SECANT DEGREE OF TORIC SURFACES AND DELIGHTFUL PLANAR TORIC DEGENERATIONS

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Abstract. The k-secant degree is studied with a combinatorial approach. A planar toric degeneration of any projective toric surface X corresponds to a regular unimodular triangulation D of the polytope defining X. If the secant ideal of the initial ideal with respect to D coincides with the initial ideal of the secant ideal, then D is said to be delightful and the k-secant degree of X can be easily computed. All delightful triangulations of toric surfaces having sectional genus $g \leq 1$ are completely classified and, for $g \geq 2$, a lower bound for the 2- and 3-secant degree, by means of the combinatorial geometry and the singularities of non-delightful triangulations, is established.

Introduction

There is a long tradition within algebraic geometry that studies the dimension and the degree of k-secant varieties. Let $X \subseteq \mathbb{P}^r$ be a projective, irreducible variety of dimension n. Its *k-secant variety* $\mathrm{Sec}_k(X)$ is defined to be the closure of the union of all the $\mathbb{P}^{k-1}$'s in $\mathbb{P}^r$ meeting X in k independent points. If $\mathrm{Sec}_k(X)$ has the expected dimension $kn+k-1$, what is the number $\nu_k(X)$ of k-secant $\mathbb{P}^{k-1}$'s to X intersecting a general subspace of codimension $kn+k-1$ in $\mathbb{P}^r$? This is a problem which is unsolved in general.

Our approach to the problem of computing the number ν_k for toric varieties is the one of Ciliberto, Dumitrescu and Miranda [6] that is close to that of Sturmfels and Sullivant [16]. Given a projective toric surface X, we perform *planar toric degenerations*, i.e., we consider *regular unimodular triangulations* D of the polytope P which defines X. The ideal $\mathcal{I}_0$ of the central fiber is the monomial initial ideal of the ideal $\mathcal{I}_X$ of X with respect to a suitable term order $\prec$ which corresponds to the triangulation D (see [15, Theor. 8.3]): $\mathcal{I}_0 = \mathrm{in}_\prec(\mathcal{I}_X)$.

In Section 1 and Section 2 we introduce the objects of our study: convex lattice polytopes, toric varieties, toric degenerations and k-secant varieties, with particular attention to the problem of computing the k-secant degree of toric surfaces.

In Section 3 we introduce the notion of k-delightful planar toric degenerations of toric varieties: if the k-secant ideal of the initial ideal $\mathcal{I}_0$ of X with respect to the degeneration coincides with the initial ideal of the k-secant ideal of X, then the degeneration is k-delightful. Sturmfels and Sullivant proved in [16, Theor. 5.4] that if there exists a triangulation D of the polytope P defining X with at least one *skew k-set*, i.e., a subset of k triangles of D that are pairwise disjoint, then the k-secant variety of X has the expected dimension. Moreover the number of such skew k-sets is a lower bound for the number $\nu_k(X)$, see Theorem 3.2.

2010 *Mathematics Subject Classification.* Primary 14M25; Secondary 14D06, 51N35.

Key words and phrases. Toric varieties, secant varieties, degenerations, polytopes, delightful triangulations.

The author was partially supported by Marie-Curie IT Network SAGA, [FP7/2007-2013] grant agreement PITN-GA- 2008-214584.

If equality holds, then D is k-delightful and the flat limit of the k-secant variety is a union of linear subspaces of dimension $kn+k-1$, hence the k-secant degree is computed. This bound is almost never sharp, indeed k-delightful degenerations are rare.

In Section 4 we approach the secant degree computation and we give a lower bound for ν_k, for $k=2,3$. The main tool is keeping into account the singularities of the configuration D and explaining how they produce k-delightfulness defect. Our results can be regarded as the beginning of a similar study for the k-secant varieties of toric surfaces for $k \geq 4$ and, in higher dimension, for $k \geq 2$.

The problem of finding delightful triangulations of polytopes was raised by Sturmfels and Sullivant [16, Sect. 5]. They explored the existence of such triangulations for Veronese varieties, Segre varieties and rational normal scrolls. In Section 5 we provide a classification of all delightful triangulations for toric surfaces with sectional genus 0 and 1.

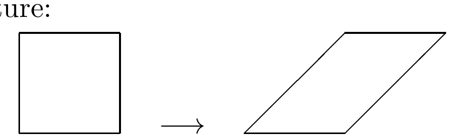
1. Convex lattice polytopes and toric varieties

1.1. **Census of polytopes in $\mathbb{R}^2$ with $g \leq 1$.** A lattice point in $\mathbb{R}^n$ is a point with integral coordinates. A lattice polytope in $\mathbb{R}^n$ is a polytope whose vertices are lattice points. The *normalized Ehrhart polynomial* of a lattice polytope P in $\mathbb{R}^n$ is the numerical function $E_P : \mathbb{N} \to \mathbb{N}$, $t \mapsto \#(tP \cap \mathbb{Z}^n)$. It is known that E_P is a polynomial of degree $\dim(P)$: $E_P = \sum_{i=0}^{\dim(P)} \frac{c_i}{i!} t^i$. The leading coefficient $c_{\dim(P)}$ is denoted by $\mathrm{Vol}(P)$ and it is called the *(normalized) volume* of P. If $\dim(P) = n$, we have $\mathrm{Vol}(P) = n! \cdot V(P)$, where $V(P)$ is the usual Euclidean volume of P (see [15, p. 36]). If $\dim(P) = 1$, $Vol(P) + 1$ turns out to be equal to the number of lattice points enclosed by P. If $\dim(P) = n = 2$, we denote by $Area(P)$ the normalized volume of P.

Set $n = 2$ and denote by g the number of interior lattice points of a plane polytope. In this section we recall the classification of all convex lattice polytopes in $\mathbb{R}^2$ with $g \leq 1$, due to Rabinowitz [14]. To this end, we need to define an equivalence relation between planar polytopes (see [9, p. 18] or [14, p. 1]). An integral unimodular affine transformation, also known as an *equiaffinity*, in the plane is a linear transformation followed by a translation such that, furthermore, the corresponding matrix has determinant 1 and integral entries. For example the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

acts on a polytope by sending the point $(x,y)^T \in \mathbb{R}^2$ to the point $(x+y,y)^T \in \mathbb{R}^2$: the points on the x-axis are fixed, while the points on the axis $y = k$ are shifted by k on the right as for example in the picture:

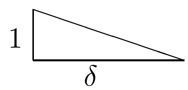


Normalized area, number of lattice points and convexity of a plane polyotope are preserved under these transformations. Two plane polytopes are said to be *lattice equivalent* if one can be transformed into the other via an equiaffinity, look for example to the above picture.

