ON MEAGER FUNCTION SPACES, NETWORK CHARACTER AND MEAGER CONVERGENCE IN TOPOLOGICAL SPACES

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Abstract. For a non-isolated point x of a topological space X let $\mathrm{nw}_\chi(x)$ be the smallest cardinality of a family $\mathcal{N}$ of infinite subsets of X such that each neighborhood $O(x)\subset X$ of x contains a set $N\in\mathcal{N}$. We prove that

- each paracompact space X admitting a closed map onto a non-discrete Fréchet-Urysohn space contains a non-isolated point x with $\mathrm{nw}_\chi(x)=\aleph_0$;
- for each point $x\in X$ with $\mathrm{nw}_\chi(x)=\aleph_0$ there is an injective sequence $(x_n)_{n\in\omega}$ in X that $\mathcal{F}$-converges to x for some meager filter $\mathcal{F}$ on ω;
- if a functionally Hausdorff space X contains an $\mathcal{F}$-convergent injective sequence for some meager filter $\mathcal{F}$, then for every T_1-space Y that contains two non-empty open sets with disjoint closures, the function space $C_p(X,Y)$ is meager.

This paper was motivated by a question of the second author who asked if the function space $C_p(\omega^*,2)$ is meager. Here $\omega^*=\beta\omega\setminus\omega$ is the remainder of the Stone-Čech compactification of the discrete space of finite ordinals ω and $2=\{0,1\}$ is the doubleton endowed with the discrete topology. According to Theorem 4.1 of [6] this question is tightly connected with the so-called meager convergence of sequences in ω^*.

A filter $\mathcal{F}$ on ω is *meager* if it is meager (i.e., of the first Baire category) in the power-set $\mathcal{P}(\omega)=2^\omega$ endowed with the usual compact metrizable topology. By the Talagrand characterization [8], a free filter $\mathcal{F}$ on ω is meager if and only if $\xi(\mathcal{F})=\mathfrak{Fr}$ for some finite-to-one function $\xi:\omega\to\omega$. A function $\xi:\omega\to\omega$ is *finite-to-one* if for each point $y\in\omega$ the preimage $\xi^{-1}(y)$ is finite and non-empty. A filter $\mathcal{F}$ on ω is defined to be ξ-*meager* for a surjective function $\xi:\omega\to\omega$ if $\xi(\mathcal{F})=\mathfrak{Fr}$.

We shall say that for a filter $\mathcal{F}$ on ω, a sequence $(x_n)_{n\in\omega}$ of points of a topological space X $\mathcal{F}$-*converges* to a point $x_\infty\in X$ if for each neighborhood $O(x_\infty)\subseteq X$ of x_∞ the set $\{n\in\omega : x_n\in O(x_\infty)\}$ belongs to the filter $\mathcal{F}$. Observe that the usual convergence of sequences coincides with the $\mathfrak{Fr}$-convergence for the Fréchet filter $\mathfrak{Fr}=\{A\subseteq\omega : \omega\setminus A \text{ is finite}\}$ that consists of all cofinite subsets of ω. The filter convergence of sequences has been actively studied both in Analysis [1], [2] and Topology [3]. A sequence $(x_n)_{n\in\omega}$ will be called *meager-convergent* if it is $\mathcal{F}$-convergent for some meager filter $\mathcal{F}$ on ω. A sequence $(x_n)_{n\in\omega}$ is called *injective* if $x_n\neq x_m$ for all $n\neq m$.

We shall prove that the function space $C_p(X,2)$ is meager if X is functionally Hausdorff and contains an injective meager-convergent sequence. We recall that a topological space X is *functionally Hausdorff* if for any distinct points $x,y\in X$ there is a continuous function $\lambda:X\to\mathbb{I}$ such that $\lambda(x)\neq\lambda(y)$. Here $\mathbb{I}=[0,1]$ is the unit interval.

Theorem 1. *Let X be a functionally Hausdorff space and Y be a topological T_1-space that contains two open non-empty subsets with disjoint closures. Assume that X is zero-dimensional or Y is path-connected. If X contains an injective meager-convergent sequence, then the function space $C_p(X,Y)$ is meager.*

Proof. Let $(x_n)_{n\in\omega}$ be a sequence in X that $\mathcal{F}$-converges to $x_\infty\in X$ for some meager filter $\mathcal{F}$ in ω. Then there is a finite-to-one surjection $\xi:\omega\to\omega$ such that $\xi(\mathcal{F})=\mathfrak{Fr}$. By our assumption, Y contains two non-empty open subsets W_0,W_1 with disjoint closures.

