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ARENS REGULARITY AND MODULE ARENS REGULARITY OF MODULE ACTIONS

KAZEM HAGHNEJAD AZAR

Abstract. In this paper, we extend some problems from Arens regularity and module Arens regularity of Banach algebras to module actions.

1.Introduction and Preliminaries

In [13], for Banach algebras A and $\mathcal{O}$, authors extended the concept of Arens regularity of Banach algebra A to the case that there is an $\mathcal{O}$-module structure of A which is called module Arens regularity of A as $\mathcal{O}$-module. In this note, we study this problem for a right module action $\pi_r : B \times A \to B$ where B is a Banach A-bimodule and we extend some problems from Arens regularity of Banach algebras to the left and right module actions.

Let X, Y, Z be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as follows:

1. $m^* : Z^* \times X \to Y^*$, given by $\langle m^*(z', x), y\rangle = \langle z', m(x, y)\rangle$ where $x \in X$, $y \in Y$, $z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \to X^*$, given by $\langle m^{**}(y'', z'), x\rangle = \langle y'', m^*(z', x)\rangle$ where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $\langle m^{***}(x'', y''), z'\rangle = \langle x'', m^{**}(y'', z')\rangle$ where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.

The mapping m^{***} is the unique extension of m such that $x'' \to m^{***}(x'', y'')$ from X^{**} into Z^{**} is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^* - to - weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{x'' \in X^{**} :\ y'' \to m^{***}(x'', y'')\ is\ weak^* - to - weak^* - continuous\}.$$

Let $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1], if $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$. So we define

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the second topological center of m as

$$Z_2(m)=\{y''\in Y^{**}:\ x''\to m^{t***t}(x'',y'')\ is\ weak^*-to-weak^*-continuous\}.$$

It is clear that m is Arens regular if and only if $Z_1(m)=X^{**}$ or $Z_2(m)=Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i\lim_j\langle z',m(x_i,y_j)\rangle=\lim_j\lim_i\langle z',m(x_i,y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i\subseteq X$, $(y_i)_i\subseteq Y$ and $z'\in Z^*$, see [5, 12].

The regularity of a Banach algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping such as m. Let a'' and b'' be elements of A^{**}, the second dual of A. By *Goldstin's* Theorem [6, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a''=weak^*-\lim_\alpha a_\alpha$ and $b''=weak^*-\lim_\beta b_\beta$. So it is easy to see that for all $a'\in A^*$, we have

$$\lim_\alpha\lim_\beta\langle a',m(a_\alpha,b_\beta)\rangle=\langle a''b'',a'\rangle$$

and

$$\lim_\beta\lim_\alpha\langle a',m(a_\alpha,b_\beta)\rangle=\langle a''ob'',a'\rangle,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of A^{**}, respectively, see [10, 12].

The mapping m is left strongly Arens irregular if $Z_1(m)=X$ and m is right strongly Arens irregular if $Z_2(m)=Y$.

2. The topological centers of module actions

Let B be a Banach $A-bimodule$, and let

$$\pi_\ell:\ A\times B\to B\ and\ \pi_r:\ B\times A\to B.$$

be the left and right module actions of A on B. Then B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{***}:\ A^{**}\times B^{**}\to B^{**}\ and\ \pi_r^{***}:\ B^{**}\times A^{**}\to B^{**}.$$

Similarly, B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{t***t}:\ A^{**}\times B^{**}\to B^{**}\ and\ \pi_r^{t***t}:\ B^{**}\times A^{**}\to B^{**}.$$

We may therefore define the topological centers of the right and left module actions of A on B as follows:

$$\begin{aligned}Z_{A^{**}}(B^{**})=Z(\pi_r)=\{b''\in B^{**}:\ &the\ map\ a''\to\pi_r^{***}(b'',a'')\ :\ A^{**}\to B^{**}\\ &is\ weak^*-to-weak^*\ continuous\}\\ Z_{B^{**}}(A^{**})=Z(\pi_\ell)=\{a''\in A^{**}:\ &the\ map\ b''\to\pi_\ell^{***}(a'',b'')\ :\ B^{**}\to B^{**}\\ &is\ weak^*-to-weak^*\ continuous\}\\ Z^t_{A^{**}}(B^{**})=Z(\pi^t_\ell)=\{b''\in B^{**}:\ &the\ map\ a''\to\pi_\ell^{t***}(b'',a'')\ :\ A^{**}\to B^{**}\\ &is\ weak^*-to-weak^*\ continuous\}\end{aligned}$$

$$Z^t_{B^{**}}(A^{**})=Z(\pi_r^t)=\{a''\in A^{**}:\ the\ map\ b''\rightarrow \pi_r^{t***}(a'',b'')\ :\ B^{**}\rightarrow B^{**}$$
$$is\ weak^*-to-weak^*\ continuous\}$$