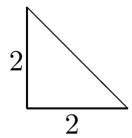
We will refer to [14] for the proofs of the following results.

Lemma 1.1 (The x-axis Lemma). *Let q_1, q_2 be the vertices of an edge of length m of a polytope. There exists an equiaffinity that maps q_1 into the origin, maps q_2 into the point $(m, 0)$ on the positive x-axis, and maps all the other vertices into points above the x-axis.*

Theorem 1.2 (Characterization of polytopes with no interior lattice point)**.** *If P is a polytope with $g = 0$, then P is lattice equivalent to one of the following:*



$\delta \geq 1$,



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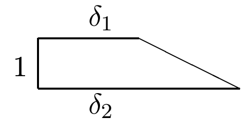
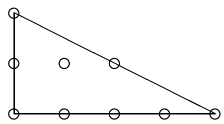


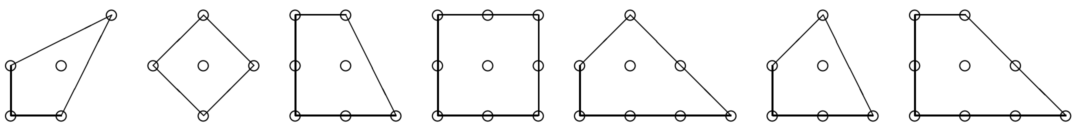
$\delta_2 \geq \delta_1 \geq 1$.

Theorem 1.3 (Characterization of polytopes with one interior lattice point)**.** *If P is a polytope with $g = 1$, then P is lattice equivalent to one of the following:*

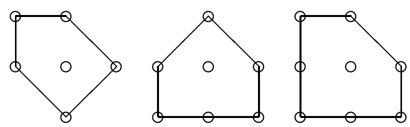
- *Triangles:*



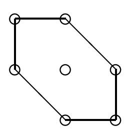
- *Quadrilaterals:*



- *Pentagons*



- *Hexagons*



The last quadrilateral was missing in the published paper [14] and was later added to the classification.

We will use the notation $P^g(l, d, m)$ for these polytopes, where g is the number of interior lattice points, l is the number of edges (or vertices), d is the normalized area and m is the normalized maximal edge length. Actually we will denote in this way both the equiaffinity class and the representatives of the class, each time specifying what representative we are dealing with. The two quadrilaterals with $g = 1$ and $l = d = 4$ are not distinguished by this notation, because they both have $m = 1$. So one could write $P^1(4, 4, 1)$ for the first one and $\tilde{P}^1(4, 4, 1)$ for the second one, but actually it does not matter since we will not deal with them in this paper.

1.2. **Toric varieties via polytopes and toric degenerations.** A convex lattice polytope P in $\mathbb{R}^n$ defines a *toric variety* X_P of dimension n endowed with an ample line bundle $\mathscr{L}$ and therefore a morphism in $\mathbb{P}^r$, where $r + 1$ equals the number of lattice points of P. Let $P \cap \mathbb{Z}^n = \{\underline{m}_0, \ldots, \underline{m}_r\}$ be the set of the lattice points of P, with $\underline{m}_i = (m_{i1}, \ldots, m_{in})$, $i = 0, \ldots, r$. Consider the monomial map

$$\begin{array}{rccc} \Phi_P: & (\mathbb{C}^*)^n & \to & \mathbb{P}^r \\ & \underline{x} & \mapsto & [\underline{x}^{\underline{m}_0}, \ldots, \underline{x}^{\underline{m}_r}] \end{array}$$

where $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{x}^{\underline{m}_i} = x_1^{m_{i1}} \cdots x_n^{m_{in}}$. The projective toric variety $X_P \in \mathbb{P}^r$ is defined to be the closure of the image of Φ_P. The degree of X_P equals the normalized volume $\mathrm{Vol}(P)$. Lattice equivalent polytopes in $\mathbb{R}^2$ define the same toric surface.

A *subdivision* D of P is a partition of P given by a finite family $\{Q_i\}_{i \in I}$ of convex sub-polytopes of maximal dimension such that

- $\bigcup_{i \in I} Q_i = P$,
- $Q_i \cap Q_j$, with $i \neq j$, is either a common face or it is empty.

A subdivision D is said to be *regular* if there exists a piecewise linear positive function F with values in $\mathbb{R}$ defined over P, verifying the following requests:

- each Q_i is the orthogonal projection of the n-dimensional faces of the graph polytope $G(F) := \{(x, z) \in P \times \mathbb{R} : 0 \leq z \leq F(x)\}$ of F on $z = 0$;
- F is *strictly convex*.

We will call such an F a *lifting function* as in [10]. Given a regular subdivision D of P, we define the associated morphism as follows:

$$\begin{array}{llll} \Phi_D : & (\mathbb{C}^*)^n \times \mathbb{C}^* & \to & \mathbb{P}^r \times \mathbb{C} \\ & (\underline{x}, t) & \mapsto & ([t^{F(\underline{m}_0)} \underline{x}^{\underline{m}_0} : \cdots : t^{F(\underline{m}_r)} \underline{x}^{\underline{m}_r}], t) \end{array} \tag{1.4}$$

The closure of $\Phi_D((\mathbb{C}^*)^n \times \{t\})$, for all $t \neq 0$, is a variety X_t projectively equivalent to X_P. Let X_0 be the flat limit of X_t, when t tends to zero: such a variety is the union of the varieties X_{Q_i}, $i \in I$. Indeed, the restriction $F_{|Q_i}$ of F to Q_i has equation $a_1 x_1 + \cdots a_n x_n + b$, for some $a_1, \ldots, a_n, b \in \mathbb{R}$; we can always compose Φ_D with a reparametrization action of the torus $\mathbb{C}^*$, $x_1, \ldots, x_n, t \mapsto t^{-a_1} x_1, \ldots, t^{-a_n} x_n, t$, getting

$$\begin{array}{lll} (\mathbb{C}^*)^{n+1} & \to & \mathbb{P}^r \times \mathbb{C} \\ (\underline{x}, t) & \mapsto & ([\cdots : t^{F(\underline{m}_i) - F_{Q_i}(\underline{m}_i)} \underline{x}^{\underline{m}_i} : \cdots], t). \end{array}$$