For every $n\in\omega$ consider the subset

$$\mathcal{C}_n=\big\{f\in C_p(X,Y):\forall i\in\{0,1\}\ \big(f(x_\infty)\notin\overline{W}_i\Rightarrow\forall m\geq n\ \exists k\in\xi^{-1}(m)\ \ (f(x_k)\notin\overline{W}_i)\big)\big\}.$$

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The meager property of $C_p(X,Y)$ will follow as soon as we check that $C_p(X,Y)=\bigcup_{n\in\omega}\mathcal{C}_n$ and each set $\mathcal{C}_n$ is nowhere dense in $C_p(X,Y)$.

To show that $C_p(X,Y)=\bigcup_{n\in\omega}\mathcal{C}_n$, fix any continuous function $f\in C_p(X,Y)$. Since $Y=(Y\setminus\overline{W}_0)\cup(Y\setminus\overline{W}_1)$, there is $i\in\{0,1\}$ such that $f(x_\infty)\notin\overline{W}_i$. Since (x_n) is $\mathcal{F}$-convergent to x_∞ and $f^{-1}(Y\setminus\overline{W}_i)$ is an open neighborhood of x_∞, the set $F=\{n\in\omega: f(x_n)\notin\overline{W}_i\}$ belongs to the filter $\mathcal{F}$ and thus the image $\xi(F)$, being cofinite in ω, contains the set $\{m\in\omega: m\ge n\}$ for some $n\in\omega$. Then $f\in\mathcal{C}_n$ by the definition of the set $\mathcal{C}_n$.

Next, we show that each set $\mathcal{C}_n$ is nowhere dense in $C_p(X,Y)$. Fix any non-empty open set $\mathcal{U}\subseteq C_p(X,Y)$. Without loss of generality, $\mathcal{U}$ is a basic open set of the following form:

$$\mathcal{U}=\{f\in C_p(X,Y):\forall z\in Z\;\; f(z)\in U_z\}$$

for some finite set $Z\subseteq X$ and non-empty open sets $U_z\subseteq Y$, $z\in Z$. We can additionally assume that $x_\infty\in Z$. We need to find a non-empty open set $\mathcal{V}\subseteq C_p(X,Y)$ such that $\mathcal{V}\subseteq\mathcal{U}\setminus\mathcal{C}_n$. If $\mathcal{U}\cap\mathcal{C}_n$ is empty, then put $\mathcal{V}=\mathcal{U}$. So we assume that $\mathcal{U}\cap\mathcal{C}_n$ contains some function f_0. For this function we can find $i\in\{0,1\}$ such that $f_0(x_\infty)\notin\overline{W}_i$. Since $f_0(x_\infty)\in U_{x_\infty}$, we lose no generality assuming that $U_{x_\infty}\subseteq Y\setminus\overline{W}_i$.

Since the sequence $(x_n)_{n\in\omega}$ is injective, we can find $m\ge n$ such that the set $X_m=\{x_k: k\in\xi^{-1}(m)\}$ does not intersect the finite set Z. Choose any function $g: Z\cup X_m\to Y$ such that $g(z)=f_0(z)$ for all $z\in Z$ and $g(x)\in W_{1-i}$ for all $x\in X_m$.

We claim that the function g has a continuous extension $\bar{g}: X\to Y$. By our assumption, X is zero-dimensional or Y path-connected. In the first case we can find a retraction $r: X\to Z\cup X_m$ and put $\bar{g}=g\circ r$. If Y is path-connected, then take any topological embedding $\phi: g(Z\cup X_m)\to\mathbb{I}$ and extend the function $\phi\circ g: Z\cup X_m\to\mathbb{I}$ to a continuous function $\lambda: X\to\mathbb{I}$ using the functional Hausdorff property of X. Since Y is path-connected, the map $\phi^{-1}:(\phi\circ g)(Z\cup X_m)\to Y$ extends to a continuous map $\psi:\mathbb{I}\to Y$. Then the continuous map $\bar{g}=\psi\circ\lambda: X\to Y$ is a required continuous extension of g.

In both cases the set

$$\mathcal{V}=\{f\in C_p(X,Y):\forall z\in Z\; f(z)\in U_z,\text{ and }\forall x\in X_m\; f(x)\in W_{1-i}\}$$

is an open neighborhood of $\bar{g}$ that lies in $\mathcal{U}\setminus\mathcal{C}_n$, witnessing that the set $\mathcal{C}_n$ is nowhere dense in $C_p(X,Y)$. □

In light of Theorem 1 it is important to detect topological spaces that contains injective meager-convergent sequences. This will be done for spaces containing a point with countable network character.

