Hence, there exists $\alpha \in (0 : s)$ such that $x = \alpha x$ and so $x \in (0 : s)I$. Therefore, I is P-flat, as desired. $\square$

By Theorem 2.1, we obtain:

Corollary 2.2. *Let R be a ring. The following conditions are equivalent:*
(1) *Every ideal of R is P-flat.*
(2) *Every ideal of R_Q is P-flat for every prime ideal Q of R.*
(3) *Every ideal of R_m is P-flat for every maximal ideal m of R.*
(4) *R_Q is a domain for every prime ideal Q of R.*
(5) *R_m is a domain for every maximal ideal m of R.*

Proof. By Theorem 2.1 and [12, Theorem 4.2.2]. $\square$

Recall that a ring R is called an arithmetical ring if the lattice formed by its ideals is distributive. If $wgldim(R) \leq 1$, then R is an arithmetical ring. See for instance [2, 3].

Now, we add a condition with arithmetical in order to have equivalence between arithmetical and $wgldim(R) \leq 1$.

Proposition 2.3. *Let R be a ring. Then the following conditions are equivalent:*
(1) *$wgldim(R) \leq 1$.*
(2) *R is arithmetical and a PF-ring.*
(3) *R is arithmetical and every principal ideal of R is flat.*
(4) *R is arithmetical and every principal ideal of R is P-flat.*
(5) *R is arithmetical and every ideal of R is P-flat.*

Proof. $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 5)$. By Theorem 2.1.
$5) \Rightarrow 1)$. Assume that the ring R is arithmetical and every ideal of R is P-flat. Our aim is to show that $wgldim(R) \leq 1$. Let I be a finitely generated ideal of R. Hence, I is P-flat and so I is flat (since R is arithmetical by [5, p. 236]) and this completes the proof. $\square$

Now we show that the localization of a PF-ring is always a PF-ring.

Proposition 2.4. *Let R be a PF-ring and let S be a multiplicative subset of R. Then $S^{-1}(R)$ is a PF-ring.*

Proof. Assume that R is a PF-ring and let J be a principal ideal of $S^{-1}(R)$. We claim that J is flat. Indeed, since J is a principal ideal of $S^{-1}R$, then there exists an element $\frac{a}{b}$ of J such that $J = S^{-1}(R)\frac{a}{b}$. Set $I = Ra$. Hence, I is flat since R is a PF-ring and so $J(= S^{-1}(I))$ is a flat ideal of $S^{-1}R$. It follows that $S^{-1}(R)$ is a PF-ring. $\square$

Now, we study the transfer of PF-ring property to the direct product.

Proposition 2.5. *Let $(R_i)_{i\in I}$ be a family of commutative rings. Then $R = \prod_{i\in I} R_i$ is a PF-ring if and only if R_i is a PF-ring for all $i \in I$.*

Proof. Assume that R_i is a PF-ring for each $i \in I$ and set $R = \prod_{i\in I} R_i$. Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be two elements of R such that $xy = 0$. Then, for every $i \in I$, there exists $\alpha_i \in (0 : x_i)$ such that $y_i = \alpha_i y_i$ (since R_i is a PF-ring). Hence, $(y_i)_{i\in I} = (\alpha_i)_{i\in I}(y_i)_{i\in I}$ and $(\alpha_i)_{i\in I}(x_i)_{i\in I} = (\alpha_i x_i)_{i\in I} = 0$. Therefore, R is a PF-ring.

Conversely, assume that $R = \prod_{i\in I} R_i$ is a PF-ring and we claim that R_i is a PF-ring for every $i \in I$.
Indeed, let $i \in I$ and let x_i, y_i be two elements of R_i such that $x_i y_i = 0$. Consider $x = (a_j)_{j\in I}$, with $\begin{cases} a_i = x_i, \\ a_j = 0 \quad for\ j \neq i. \end{cases}$ and $y = (b_j)_{j\in I}$, with $\begin{cases} b_i = y_i. \\ b_j = 0 \quad for\ i \neq j. \end{cases}$ Since R is a PF-ring, then there exists $\alpha \in (0 : x)$ such that $y = \alpha y$ (that is, for all $j \in I$, $b_j = \alpha_j b_j$ and $\alpha_j a_j = 0$). Hence, $y_i = \alpha_i y_i$ with $\alpha_i \in (0 : x_i)$. Therefore, R_i is a PF-ring for all $i \in I$ and this completes the proof. $\square$

Next we study the transfer of PF-ring property to homomorphic image. First, the following example shows that the homomorphic image of a PF-ring is not always a PF-ring.