By letting $t \to 0$, one sees that X_{Q_i} sits in the flat limit X_0 of X_t. The map (1.4) can be extended to a map

$$\begin{array}{lll} X_P \times \mathbb{C}^* & \to & \mathbb{P}^r \times \mathbb{C} \\ (\underline{x}, t) & \mapsto & ([t^{F(\underline{m}_0)} \underline{x}^{\underline{m}_0} : \cdots : t^{F(\underline{m}_r)} \underline{x}^{\underline{m}_r}], t) \end{array}$$

and the flat morphism $\pi_D : ([t^{F(\underline{m}_0)} \underline{x}^{\underline{m}_0} : \cdots : t^{F(\underline{m}_r)} \underline{x}^{\underline{m}_r}], t) \mapsto t$ provides a 1-dimensional embedded degeneration of X to X_0. π_D is said to be a *toric degeneration* of the toric variety X_P and we will use the notation $X_0 = \lim_D X$. The reducible central fiber X_0 is given by the subdivision D of P: the irreducible components of X_0 are the X_{Q_i}'s. Notice that if $i \neq j$ and Q_i and Q_j have a common face $Q_i \cap Q_j$, then X_{Q_i} and X_{Q_j} intersect along $X_{Q_i \cap Q_j}$.

If $n = 2$ and the reducible central fiber X_0 is a union of planes, i.e. if the subdivision D of the polytope P is a regular unimodular triangulation of it, we say that π_D is a *planar toric degeneration* of X_P. In this case the family D of sub-polytopes of P is a simplicial complex, whose maximal simplices are the Q_i's. The notion of toric degeneration to union of $\mathbb{P}^n$'s leads to the notion of term order. In fact there is a one-to-one correspondence between regular triangulations and term orders. Let $\prec$ be any term order in $\mathbb{C}[x_0, \ldots, x_r]$ and let $\mathcal{I}_0 := \mathrm{in}_\prec(\mathcal{I})$ be the initial ideal of the ideal $\mathcal{I}$ of X. The radical of $\mathcal{I}_0$ is a squarefree monomial ideal whose corresponding simplicial complex $\Delta_\prec(\mathcal{I}_0)$ is a regular triangulation of the polytope P defining X. Conversely any regular triangulation of P is of that form, for some $\prec$, see [15, Theor. 8.3].

2. Secant varieties

Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective variety of dimension n. Fix an integer $k \geq 2$ and consider the k-th symmetric product $\mathrm{Sym}^k(X)$. We define the *abstract k-th secant variety* of X, $S^k_X \subseteq \mathrm{Sym}^k(X) \times \mathbb{P}^r$, as the Zariski closure of the set

$$\{((x_1, \dots, x_k), z) \in \mathrm{Sym}^k(X) \times \mathbb{P}^r : \dim(\pi) = k-1 \text{ and } z \in \pi\}$$

where $\pi = \langle x_1, \dots, x_k \rangle$. It is irreducible of dimension $kn + k - 1$. Consider the projection p^k_X on the second factor and define the *k-th secant variety* of X, $\mathrm{Sec}_k(X) := p^k_X(S^k_X)$, as the image of S^k_X in $\mathbb{P}^r$. It is an irreducible algebraic variety of dimension $\dim(\mathrm{Sec}_k(X)) \leq \min\{kn + k - 1, r\}$. The right hand side is called the *expected dimension* of $\mathrm{Sec}_k(X)$. If strict inequality holds, X is said to be *k-defective.*

The general fiber of p^k_X is pure of dimension $kn+k-1-\dim(\mathrm{Sec}_k(X))$. Denote by $\mu_k(X)$ the number of irreducible components of this fiber. If $\dim(\mathrm{Sec}_k(X)) = kn+k-1 \leq r$, then p^k_X is generically finite and $\mu_k(X) = \deg(p^k_X)$, i.e., $\mu_k(X)$ is the number of k-secant $\mathbb{P}^{k-1}$'s to X passing through the general point of $\mathrm{Sec}_k(X)$ and it is called the *k-secant order* of X, see [5]. This number is equal to one unless X is *k-weakly defective.* The weakly defective surfaces are classified in [4]. Let L be a general linear subspace of $\mathbb{P}^r$ of codimension $kn + k - 1$: X has

$$\nu_k(X) = \mu_k(X) \cdot \deg(\mathrm{Sec}_k(X))$$

k-secant $\mathbb{P}^{k-1}$'s meeting L. Let π_L be the projection of X from L to $\mathbb{P}^{kn+k-2}$: the image of X has $\nu_k(X)$ new k-secant $\mathbb{P}^{k-2}$'s that X did not have. The number $\nu_k(X)$ is called the *number of apparent k-secant $\mathbb{P}^{k-2}$'s to X.* In particular $\nu_2(X)$ corresponds to the number of double points that X acquires in a general projection to $\mathbb{P}^{2n}$, $\nu_3(X)$ is the number of trisecant lines in a general projection of X to $\mathbb{P}^{3n+1}$ and so on. Notice that if $\nu_k(X) = 1$, then $\mathrm{Sec}_k(X) = \mathbb{P}^r$ and $\mu_k(X) = 1$ which means that for a general points of $\mathrm{Sec}_k(X)$ there is a unique k-secant $\mathbb{P}^{k-1}$.

Let X be a smooth surface. Severi's *double point formula* gives the number of nodes of a general projection of X to $\mathbb{P}^4$:

$$\nu_2(X) = \frac{d(d-5)}{2} - 5g + 6p_a - K^2 + 11,$$

where d is the degree, g is the sectional genus, p_a is the arithmetic genus and K is the canonical divisor of X. In particular, if $X = X_P$ is a projective toric surface, then

$$\nu_2(X) = \frac{1}{2}(d^2 - 10d + 5B + 2V - 12),$$

where d is the normalized area of the polytope P, B is the number of lattice points on the boundary and V is the number of vertices of P, see [8, Cor. 1.6].

If X is a surface not containing lines, a formula for $\nu_3(X)$, known as *LeBarz' trisecant formula for surfaces in* $\mathbb{P}^7$ (see [11, p. 7] or [12, p. 202]), is

$$\nu_3(X) = \frac{1}{6}(d^3 - 30d^2 + 224d - 3d(5HK + K^2 - c_2) + 192HK + 56K^2 - 40c_2)$$

where H is the hyperplane divisor and c_2 is the second Chern class of X. Moreover, if X contains a finite number of lines, the contribution of each line to $\nu_2(X)$ is $-\binom{4+a}{3}$, where $a \in \mathbb{Z}$ is its self-intersection. There are similar, but more complicated, formulas for the number $\nu_k(X)$ in the curve case (see [1, Chapt. VIII]), and in the surface case, if X does not contain any line, for $k \leq 5$ (see [11, 12]).