A family $\mathcal{N}$ of subsets of a topological space X is called a *π-network* at a point $x\in X$ if each neighborhood $O(x)\subset X$ of x contains some set $N\in\mathcal{N}$. If each set $N\in\mathcal{N}$ is infinite, then $\mathcal{N}$ will be called an *i-network* at x. An i-network at x exists if and only if each neighborhood of x in X is infinite. In this case let $\mathrm{nw}_\chi(x;X)$ denote the smallest cardinality $|\mathcal{N}|$ of an i-network $\mathcal{N}$ at x. If some neighborhood of x in X is finite, then let $\mathrm{nw}_\chi(x;X)=1$. If the space X is clear from the context, then we write $\mathrm{nw}_\chi(x)$ instead of $\mathrm{nw}_\chi(x;X)$ and call this cardinal the *network character* of x in X. If X is a T_1-space, then $\mathrm{nw}_\chi(x)\ge\aleph_0$ if and only if the point x is not isolated in X. The cardinal $\mathrm{hnw}_\chi(x)=\sup\{\mathrm{nw}_\chi(x;A): x\in A\subset X\}$ is called the *hereditary network character* at x. Points $x\in X$ with $\mathrm{hnw}_\chi(x)\le\aleph_0$ are called *Pytkeev points*, see [9].

Theorem 2. *If some point x of a topological space X has $\mathrm{nw}_\chi(x)=\aleph_0$, then for each finite-to-one function $\xi:\omega\to\omega$ with $\lim_{n\to\infty}|\xi^{-1}(n)|=\infty$ there is an injective sequence $(x_n)_{n\in\omega}$ in X that $\mathcal{F}$-converges to x for some ξ-meager filter $\mathcal{F}$.*

Proof. Let $(N_i)_{i\in\omega}$ be a countable i-network at x. Since each set N_i is infinite, we can choose an injective sequence $(x_k)_{k\in\omega}$ in X such that for every $n\in\omega$ and $0\le i<|\xi^{-1}(n)|$ the set N_i meets the set $\{x_k: k\in\xi^{-1}(n)\}$.

It is clear that the sequence $(x_n)_{n\in\omega}$ $\mathcal{F}$-converges to x for the filter

$$\mathcal{F}=\big\{\{n\in\omega: x_n\in O(x)\}: O(x)\text{ is a neighborhood of }x\text{ in }X\big\}.$$

It remains to check that the filter $\mathcal{F}$ is ξ-meager. Given any neighborhood $O(x)\subset X$ of x we need to find $n\in\omega$ such that for every $m\ge n$ there is $k\in\xi^{-1}(m)$ with $x_k\in O(x)$. Since $(N_i)_{i\in\omega}$ is a network at x, there is $i\in\omega$ such that $N_i\subset O(x)$. Taking into account that $\lim_{n\to\infty}|\xi^{-1}(n)|=\infty$, find $n\in\omega$ such that $|\xi^{-1}(m)|>i$ for all $m\ge n$. Now the choice of the sequence (x_k) guarantees that for every $m\ge n$ there is $k\in\xi^{-1}(m)$ with $x_k\in N_i\subset O(x)$. □

In light of Theorem 2 it is important to detect points x with countable network character $\mathrm{nw}_\chi(x)$. Let us recall that the *character* $\chi(x)$ (resp. the *π-character* $\pi\chi(x)$) of a point x in a topological space X is equal to the smallest cardinality of a neighborhood base (resp. a π-base) at x. A π-*base* at x is any π-network at x consisting of non-empty open subsets of X. These definitions imply the following simple:

Proposition 1. *For any non-isolated point x of a T_1-space X,*

(1) $\mathrm{nw}_\chi(x) \le \chi(x)$*;*
(2) $\mathrm{nw}_\chi(x) \le \pi\chi(x)$ *provided that x has a neighborhood containing no isolated point of X;*
(3) $\mathrm{nw}_\chi(x) = \aleph_0$ *if x is a limit of an injective sequence in X.*

The following simple example shows that the usual convergence of the injective sequence in Proposition 1(3) cannot be replaced by the meager convergence. Also it shows that Theorem 2 cannot be reversed.