We note also that if B is a left(resp. right) Banach $A-module$ and $\pi_\ell:\ A\times B\rightarrow B$ (resp. $\pi_r:\ B\times A\rightarrow B$) is left (resp. right) module action of A on B, then B^* is a right (resp. left) Banach $A-module$.
We write $ab=\pi_\ell(a,b)$, $ba=\pi_r(b,a)$, $\pi_\ell(a_1a_2,b)=\pi_\ell(a_1,a_2b)$, $\pi_r(b,a_1a_2)=\pi_r(ba_1,a_2)$, $\pi_\ell^*(a_1b',a_2)=\pi_\ell^*(b',a_2a_1)$, $\pi_r^*(b'a,b)=\pi_r^*(b',ab)$, for all $a_1,a_2,a\in A$, $b\in B$ and $b'\in B^*$ when there is no confusion.
A functional a' in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a\rightarrow a'a$ from A into A^* is weakly compact. In [12], Pym showed that this definition to the equivalent following condition
For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a\in A:\parallel a\parallel\leq 1\}$, we have

$$lim_\alpha lim_\beta\langle a',a_\alpha b_\beta\rangle=lim_\beta lim_\alpha\langle a',a_\alpha b_\beta\rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on A is denoted by $wap(A)$. Also we have $a'\in wap(A)$ if and only if $\langle a''b'',a'\rangle=\langle a''ob'',a'\rangle$ for every $a'',\ b''\in A^{**}$.
Let B be a Banach left $A-module$. Then, $b'\in B^*$ is said to be left weakly almost periodic functional if the set $\{\pi_\ell^*(b',a)\ :\ \ a\in A,\ \parallel a\parallel\leq 1\}$ is relatively weakly compact. We denote by $wap_\ell(B)$ the closed subspace of B^* consisting of all the left weakly almost periodic functionals in B^*.
The definition of the right weakly almost periodic functional ($=wap_r(B)$) is the same. By [12], $b'\in wap_\ell(B)$ is equivalent to the following

$$\langle\pi_\ell^{***}(a'',b''),b'\rangle=\langle\pi_\ell^{t***t}(a'',b''),b'\rangle$$

for all $a''\in A^{**}$ and $b''\in B^{**}$. Thus, we can write

$$wap_\ell(B)=\{b'\in B^*:\ \langle\pi_\ell^{***}(a'',b''),b'\rangle=\langle\pi_\ell^{t***t}(a'',b''),b'\rangle$$
$$for\ all\ a''\in A^{**},\ b''\in B^{**}\}.$$

Theorem 2-1. Suppose that B is a left Banach $A-module$. Then the following assertions are equivalents.

(1) The mapping $a\rightarrow\pi_\ell^*(b',a)$ from A into B^* is weakly compact.
(2) $Z^\ell_{B^{**}}(A^{**})=A^{**}$.
(3) There are a subset E of B^* with $\overline{lin}E=B^*$ such that for each sequence $(a_n)_n\subseteq A$ and $(b_m)_m\subseteq B$ and each $b'\in B^*$, we have

$$\lim_m\lim_n\langle b',a_nb_m\rangle=\lim_n\lim_m\langle b',a_nb_m\rangle,$$

whenever both the iterated limits exist.

(4) Suppose that $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \overset{w^*}{\to} a''$. Then we have

$$\pi_\ell^*(b', a_\alpha) \overset{w}{\to} \pi_\ell^{t**t}(b', a''),$$

for each $b' \in B^*$.

Proof. (1) $\Rightarrow$ (2)
Suppose that $b' \in B^*$. Take $T(a) = \pi_\ell^*(b', a)$ where $a \in A$. By easy calculation, we have $T^{**}(a'') = \pi_\ell^{****}(b', a'')$ for each $a'' \in A^{**}$. Now let T be a weakly compact mapping. Then by using Theorem VI 4.2 and VI 4.8, from [6], we have $\pi_\ell^{****}(b', a'') \in B^*$ for each $a'' \in A^{**}$. Suppose that $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \overset{w^*}{\to} b''$ on B^{**}. Then for every $a'' \in A^{**}$, we have

$$\langle \pi_\ell^{***}(a'', b''_\alpha), b' \rangle = \langle a'', \pi_\ell^{**}(b''_\alpha, b') \rangle = \langle \pi_\ell^{*****}(b''_\alpha, b'), a'' \rangle$$

$$= \langle b''_\alpha, \pi_\ell^{****}(b', a'') \rangle \to \langle b'', \pi_\ell^{****}(b', a'') \rangle = \langle \pi_\ell^{***}(a'', b''), b' \rangle.$$