Unfortunately, the Severi's formula for $\nu_2(X)$ does not apply if X is a singular surface. Moreover, in order to apply the formulas for $\nu_k(X)$, $k \geq 3$, one needs to know how many lines are contained in X. In this paper we present a combinatorial framework for the study of the k-secant varieties to any projective toric surface, that makes the computation of ν_2 and ν_3 easier.

2.1. **The k-secant degree of toric surfaces with $g \leq 1$.** In this section we will deal with the toric surfaces defined by the polytopes of Theorem 1.2 and Theorem 1.3. They are all minimal k-secant degree surfaces, $\mathcal{M}^k$-surfaces (see [7]), i.e.

$$\deg(\mathrm{Sec}_k(X)) = \binom{r - \dim(\mathrm{Sec}_k(X)) + k}{k}.$$

2.1.1. *$g = 0$.* The Veronese surface V_2 in $\mathbb{P}^5$ is described by the triangle $P^0(3, 4, 2)$. Its 2-secant variety is a hypersurface of degree 3. Moreover $\mathrm{Sec}_k(V_2) = \mathbb{P}^5$, $k \geq 3$.

Consider the rational normal surface scroll $S = S(\delta_1, \delta_2) \subseteq \mathbb{P}^{\delta_1+\delta_2+1}$, $\delta_1 \leq \delta_2$, whose polytope is either the triangle $P^0(3, \delta_2, \delta_2)$ or the trapezium $P^0(4, \delta_1 + \delta_2, \delta_2)$. If $k \leq \delta_1$ and $3k - 1 \leq \delta_1 + \delta_2 + 1$ then S is non k-defective and has minimal k-secant degree, namely $\deg(\mathrm{Sec}_k(S)) = \binom{\delta-2k+2}{k}$ and $\mu_k(S) = 1$, $k \geq 2$. The ideal of these surfaces is generated by the 2×2-minors of a Hankel matrix. A determinantal presentation for the ideals of their k-secant varieties is known, see [2, Prop. 2.2].

2.1.2. *$g = 1$.* The k-secant varieties of the three quartic toric surfaces in $\mathbb{P}^4$ defined by $P^1(3, 4, 2)$, $P^1(4, 4, 1)$ and $\tilde{P}^1(4, 4, 1)$ fill up $\mathbb{P}^4$, for each $k \geq 2$.

Let V_3 be the 3-ple Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$, described by the polytope $P^1(3, 9, 3)$. It is well known that it is non k-defective and is minimal k-secant degree for $k = 2, 3$. In particular $\mathrm{Sec}_2(V_3)$ has dimension 5 and degree 15, while $\mathrm{Sec}_3(V_3)$ has dimension 8 and degree 4. Moreover $\mathrm{Sec}_k(V_3) = \mathbb{P}^9$, $k \geq 4$.

The i-internal projections of V_3, i.e., the surfaces obtained from V_3 as projections from i general points on it, $1 \leq i \leq 4$, are del Pezzo surfaces of degree $9 - i$ in $\mathbb{P}^{9-i}$. They are the ones defined by the subpolytopes of $P^1(3, 9, 3)$: $P^1(4, 8, 3)$, $P^1(4, 7, 3)$, $P^1(5, 7, 2)$, $P^1(3, 6, 3)$, $P^1(4, 6, 2)$, $P^1(5, 6, 2)$, $P^1(6, 6, 1)$, $P^1(4, 6, 2)$, $P^1(5, 5, 1)$. For $k = 2$, we have $\dim(\mathrm{Sec}_2(X)) = 5$ and $\nu_2(X) = \binom{d-3}{2}$. For $k \geq 3$, $\mathrm{Sec}_3(X) = \mathbb{P}^{9-i}$. In particular for the del Pezzo surface of degree 8 in $\mathbb{P}^8$, that corresponds to $P^1(4, 8, 3)$, we have $\nu_3(X) = 1$. All of them have ideals which are generated by quadrics and given by the 2×2 minors of a known matrix. Also the k-secant varieties, for $k = 2, 3$, have a nice determinantal presentation: the equations are given by the $(k+1) \times (k+1)$ minors of the same matrix. For an overview see [3, 13].

Let now $X, Y \subseteq \mathbb{P}^8$ be respectively the embedding of the smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ and of the cone in $\mathbb{P}^3$ over a rational normal conic via the 2-Veronese embedding. They correspond to $P^1(4, 8, 2)$ and $P^1(3, 8, 4)$ respectively. They both have 2-secant variety of dimension 5 and degree 10. Moreover for both of them, the 3-secant variety has dimension 7 and degree 4, see [7, Theor. 9.1].

3. k-delightful planar toric degenerations

Let $\mathcal{I}$ be an ideal in the polynomial ring $K[x_0, \ldots, x_r]$. The *secant* $\mathcal{I}^{\{2\}} = \mathcal{I} * \mathcal{I}$ of $\mathcal{I}$ is an ideal in $K[x_0, \ldots, x_r]$ defined in the following way: take the polynomial ring $K[\underline{x}, \underline{y}, \underline{z}] = K[x_0, \ldots, x_r, y_0, \ldots, y_r, z_0, \ldots, z_r]$ and let $\mathcal{I}(\underline{y})$ and $\mathcal{I}(\underline{z})$ be the ideals obtained as images of $\mathcal{I}$ in $K[\underline{x}, \underline{y}, \underline{z}]$ via the maps $x_i \mapsto y_i$ and $x_i \mapsto z_i$, for $i = 0, \dots, r$. Then $\mathcal{I}^{\{2\}}$ is the elimination ideal $\big(\mathcal{I}(\underline{y}) + \mathcal{I}(\underline{z}) + \langle y_i + z_i - x_i : 0 \le i \le r \rangle\big) \cap K[x_0, \dots, x_r]$. Similarly, we define the *k-secant* of $\mathcal{I}$ as $\mathcal{I}^{\{k\}} = \mathcal{I} * \cdots * \mathcal{I}$.

For homogeneous prime ideals, the k-secant ideals represent the prime ideals of the k-secant varieties of irreducible projective varieties.

Let now $\prec$ be any term order. The initial ideal of the k-secant ideal $\mathcal{I}^{\{k\}}$ of $\mathcal{I}$ is contained in the k-secant of the initial ideal of $\mathcal{I}$, for $k \ge 1$:

$$\mathrm{in}_\prec(\mathcal{I}^{\{k\}}) \subseteq (\mathrm{in}_\prec(\mathcal{I}))^{\{k\}}. \tag{3.1}$$

For a reference see [16, Cor. 4.2]. If equality holds in (3.1), then $\prec$ is said to be *k-delightful* for the ideal $\mathcal{I}$. It is said to be *delightful* for $\mathcal{I}$ if it is k-delightful for $\mathcal{I}$, for every $k \ge 1$.