Example 2.6. *Let A be a domain and let $R = A[X]$. Then:*
(1) *R is a PF-ring since it is a domain.*
(2) *$R/(X^n)$ (for $n \geq 2$) is not a PF-ring since $\overline{X^n} = 0$ and $\overline{X} \neq 0$.*

The converse is not generally true as the following example shows.

Example 2.7. *Let R be a non-PF-ring and let P be a prime ideal of R. Then R/P is always a PF-ring.*

Recall that if R is a Dedekind domain and I is a nonzero ideal of R, then $I = P_1^{\alpha_1}...P_n^{\alpha_n}$ for some distinct prime ideals $P_1, ..., P_n$ uniquely determined by I and some positive integers $\alpha_1, ..., \alpha_n$ uniquely determined by I (by [11, Theorem 3.14]).

Now, when R is a Dedekind domain or I is a primary ideal, we give a characterization of R and I such that R/I is a PF-ring.

Theorem 2.8. *Let R be a ring and let I be an ideal of R. Then:*
(1) *Assume that R is a Dedekind domain and $I = P_1^{\alpha_1}...P_n^{\alpha_n}$ a nonzero ideal of R, where $P_1, .., P_n$ are the prime ideals defined by I. Then R/I is a PF-ring if and only if $\alpha_i = 1$ for all $i \in \{1, ..., n\}$.*
(2) *Assume that I is a primary ideal of R. Then R/I is a PF-ring if and only if I is a prime ideal of R.*

Proof. 1) Let R be a Dedekind domain and let $I = P_1^{\alpha_1}...P_n^{\alpha_n}$, where $P_1, ..., P_n$ are nonzero prime ideals of R, then $R/I = \prod_{i=1}^{n}(R/P_i^{\alpha_i})$.
Assume that $\alpha_i = 1$ for all $1 \leq i \leq n$. Hence, R/P_i is a PF-ring since R/P_i is an integral domain (as $\alpha_i = 1$), and so $R/I = \prod_{i=1}^{n}(R/P_i^{\alpha_i})$ is a PF-ring by Proposition 2.5.

Conversely, assume that $R/I = \prod_{i=1}^{n}(R/P_i^{\alpha_i})$ is a PF-ring. Let $i \in \{1, ..., n\}$. Then $R/P_i^{\alpha_i}$ is a PF-ring by Proposition 2.5 since R/I is a PF-ring. Hence, $R/P_i^{\alpha_i}$ is reduced and so the intersection of all prime ideals Q of $R/P_i^{\alpha_i}$ is zero (i.e $\bigcap_{Q \in spect(R/P_i^{\alpha_i})} Q = \{0\}$) by [1, Proposition 1.8]. For all prime ideals Q of $R/P_i^{\alpha_i}$, there exists a prime ideal Q' of R such that $P_i^{\alpha_i} \subset Q'$ and $Q = Q'/P_i^{\alpha_i}$. Then $P_i \subset Q'$ and so $P_i/P_i^{\alpha_i} \subset Q'/P_i^{\alpha_i} = Q$. It follows that $P_i/P_i^{\alpha_i} = 0$ and so $P_i = P_i^{\alpha_i}$ since R is a Dedekind domain. Hence, $\alpha_i = 1$.

2) It's obvious that if I is a prime ideal, then R/I is a PF-ring and I is a primary ideal.
Conversely, assume that I is a primary ideal and R/I is a PF-ring. Our aim is to show that I is a prime ideal of R. Let $x, y \in R$ such that $xy \in I$. We claim that $x \in I$ or $y \in I$. Without loss of generality, we may assume that $x \notin I$. Since $xy \in I$, then there exists an integer $n > 0$ such that $y^n \in I$ (as I is a primary ideal). Hence, $\overline{y}^n = 0$ and so $\overline{y} = 0$ since R/I is a PF-ring; that is $y \in I$. Therefore, $x \in I$ or $y \in I$ and so I is a prime ideal of R, as desired. □

Now, we are able to give examples of PF-rings and non-PF-rings.

Example 2.9. *(1) $\mathbb{Z}/4\mathbb{Z}$ is not a PF-ring by Theorem 2.8(1).*
(2) $\mathbb{Z}/30\mathbb{Z}$ is a PF-ring by Theorem 2.8(1).