Example 1. Let $\mathcal{F}$ be the meager filter on ω consisting of the sets $F \subset \omega$ such that

$$\lim_{n\to\infty} \frac{|F \cap [2^n, 2^{n+1})|}{2^n} = 1.$$

On the space $X = \omega \cup \{\omega\}$ consider the topology in which all points $n \in \omega$ and isolated while the sets $F \cup \{\omega\}$, $F \in \mathcal{F}$, are neighborhoods of ∞. It is clear that the sequence $x_n = n$, $n \in \omega$, $\mathcal{F}$-converges to ∞ in X. On the other hand, a simple diagonal argument shows that $\mathrm{nw}_\chi(\infty; X) > \aleph_0$.

A continuous map $f : X \to Y$ between two topological spaces is called *irreducible* (*at a point* x) if $f(X) = Y$ but $f(A) \ne Y$ for each closed subset $A \subsetneq X$ (with $x \notin A$).

Proposition 2. *Assume that a closed surjective map $f : X \to Y$ between topological T_1-spaces is irreducible at a point $x \in X$ and let $y = f(x)$. Then $\pi\chi(x) \le \pi\chi(y)$ and $\mathrm{nw}_\chi(x) \le \mathrm{nw}_\chi(y)$.*

Proof. If the point y is isolated, then so is the point x and hence $\mathrm{nw}_\chi(x) = 1 = \mathrm{nw}_\chi(y)$.

So, we assume that y is not isolated in Y. Let $\mathcal{N}_Y$ be an i-network at y with $|\mathcal{N}_Y| = \mathrm{nw}_\chi(y)$. We claim that the family $\mathcal{N}_X = \{f^{-1}(N) : N \in \mathcal{N}_Y\}$ is an i-network at x. Since f is surjective, each set $f^{-1}(N)$, $N \in \mathcal{N}_Y$, is infinite.

Given an open neighborhood $O(x) \subset X$ of x, consider the closed set $X \setminus O(x)$. Since f is closed and irreducible at x, the image $f(X \setminus O(x))$ is a closed subset in Y that does not contain the point y. Then its complement $O(y) = Y \setminus f(X \setminus O(x))$ is an open neighborhood of y. Since $\mathcal{N}_Y$ is a network at y, there is a set $N \in \mathcal{N}_Y$ with $N \subset O(y)$. Then the preimage $f^{-1}(N)$ lies in $O(x)$ and witnesses that $\mathcal{N}_X$ is an i-network at x and hence $\mathrm{nw}_\chi(x) \le |\mathcal{N}_X| = |\mathcal{N}_Y| = \mathrm{nw}_\chi(y)$.

The proof of the inequality $\pi\chi(x) \le \pi\chi(y)$ is analogous. □

Theorem 3. *A paracompact space X contains a point $x \in X$ with $\mathrm{nw}_\chi(x) = \aleph_0$ provided that X admits a closed map $f : X \to Y$ onto a non-discrete Fréchet-Urysohn space Y.*

Proof. By the Lašnev Theorem [5] (see also [4, 5.5.12]), there is a closed subset $A \subset X$ such that the restriction $f|A : A \to Y$ is irreducible. The space Y, being non-discrete and Fréchet-Urysohn, contains an injective sequence that converses to some point $y \in Y$. Consequently, this point has $\mathrm{nw}_\chi(y) = \aleph_0$. Take any point $x \in A$ with $f(x) = y$. Since $f|A : A \to Y$ is closed and irreducible, x is not isolated in A and hence $\mathrm{nw}_\chi(x) \ge \aleph_0$. Now Proposition 2 implies that $\mathrm{nw}_\chi(x) = \mathrm{nw}_\chi(y) = \aleph_0$. □

Since each infinite compact Hausdorff space admits a closed map onto an infinite metric compact space, Theorems 3 implies:

Corollary 1. *Each infinite compact space X contains a point $x \in X$ with $\mathrm{nw}_\chi(x) = \aleph_0$.*

Theorems 1 and 3 imply:

Corollary 2. *If a zero-dimensional paracompact space X admits a closed map onto a non-discrete Fréchet-Urysohn space, then for each T_1-space Y containing two non-empty open sets with disjoint closures the function space $C_p(X, Y)$ is meager. In particular, $C_p(\omega^*, 2)$ is meager.*

Finally, we show that the condition $\lim_{n\to\infty} |\xi^{-1}(n)| = \infty$ in Theorem 2 can not be weakened.

Let us recall that an infinite subset $A \subseteq \omega$ is called a *pseudointersection* of a filter $\mathcal{F}$ on ω if $A \subseteq^* F$ for all $F \in \mathcal{F}$ where $A \subseteq^* F$ means that $A \setminus F$ is finite. If a sequence $(x_n)_{n\in\omega}$ in a topological space $\mathcal{F}$-converges to a point x_∞ for some filter $\mathcal{F}$ with infinite pseudointesection $A \subseteq \omega$ then the subsequence $(x_k)_{k\in A}$ converges to x_∞ in the standard sense.