For toric varieties this leads to the notion of delightful triangulations of polytopes. Let π_D be a toric degeneration of a toric variety X of dimension n to a union of $\mathbb{P}^n$'s. Any subset of D of k pairwise skew $\mathbb{P}^n$'s, i.e. $k(n+1)$ vertices of D such that they form the vertices of k disjoint tetrahedra of D, $k \ge 1$, will span a linear subspace of $\mathbb{P}^r$ of dimension $kn + k - 1$. A subset of this type is said to be a *skew k-set*; we denote by $N_k(D)$ the set of such skew k-sets and by $\bar{\nu}_k(D)$ its cardinality, see [6, 16]. Consider the following result, due to Sturmfels and Sullivant, which gives a lower bound to the number $\nu_k(X)$ for toric varieties.

Theorem 3.2. [16, Theor. 5.4] *If there exists a toric degeneration π_D of X to a union of $\mathbb{P}^n$'s for which there exists at least one skew k-set, then $Sec_k(X)$ has the expected dimension and $\nu_k(X)$ is bounded below by the number of skew k-sets:*

$$\nu_k(X) \ge \bar{\nu}_k(D). \tag{3.3}$$

Proof. Notice first of all that $kn + k - 1 \le r$. Let $\mathcal{I}$ be the ideal of X and let $\mathcal{I}_0$ be the ideal of the central fiber X_0 with respect to the toric degeneration π_D. The simplicial complex of X_0 is D; let $D^{\{k\}}$ be the simplicial complex of $\mathcal{I}_0^{\{k\}}$: the simplices in $D^{\{k\}}$ are the unions of k simplices in D, see [16, Remark 2.9]. Notice that the simplices of $D^{\{k\}}$ of maximal dimension are the skew k-sets and the subspaces they span sit in the flat limit of $\mathrm{Sec}_k(X)$. Therefore, if there exists at least one skew k-set in D, then $\mathrm{Sec}_k(X)$ has the expected dimension $kn + k - 1$.

Notice that different skew k-sets could span the same subspace π of $\mathbb{P}^r$ and that for the general point of π there is a unique subspace of dimension $k-1$ meeting the k planes each in a point, for each skew k-set spanning π. The toric variety described by $D^{\{k\}}$ is the reduced union of the coordinate subspaces in $\mathbb{P}^r$ given by the skew k-sets. Furthermore, the limit of the k-secant variety of X contains the variety defined by the k-secant of $\mathcal{I}_0$ by (3.1). This concludes the proof. □

Sturmfels and Sullivant in [16] conjectured that if equality holds in the lower bound in (3.3), then the term order corresponding to the triangulation D is k-delightful. We will call such degenerations k-delightful, according to [6].

Definition 1. *Let P and D be as above. If $\dim(\mathrm{Sec}_k(X_P)) = kn + k - 1 \le r$ and equality holds in (3.3), then D is said to be k-delightful. Moreover D is said to be delightful if it is k-delightful for every k.*

Now, consider the examples in Figure 1. The first picture represents a triangulation D of the hexagon $P^1(6,6,1)$, i.e., a degeneration of the smooth del Pezzo surface $X \subseteq \mathbb{P}^6$ to

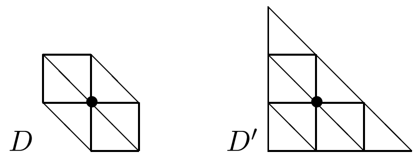


FIGURE 1. Non-2-delightful triangulations

a union of six planes intersecting at a point. Since $\bar{\nu}_2(D) = 0$ and $\nu_2(X) = 3$, D is not 2-delightful. The second one represents a triangulation of the polytope $P^1(3,9,3)$ defining the Veronese surface X' in $\mathbb{P}^9$. $\bar{\nu}_2(D') = 12$ and $\nu_2(X') = 15$ hence D' is not 2-delightful. Notice that in both cases there is a 2-delightfulness defect equal to 3. It is natural to wonder if the cause has to be sought in the sextuple central point, marked in the figures, that of course prevents the presence of disjoint triangles in the configurations. More generally, how do the singularities of the configuration influence the delightfulness property? This question was asked by Ciliberto, Dumitrescu and Miranda [6]. Our aim is to give an explanation of this phenomenon. In the next section we will propose our results in this direction.

4. A lower bound for ν_k, $k = 2, 3$

Let $P \subseteq \mathbb{R}^2$ be the defining polytope of a projective toric surface X and let π_D be a (planar) toric degeneration of X to a union of planes X_0. Let $p \in P \cap \mathbb{Z}^n$ be a lattice point of P and let $Q^1, \dots, Q^\delta \in D$ be the triangles in D covering p: $Q^1 \cap \dots \cap Q^\delta = \{p\}$. Suppose that the union of the Q^i's is a convex planar figure, namely a sub-polytope Q_p of P. Q_p has (normalized) area δ. Let $Z = Z_p$ be the projective toric surface of degree δ defined by Q_p and let Z_0 be the union of δ planes defined by the Q^i's. If p is a boundary lattice point, i.e. Q_p has $g = 0$, we will call it a *rational singularity* for D because Z_0 is a reduced chain of planes intersecting at a point (corresponding to p). If p is an interior point, i.e. Q_p has $g = 1$, we will say that p is an *elliptic singularity* for D since the general hyperplane section of Z_0 is a cycle of lines. In Table 7 and Table 8 all these singularities are classified.

This section is devoted to the proof of the following result that improves the lower bound for ν_k of Proposition 3.2 for the case $n = 2$, $k = 2, 3$.