Now, we study the transfer of a PF-property to amalgamated duplication of a ring R along an ideal I.

Theorem 2.10. *Let R be a ring, and I an ideal of R. Then the following conditions are equivalent.*
(1) *$R \bowtie I$ is a PF-ring;*
(2) *R is a PF-ring and $I_p \in \{0, R_p\}$ for every prime ideal p of R containing I;*
(3) *R is a PF-ring and $I_m \in \{0, R_m\}$ for every maximal ideal m of R containing I.*

We need the following lemma before proving this Theorem.

Lemma 2.11. *Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism making R a module retract of S. If S is a PF-ring, then so is R.*

Proof. Let $\varphi : R \to S$ be a ring homomorphism and let $\psi : S \to R$ be a ring homomorphism such that $\psi o \varphi = id_R$. Let $(x, y) \in R^2$ such that $xy = 0$. Then $\varphi(x)\varphi(y) = \varphi(xy) = 0$. Hence, there exists an element $\alpha \in S$ such that $\alpha\varphi(x) = 0$ and $\varphi(y) = \alpha\varphi(y)$ (since S is a PF-ring) and so $y = \psi(\varphi(y)) = \psi(\alpha\varphi(y)) = \psi(\alpha)y$ and $\psi(\alpha)x = \psi(\alpha\varphi(x)) = \psi(0) = 0$, as desired. □

Proof. of Theorem 2.10.
(1) $\Rightarrow$ (2) Assume that $R \bowtie I$ is a PF-ring and we must to show that R is a PF-ring and $I_p \in \{0, R_p\}$ for every prime ideal p of R containing I. We can easily show that R is a module retract of $R \bowtie I$ where the retraction map φ is defined by $\varphi(r, r + i) = r$ and so R is a PF-ring by Lemma 2.11. We claim that $I_p \in \{0, R_p\}$ for every prime ideal p of R containing I. Deny. Then there exists a prime ideal p of R such that $I \subseteq p$ and $I_p \notin \{0, R_p\}$ and so $(R \bowtie I)_P = R_p \bowtie I_p$, where P is a prime ideal of $R \bowtie I$ such that $P \cap R = p$. Since R_p is a domain (as it is a PF-ring), then $R_p \bowtie I_p$ is reduced and $O_1(= \{0\} \times I_p)$ and $O_2(= I_p \times \{0\})$ are the only minimal prime ideals of $(R \bowtie I)_P$ by [8, Proposition 2.1]; hence it is not a PF-ring by [12, Theorem 4.2.2] (since $(R \bowtie I)_P$ is local), a desired contradiction. Therefore, $I_p \in \{0, R_p\}$ for every prime ideal p of R containing I.
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (1) Assume that R is a PF-ring and $I_m \in \{0, R_m\}$ for every maximal ideal m of R containing I. Our aim is to prove that $R \bowtie I$ is a PF-ring. Using Corollary 2.2, we need to prove that $(R \bowtie I)_M$ is a PF-ring whenever M is a maximal ideal of $R \bowtie I$. Let M be an arbitrary maximal ideal

of $R \bowtie I$ and set $m = M \cap R$. Then, necessarily $M \in \{M_1, M_2\}$, where $M_1 = \{(r, r+i)/r \in m, i \in I\}$ and $M_2 = \{(r+i, r)/r \in m, r \in I\}$, by [7, Theorem 3.5]. On the other hand, $I_m \in \{0, R_m\}$. Then, testing all cases of [6, Proposition 7], we have two cases:
(a) $(R \bowtie I)_M \cong R_m$ if $I_m = 0$ or $I \nsubseteq m$.
(b) $(R \bowtie I)_M \cong R_m \times R_m$ if $I_m = R_m$ and $I \subseteq m$.
Since R_m is a PF-ring (by Corollary 2.2), then so is $R_m \times R_m$ by Proposition 2.5 and hence $(R \bowtie I)_M$ is a PF-ring. □

Corollary 2.12. *Let R be a domain and let I be a nonzero ideal of R. Then $R \bowtie I$ is never a PF-ring.*

Corollary 2.13. *Let (R, m) be a local ring and let I be a nonzero ideal of R. Then $R \bowtie I$ is never a PF-ring.*

Corollary 2.14. *Let R be a ring and let I be a pure ideal of R. Then R is a PF-ring if and only if $R \bowtie I$ is a PF-ring.*

Proof. The sufficient condition holds by Theorem 2.10.
The converse follows immediately from Theorem 2.10.(3) (since I is pure and m is a maximal ideal in R, $I_m \in \{0, R_m\}$ by [12, Theorem 1.2.15]). □

Now we are able to construct a class of PF-rings.