It follows that $\pi_\ell^{***}(a'', b''_\alpha) \overset{w^*}{\to} \pi_\ell^{***}(a'', b'')$, and so $a'' \in Z^\ell_{B^{**}}(A^{**})$.
(2) $\Rightarrow$ (1)
Let $Z^\ell_{B^{**}}(A^{**}) = A^{**}$. Suppose that $(b''_\alpha)_\alpha \subseteq B^{**}$ such that $b''_\alpha \overset{w^*}{\to} b''$ in B^{**}. Then for every $a'' \in A^{**}$, we have $\pi_\ell^{***}(a'', b''_\alpha) \overset{w^*}{\to} \pi_\ell^{***}(a'', b'')$. It follows that

$$\langle T^{**}(a''), b''_\alpha \rangle \to \langle T^{**}(a''), b'' \rangle,$$

for each $a'' \in A^{**}$. Consequently, $T^{**}(a'') \in B^*$ for each $a'' \in A^{**}$, and so $T^{**}(A'') \subseteq B^*$. By another using Theorem VI 4.2 and VI 4.8, from [6], we conclude that the mapping $a \to \pi_\ell(b', a)$ from A into B^* is weakly compact.
(2) $\Rightarrow$ (3)
By definition of $Z^\ell_{B^{**}}(A^{**})$, since $Z^\ell_{B^{**}}(A^{**}) = A^{**}$, proof hold.
(3) $\Rightarrow$ (1)
Proof is similar to Theorem 2.6.17 from [5].
(1) $\Rightarrow$ (4)
Let $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \overset{w^*}{\to} a''$. Then for each $b'' \in B^{**}$, we have

$$\lim_\alpha \langle b'', \pi_\ell^*(b', a_\alpha) \rangle = \lim_\alpha \langle \pi_\ell^{**}(b'', b'), a_\alpha \rangle = \langle \pi_\ell^{***}(a'', b''), b' \rangle$$

$$= \langle \pi_\ell^{t***t}(a'', b''), b' \rangle = \langle b'', \pi_\ell^{t**}(a'', b') \rangle.$$

It follows that $\pi_\ell^*(b', a_\alpha) \overset{w}{\to} \pi_\ell^{t**t}(b', a'')$, so this completes the proof.
(4) $\Rightarrow$ (2)
Let $b' \in B^*$ and suppose that $a'' \in A^{**}$ and $b'' \in B^{**}$. Let $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \overset{w^*}{\to} a''$. Since

$$\pi_\ell^*(b', a_\alpha) \overset{w}{\to} \pi_\ell^{t**t}(b', a''),$$

for each $b' \in B^*$, we have the following equality

$$\langle \pi_\ell^{***}(a'', b''), b' \rangle = \langle a'', \pi_\ell^{**}(b'', b') \rangle = \lim_\alpha \langle \pi_\ell^{**}(b'', b'), a_\alpha \rangle$$

$$= \lim_\alpha \langle b'', \pi_\ell^*(b', a_\alpha) \rangle = \langle b'', \pi_\ell^{t**t}(b', a'')$$

$$= \langle \pi_\ell^{t***t}(a'', b''), b' \rangle.$$

It follows that $b' \in wap_\ell(B)$, and so $Z^\ell_{B^{**}}(A^{**}) = A^{**}$. $\square$

Corollary 2-2. Suppose that B is a left Banach $A-module$. Then $B^*A^{**}\subseteq B^*$ if and only if $Z^\ell_{B^{**}}(A^{**})=A^{**}$.

Example 2-3. Suppose that G is a locally compact group. In the preceding corollary, take $A=B=c_0(G)$. Therefore we conclude that $Z^\ell_1(\ell^1(G)^{**})=\ell^1(G)^{**}$, see [5, Example 2.6.22(iii)].

Theorem 2-4. Suppose that B is a right Banach $A-module$. Then the following assertions are equivalents.

(1) $Z^\ell_{A^{**}}(B^{**})=B^{**}$.
(2) The mapping $b\rightarrow\pi_r^*(b',b)$ from B into A^* is weakly compact.
(3) There are a subset E of B^* with $\overline{lin}E=B^*$ such that for each sequence $(a_n)_n\subseteq A$ and $(b_m)_m\subseteq B$ and each $b'\in B^*$,

$$\lim_m\lim_n\langle b',b_ma_n\rangle=\lim_n\lim_m\langle b',b_ma_n\rangle,$$

whenever both the iterated limits exist.