Theorem 4.1. *Let $k \in \{2,3\}$. Let $X = X_P$ be a projective toric surface such that $\dim(Sec_k(X)) = 3k - 1$. Let D be any triangulation of P. Let $\{p_i\}_{i\in I} \subseteq P \cap \mathbb{Z}^n$, $\{Q_{p_i}\}_{i\in I}$ and $\{Z_{p_i}\}_{i\in I}$ be as above. Assume that*

1. $\dim Sec_k(Z_{p_i}) = 3k - 1$*, for $i \in I$,*
2. *there exists a regular subdivision D_i^1 of P containing Q_{p_i}.*

Then D is not k-delightful. Moreover

$$\nu_k(X) \geq \bar{\nu}_k(D) + \sum_{i \in I} \nu_k(Z_{p_i}). \tag{4.2}$$

Remark 4.3. *This result can not be generalized to the higher-order secant case. Let $k \geq 4$. The expected dimension of $Sec_k(X)$ is $\min\{3k-1, r\}$, when $X \subseteq \mathbb{P}^r$ is a projective toric surface. None of the rational or elliptic sub-polytopes is interesting in this case, because $\dim(Sec_k(Z_p)) < \dim(Sec_k(X))$, for any Z_p as in Table 7 or Table 8.*

4.1. **Proof of Theorem 4.1.**

4.1.1. $k = 2$. Let $X = X_P$ be a projective toric surface such that $\dim \mathrm{Sec}(X) = 5$. Let π_D be a planar toric degeneration of X and let p be a rational or elliptic singularity for D. Let $Q = Q_p = P^0(l, \delta, m)$ be the sub-polytope of P corresponding to p and let $Z = Z_p$ be the projective toric surface of degree δ defined by Q: $Z \subseteq \mathbb{P}^{\delta'} \subseteq \mathbb{P}^r$, where

$$\delta' = \begin{cases} \delta + 1 & \text{if } p \text{ is rational} \\ \delta & \text{if } p \text{ is elliptic.} \end{cases}$$

We are going to prove that the flat limit of the secant variety of X has a 5-dimensional component of degree $\nu_2(Z)$. For this reason, we assume that $\delta' \geq 5$ so that $\dim(\mathrm{Sec}_2(Z_p)) = 5$ (cf. Section 2.1). Furthermore we assume that a lifting function F_{D^1} over an intermediate partition D^1 of P, that contains Q and other polytopes obtained as union of triangles of D, exists. We propose a couple of examples in Figure 2 and in Figure 3. The existence of such an F_{D^1} will be discussed in Subsection 4.1.3. D^1 defines a degeneration π_{D^1} of X to a reducible surface that has Z as component.

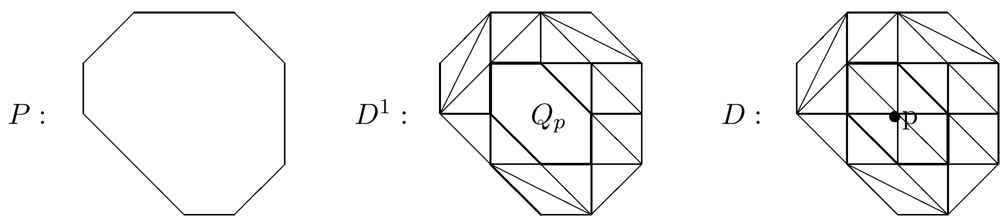


FIGURE 2. An example of decomposed degeneration, $Q_p = P^1(6, 6, 1)$.

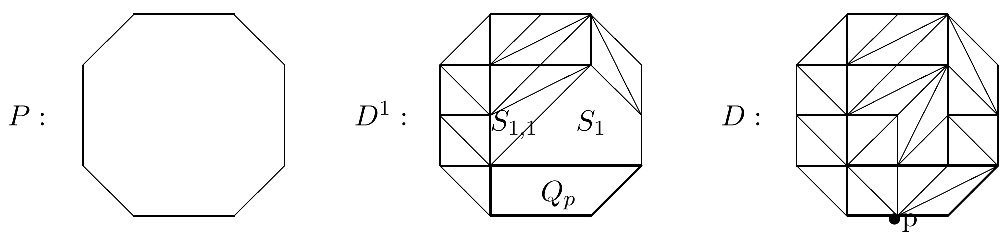


FIGURE 3. An example of decomposed degeneration, $Q_p = P^0(4, 5, 3)$.

Let π_{D^2} be the degeneration of the central fiber of π_{D^1} to X_0.

Proposition 4.4. *Keeping the same setting as above, if there exists in D a singularity p as in Table 7 or Table 8 and if there exists a regular subdivision D^1 of P as above, then*

$$\nu_2(X) \geq \bar{\nu}_2(D) + \nu_2(Z). \tag{4.5}$$

Proof. Consider first the degeneration D^1 of X. Let X_t^1 be the fiber of D^1: $X_t^1 \cong X$, for $t \neq 0$, while X_0^1 is the reduced union of the toric surfaces given by D^1. We have that the secant variety of Z and all the joins between components of X_0^1 sit in the flat limit $\lim_{D^1} \mathrm{Sec}(X)$ of the secant variety of X, with respect to D^1.

We consider now the second degeneration D^2 which has as general fiber $X_s^2 \cong X_0^1$, $s \neq 0$, and as central fiber the reduced union of planes $X_0^2 \cong X_0$. The flat limit, with respect to D^2, of $\lim_{D^1} \mathrm{Sec}(X)$, that is $\lim_D \mathrm{Sec}(X)$, contains as component the flat limits, with respect to D^2, of all the components of $\lim_{D^1} \mathrm{Sec}_2(X)$, namely the following: $\lim_{D^2} \mathrm{Sec}_2(Z)$, which is a 5-dimensional component of degree $\nu_2(Z)$ and the flat limit, with respect to D^2, of all the joins between components of X_s^2, $s \neq 0$. The union of these components contains the $\mathbb{P}^5$'s spanned by the elements of $N_2(D)$.

The contributions in terms of degree given by these components can be summed up. Indeed none of the $\mathbb{P}^5$'s spanned by the skew 2-sets are contained in $\lim_{D^2} \mathrm{Sec}_2(Z)$. □

If $\{p_i\}_{i \in I}$ are singularities of D satisfying the hypotheses of Theorem 4.4, then the contributions given by $\nu_2(Z_{p_1})$'s do not interfere with each other. To see this, let us decompose the degeneration D by taking subdivisions D_i^1 and D_i^2, for each i. The flat limit of the secant variety of Z_{p_i} with respect to D_i^2 sits in the flat limit of the secant variety of X with respect to D, for every i, by Theorem 4.4. Furthermore, let $\mathbb{P}_i \subseteq \mathbb{P}^r$ be the projective subspace where Z_{p_i}, $\mathrm{Sec}_k(Z_{p_i})$ and their limits live, namely the space whose coordinate are given by the lattice points of Q_{p_i}. Notice that $\dim(\mathbb{P}_i \cap \mathbb{P}_j) \leq 3$, for all $i \neq j$. Indeed there are at most two coplanar triangles with vertices at two distinct points p_i, p_j. Since $(\lim_{D_i^2} \mathrm{Sec}_2(Z_{p_i})) \cap (\lim_{D_i^2} \mathrm{Sec}_2(Z_{p_i})) \subseteq \mathbb{P}_i \cap \mathbb{P}_j$, they have no common 5-dimensional component. Therefore these limits are distinct components of $\lim_D \mathrm{Sec}_2(X)$, for all $i, j \in I$, $i \neq j$. Furthermore all of them do not contain any element of $N_2(D)$, hence the respective degrees sum up to $\bar{\nu}_2(D)$. This proves Theorem 4.1 for the case $k = 2$.