Example 2.15. *Let R be a regular von-Neumann ring and let I be an ideal of R. Then $R \bowtie I$ is a PF-ring by Theorem 2.10.*

Example 2.16. *Let R be a PF-ring and let $I = Re$, where e is an idempotent element of R. Then $R \bowtie I$ is a PF-ring.*

The following example shows that a subring of PF-ring is not always a PF-ring. For any ring R, we denote by $T(R)$ the total ring of quotients of R.

Example 2.17. *Let R be an integral domain, I a nonzero ideal and let $S = R \bowtie I$. Then:*
(1) *$S(= R \bowtie I)$ is not a PF-ring by Corollary 2.12.*
(2) *$R \bowtie I \subseteq R \times R$ and $R \times R$ is a PF-ring by Proposition 2.5 (since R is a PF-ring).*
(3) *$T(S) = T(R \times R) = K \times K$, where $K = T(R)$.*

We end this paper by showing that the transfer of PF-ring property to Pullbacks is not always a PF-ring.

Example 2.18. *Let R be a domain and I a proper ideal of R. Then:*
(1) *The ring $R \bowtie I$ can be obtained as a pullback of R and $R \times R$ over $R \times (R/I)$.*
(2) *The ring $R \bowtie I$ is not a PF-ring by Corollary 2.12.*
(2) *The rings R and $R \times R$ are PF-rings.*

References

1. M.F. Atiyah, I.G. Macdonald; *Introduction to commutative algebra*, Addison-Weslley Publishing Company, (1969).
2. C. Bakkari, S. Kabbaj, and N. Mahdou; *Trivial extensions defined by Prüfer conditions*, J. Pure and Appl. Algebra 214 (2010) 53-60.
3. S. Bazzoni and S. Glaz; *Gaussian properties of total rings of quotients*, J. Algebra 310 (2007), 180-193.
4. J. G. Boynton; *Pullbacks of Prüfer rings*, J. Algebra 320 (6) (2008), 2559-2566.
5. F. Couchot; *Flat modules over valuation rings*, J. Pure and Appl. Algebra 211 (2007) 235-247.
6. M. D'Anna; *A construction of Gorenstein rings*, J. Algebra 306 (2006) 507-519.
7. M. D'Anna and M. Fontana; *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl.6 (2007) 443-459.
8. M. D'Anna and M. Fontana; *An amalgamated duplication of a ring along a multiplicative-canonical ideal*, Arkiv Mat. 6 (2007) 241-252.
9. M. D'Anna, C.A. Finacchiaro, and M. Fontana; *Amalgamated algebras along an ideal*, Commutative Algebra and Applications, Walter De Gruyter, (2009) 155 - 172.
10. M. D'Anna, C.A. Finacchiaro, and M. Fontana; *Properties of chains of prime ideals in amalgamated algebra along an ideal*, J. Pure and Appl. Algebra 214 (2010) 1633-1641.
11. Gerald J. Janusz; *Algebraic number fields*, American Mathematical Society, Vol. 7, (1996).
12. S. Glaz; *Commutative coherent rings*, Springer-Verlag, Lecture Notes in Mathematics, 1371 (1989).
13. J. A. Huckaba; *Commutative Rings with Zero Divizors*, Marcel Dekker, New York Basel, (1988).
14. W. Heinzer, J. Huckaba and I. Papick; *m-canonical ideals in integral domains*, Comm. Algebra 26(1998), 3021-3043.
15. S. Kabbaj and N. Mahdou ; *Trivial Extensions Defined by coherent-like condition*, Comm. Algebra 32 (10) (2004), 3937-3953.
16. H. R. Maimani and S. Yassemi; *Zero-divisor graphs of amalgamated duplication of a ring along an ideal*, J. Pure Appl. Algebra 212 (1) (2008), 168-174.
17. M. Nagata; *Local Rings*, Interscience, New york, (1962).
18. J.J. Rotman; *An introduction to homological algebra*, Academic Press, New York, (1979).

Fatima Cheniour, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.

$E-mail\ address:$ $cheniourfatima@yahoo.fr$

Najib Mahdou, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.

$E-mail\ address:$ $mahdou@hotmail.com$