(4) Suppose that $b''\in B^{**}$ and $(b_\alpha)_\alpha\subseteq B$ such that $b_\alpha\overset{w^*}{\rightarrow}b''$ on B^{**}. Then

$$\pi_r^*(b',b_\alpha)\overset{w}{\rightarrow}\pi_r^{t**t}(b',b''),$$

for all $b'\in B^*$.

Proof. Proof is similar to Theorem 2-1. □

Corollary 2-5. Suppose that B is a right Banach $A-module$. Then $B^*B^{**}\subseteq A^*$ if and only if $Z^\ell_{A^{**}}(B^{**})=B^{**}$.

Corollary 2-6. In the preceding corollary, if we take $B=A$, we obtain Lemma 3.1 (i) from [10] and in the Theorems 2-1, if we take $B=A$, we obtain Theorem 2.6.17 from [5].

Definition 2-7. The bilinear mapping $\pi_r:B\times A\rightarrow B$ is called module Arens regular if satisfies in the following conditions:
i) the mapping $T_{b'}:b\rightarrow b'b$ from B into A^* is weakly compact for any $b'\in B^*$ for which the mapping $T_{b'}$ is $A-$module homomorphism.
ii) $T_{b'a}$ from B into A^* is $A-$module homomorphism when $T_{b'}$ is $A-$module homomorphism $b'\in B^*$ and $a\in A$.

Suppose that B is a Banach $A-bimodule$. We assume that J is the closed right ideal of A generated by elements of the form $a_1ba_2 - ba_1a_2$ for all $a_1, a_2 \in A$, $b \in B$.

Theorem 2-8. Take J is defined as above. Then π_r is a module Arens regular if and only if $\pi_r^{***}(b'', a'') - \pi_r^{t****t}(b'', a'') \in J^{\perp\perp}$ for $a'' \in A^{**}$ and $b'' \in B^{**}$.

Proof. The mapping $T_{b'} : b \to b'b$ from B into A^* is an A-module homomorphism if and only if $T_{b'}(ab) = aT_{b'}(b)$ for all $b \in B$ and $a \in A$,

$$\Leftrightarrow b'ab = ab'b,$$

$$\Leftrightarrow < b', abx - bxa >= 0 \ for\ all\ x \in A.$$

In the above statements, if we replace $T_{b'y}$ with $T_{b'}$ where $y \in A$, then the last equality is equivalent with $b' \in J^{\perp}$. Consequently, the bilinear mapping π_r is module Arens regular if and only $T_{b'}$ is weakly compact for any $b' \in J^{\perp}$. By using Theorem 2.4, $T_{b'}$ is weakly compact if and only if for every $a'' \in A^{**}$ and $b'' \in B^{**}$ we have $< \pi_r^{***}(b'', a''), b' >=< \pi_r^{t****t}(b'', a''), b' >$, and so $\pi_r^{***}(b'', a'') - \pi_r^{t****t}(b'', a'') \in J^{\perp\perp}$. This complete the proof. □

Theorem 2-9. Suppose that B is a right Banach $A - module$. Then the following assertions are equivalents.

(1) π_r is module Arens regular.

(2) the mapping $T_{b'} : b \to b'b$ from B into A^* is weakly compact and $T_{b'a}$ from B into A^* is $A-$module homomorphism for which the mapping $T_{b'}$ is $A-$module homomorphism whenever $b' \in B^*$ and $a \in A$.

(3) There are a subset E of B^* with $\overline{lin}E = B^*$ such that for each sequence $(a_n)_n \subseteq A$ and $(b_m)_m \subseteq B$ and each $b' \in J^{\perp}$,
$$\lim_m \lim_n \langle b', b_m a_n \rangle = \lim_n \lim_m \langle b', b_m a_n \rangle,$$
whenever both the iterated limits exist.

(4) the mapping $b'' \to \pi_r^{***}(b'', a'')$ is $\sigma(B^{**}, J^{\perp})-$continuous for every $a'' \in A^{**}$.

Proof. Proof is similar to the preceding theorem and Theorem 2.4. □

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Department of Mathematics, Amirkabir University of Technology, Tehran, Iran
Email address: haghnejad@aut.ac.ir

Department of Mathematics, Amirkabir University of Technology, Tehran, Iran
Email address: riazi@aut.ac.ir