Example 4.6. *Let X be the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^{11}$ via $\mathcal{O}(2,3)$: $\nu_1(X) = \deg(Sec(X)) = 35$. Consider the two planar degenerations of X shown in Figure 4. In*

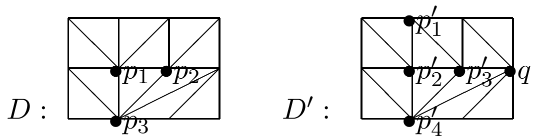


FIGURE 4. Triangulations of a rectangle

the first case, the sum of the number of skew 2-sets and of the contributions of the singularities restores the secant degree: $\bar{\nu}_2(D) + \nu_2(X_{p_1}) + \nu_2(X_{p_2}) + \nu_2(X_{p_3}) = 28 + 3 + 1 + 3 = 35$. In the second case we have: $v_2(D') + \nu_1(X_{p'_1}) + \nu_1(X_{p'_2}) + \nu_1(X_{p'_3}) + \nu_1(X_{p'_4}) = 29 + 1 + 1 + 1 + 1 = 33 < 35$. In D' there is a lattice boundary point q which is the common vertex of five triangles: certainly it causes an obstruction to the presence of skew 2-sets, but the polygon given by the triangles around it is not convex and our argument does not apply.

4.1.2. $k = 3$. Let $X = X_P$ be a toric surface such that $\dim(\mathrm{Sec}_2(X)) = 8$. Let D be any triangulation of P.

Remark 4.7. *There are only two types of elliptic singularities we are interested in, namely the ones such that Z_p is either the Veronese surface V_3 in $\mathbb{P}^9$ or the del Pezzo surface X_8 of degree eight in $\mathbb{P}^8$. Indeed in all remaining cases (see Table 8) the 3-secant variety has dimension less than 8. On the other hand, the only toric surface with $g = 0$ such that its 3-secant variety has dimension 8 and such that there exists a toric degeneration of it to a union of planes all of them intersecting at a single point is the rational normal scroll $S(2, \delta - 2) \subseteq \mathbb{P}^{\delta+1}$, with $\delta \geq 7$, (see Table 7).*

Proposition 4.8. *Let $X = X_P$ be a toric surface such that $\dim Sec_3(X) = 8$ and let D be a triangulation of P. Let p be a multiple point such that the corresponding surface Z is either*

V_3, or X_8, or $S(2,\delta-2)$, with $\delta \geq 7$. Assume furthermore that there exists an intermediate regular subdivision D^1 of P containing Q_p. Then

$$\nu_3(X) \geq \bar{\nu}_3(D) + \nu_3(Z_p). \tag{4.9}$$

Proof. It is easy to see that $\mathrm{Sec}_3(Z)$ and $J(Y_i, J(Y_j, Y_l))$, where Y_i, Y_j, Y_l are components of $\lim_{D^1} X$, are in the flat limit $\lim_{D^1} \mathrm{Sec}_2(X)$.

Then, looking at the second degeneration D^2, we see that the $\mathbb{P}^8$'s spanned by the skew 3-sets of D^2 (that are the skew 3-sets of D) and the limit $\lim_{D^2} \mathrm{Sec}_3(Z)$ are 8-dimensional of $\mathrm{Sec}_3(X)$ with respect to D.

Finally, the contributions $\bar{\nu}_3(D)$ and $\nu_3(Z)$ do not interfere with each other, following the same argument as in Theorem 4.4. □

If there are more than one singularity in D, $\{p_i\}_{i\in I}$, satisfying the hypotheses of Theorem 4.8, arguing as for the case $k=2$, we get inequality (4.2) for $k=3$.

4.1.3. *On the existence of an intermediate regular subdivision of a given triangulation.* Let P, D and Q be as previously defined. To conclude this section we explore the existence of an intermediate regular subdivision D^1 containing Q.

Assume first of all that either the edges of Q have (normalized) length equal to one or they lie on the boundary of P (under this assumption p must be an elliptic singularity). The family of sub-polytopes of P given by Q and by the $\mathrm{Area}(P)-\delta$ remaining triangles of D form a subdivision of P (see Figure 2). Such a subdivision is regular. Indeed, given a lifting function F_D over D, one can always find a lifting function F_{D^1} over D^1, exploiting the fact that strict convexity is a local property: it is enough to flatten F_D over Q. More precisely, one can always assume that $F_D(\underline{m}) \gg 2$, for $\underline{m} \notin Q$ and that

$$F_D(\underline{m}) = \begin{cases} 1-\epsilon & \text{if } \underline{m} = p \\ 1 & \text{if } \underline{m} \in Q \cap \mathbb{Z}^2 \setminus \{p\} \end{cases},$$

with $0 < \epsilon \ll 1$. Hence, a lifting function for D^1, F_{D^1}, is the following:

$$F_{D^1}(\underline{m}) := \begin{cases} 1 & \text{if } \underline{m} = p \\ F_D(\underline{m}) & \text{if } \underline{m} \neq p \end{cases}$$

Suppose now that Q has edges $L_1 \ldots, L_s$, $s \leq l$ of length > 1. Let us construct a partition of P containing Q, triangles and convex polytopes given as union of triangles of D, using the following algorithm.

INPUT: a regular unimodular triangulation D of P.
OUTPUT: a regular subdivision D^1 of P containing Q.

- Let S_i be the minimal convex union of triangles of D such that $S_i \cap Q = L_i$, for $i = 1, \ldots, s$. If all the S_i's have *external* edges (i.e., all the edges except L_i) either of length one or lying on ∂P, we stop.
- Otherwise, let $L_{i,1}, \ldots, L_{i,s_i}$ be the external edges of S_i of length > 1, for $i \in \{1, \ldots, s\}$. Let $S_{i,j}$ be the minimal convex union of triangles of D such that $S_{i,j} \cap S_i = L_{i,j}$, $i = 1, \ldots, s$, $j = 1, \ldots, s_i$. If all the $S_{i,j}$'s have external edges either of length one or contained in ∂P, then we stop.
- Otherwise we go on as above, until all the polytopes obtained in this way have external edges either of length one, or contained in ∂P.