Lemma 1. *Let I be a countable set and $C = \bigcup_{i\in I} C_i$, where the sets C_i are nonempty and mutually disjoint, and $\sup_{i\in I} |C_i| < \omega$. If $\mathcal{H}$ is a filter on C all of whose elements intersect all but finitely many C_i's, then $\mathcal{H}$ has an infinite pseudointersection.*

Proof. The proposition will be proved by induction on $n = \sup_{i\in I} |C_i|$. If $n = 1$ there is nothing to prove. Suppose that it is true for all $k < n$ and let I, $\{C_i : i \in I\}$, $\mathcal{H}$ be as above with $\max\{|C_i| : i \in I\} = n$. If for every $H \in \mathcal{H}$ the set $\{i \in I : |C_i \cap H| < n\}$ is finite, then C itself is a pseudointersection of $\mathcal{H}$. So suppose that $J = \{i \in I : |C_i \cap H_0| < n\}$ is infinite for some $H_0 \in \mathcal{H}$. In this case we may use our inductive hypothesis for J, $\{C_i \cap H_0 : i \in J\}$, $\mathcal{G} = \mathcal{H} \restriction (\bigcup_{i\in J} C_i \cap H_0)$, and $n - 1$. Thus $\mathcal{G}$ has an infinite pseudointersection, and hence so does $\mathcal{H}$. □

Proposition 3. *If $\mathcal{F}$ is a ξ-meager filter on ω for some surjective function $\xi : \omega \to \omega$ with $\underline{\lim}_{n\to\infty}|\xi^{-1}(n)| < \infty$, then any sequence $(x_n)_{n\in\omega}$ in a topological space X that $\mathcal{F}$-converges to a point $x_\infty \in X$ contains a subsequence $(x_{n_k})_{k\in\omega}$ that converges to x_∞.*

Proof. Choose infinite set $I \subseteq \omega$ such that $\sup_{i\in I} |\xi^{-1}(i)| < \omega$. Let $C_i = \xi^{-1}(i)$ for every $i \in I$, $C = \bigcup_{i\in I} C_i$ and $\mathcal{H} = \{F \cap C : F \in \mathcal{F}\}$. According to Lemma 1 there exists an infinite set $D \subseteq C$ such that $D \subseteq^* H$ for every $H \in \mathcal{H}$. Then the subsequence $(x_i)_{i\in D}$ converges to x_∞. □

Thus Theorem 2 is not true for any infinite compact Hausdorff space $X \subseteq \omega^*$ and any finite-to-one function with $\underline{\lim}_{n\to\infty}|\xi^{-1}(n)| < \infty$ because X contains no non-trivial converge sequence.

Remark 1. After writing this paper the authors learned from V.Tkachuk that the meager property of the function space $C_p(\omega^*, 2)$ was also established by E.G. Pytkeev in his Dissertation [7, 3.24].

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References

[1] A. Aviles, B. Cascales, V. Kadets, A. Leonov, *The Schur l_1 theorem for filters*, Zh. Mat. Fiz. Anal. Geom. **3**:4 (2007), 383–398.
[2] M. Ganichev, V. Kadets, *Filter convergence in Banach spaces and generalized bases*, in: General topology in Banach spaces (T.Banakh ed.), Nova Sci. Publ., Huntington, NY, 2001. p.61–69.
[3] S. García-Ferreira, V. Malykhin, A. Tamariz-Mascarúa, *Solutions and problems on convergence structures to ultrafilters*, Questions Answers Gen. Topology **13**:2 (1995), 103–122.
[4] R. Engelking, *General topology*, Heldermann Verlag, Berlin, 1989.
[5] N. Lašnev, *On continuous decompositions and closed mappings of metric spaces,* Dokl. Akad. Nauk SSSR **165** (1965), 756–758 (in Russian).
[6] V.Mykhaylyuk, *On questions connected with Talagrand's problem*, Mat. Studi. **29** (2008) 81–88.
[7] E.G.Pytkeev, *Spaces of continuous and Baire functions in weak topologies*, Doktor Sci. Dissertation, Ekaterinburg, 1993 (in Russian).
[8] M. Talagrand, *Compacts de fonctions mesurables et filtres non mesurables*, Studia Math. **67**:1 (1980), 13–43.
[9] V.I. Malykhin, G. Tironi, *Weakly Fréchet-Urysohn and Pytkeev spaces*, Topology Appl. **104** (2000) 181–190.

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