This process is finite. The output is a complex D^1 whose maximal polyhedra are Q, the S_i's, the $S_{i,j}$'s, etc., and the remaining triangles of D. If one is able to flatten the lifting function F_D over Q, the S_i's, the $S_{i,j}$'s, etc., by rescaling it in such a way that the resulting piecewise linear function is strictly convex over P, one has found a lifting function F_{D^1} for D^1 to be regular.

At this point it is not difficult to define D^2: it is sufficient to take unimodular triangulations D_Q of Q, D_{S_i} of S_i, $D_{S_{i,j}}$ of $S_{i,j}$, etc., such that, combining them, one obtains the full regular unimodular triangulation D of P. See for example Figure 3 to get an idea.

5. Classification of delightful triangulations of polytopes with $g \leq 1$

In this section we classify all delightful triangulation of $g \leq 1$ polytopes in $\mathbb{R}^2$. A necessary condition for the degeneration to be 2-delightful is that it contains no lattice point as in Table 7 or Table 8 in its configuration. Surprisingly we will see that the triangulations verifying this property turn out to be k-delightful, for any k.

5.1. The rational case.

The $g = 0$ polytopes are classified in Theorem 1.2. In this section we are going to prove the following theorem.

Theorem 5.1. *The trapezium $P^0(4, 2\delta+i, \delta+i)$ admits delightful triangulations if and only if $0 \leq i \leq 3$.*

The unique delightful triangulations of $P^0(4, 2\delta + i, \delta + i)$, up to lattice equivalence, are the ones represented in Figure 5.

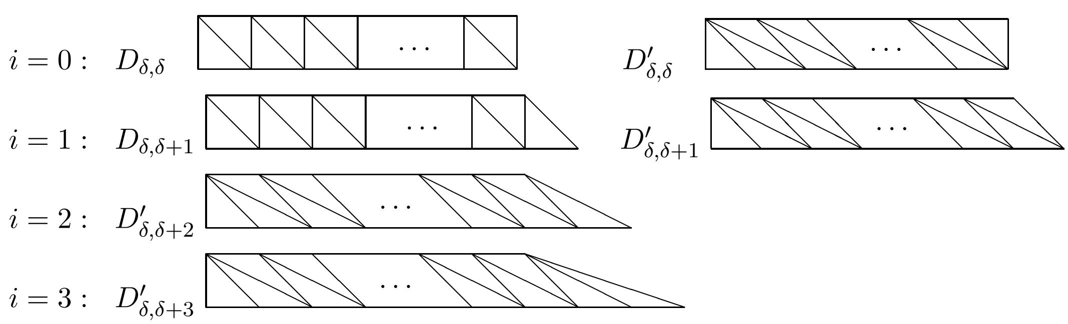
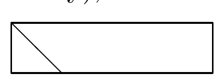


Figure 5. Delightful triangulations of $g = 0$ polytopes

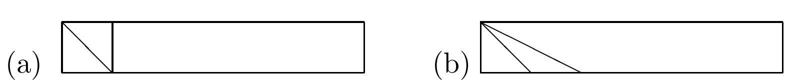
The outline of the proof will be the following. As a first step we fix $k = 2$ and we construct triangulations without rational singularities at the (boundary) lattice points for the polytopes $P^0(4, 2\delta+i, \delta+i)$, $\delta \geq 2$, $i \geq 0$. Then we will investigate their k-delightfulness.

Remark 5.2. *The unique triangulations of $P^0(4, 2\delta+i, \delta+1)$ without rational singularities occur when $0 \leq i \leq 3$ and are the ones in Figure 5.*

Proof. Consider the rectangle $P^0(4, 2\delta, \delta)$ with bases of length δ. We start the triangulation in the only possible way (up to equiaffinity), as follows



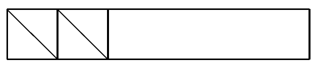
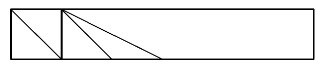
Then there are only two distinct possibilities to add a further triangle that is adjacent to the previous one:



In case (a), the ways of putting another triangle adjacent to the previous are the following:

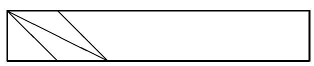
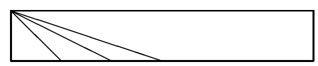
(a.1)  (a.2) 

The second possibility must be excluded, otherwise we would get at least four triangles covering the point with coordinates $(1, 0)$ and this certainly will generate a rational singularity (see Table 7). On the other hand, starting from the case (a.1) and adding a triangle in the subdivision, we get

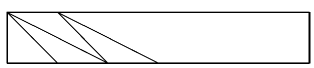
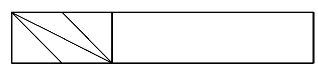
(a.1.1)  (a.1.2) 

The second configuration is excluded once again, otherwise the point $(1, 1)$ would be covered by a chain of at least four triangles. So, iterating this argument, we obtain $D_{\delta,\delta}$.

In case (b), the possibilities are:

(b.1)  (b.2) 

As above, the case (b.2) is excluded, otherwise $(0, 1)$ would be a rational singularity. Then from (b.1) we obtain

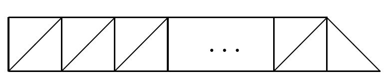
(b.1.1)  (b.1.2) 

We exclude the case (b.1.2) and iterating the process we get $D'_{\delta,\delta}$ from (b.1.1). The subdivisions $D_{\delta,\delta}$ and $D'_{\delta,\delta}$ do not contain any rational singularity and they are the unique triangulations of $P^0(4, 2\delta, \delta)$ with this property.

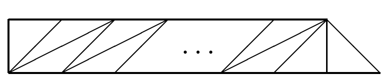
Consider $P^0(4, 2\delta + 1, \delta + 1)$. One has two distinct ways (up to equiaffinity) to start a triangulation of this polytope:

(a)  (b) 

From (a), arranging the argument of above to this case, we arrive to $D_{\delta,\delta+1}$ or $D'_{\delta,\delta+1}$. Instead, from (b) we get either



that is lattice equivalent to $D'_{\delta,\delta+1}$, or



that is excluded; in fact a singularity at the point $(\delta, 1)$ has been generated.

Finally, arguing as above, we get $D'_{\delta,\delta+2}$ for the trapezium $P^0(4, 2\delta + 2, \delta + 2)$ and $D'_{\delta,\delta+3}$ for $P^0(4, 2\delta + 3, \delta + 3)$. The details are easy and left to the reader.

If $i \geq 4$, it is not possible to find a triangulation without generating a rational singularity, because a chain of four triangles around a boundary lattice point will inevitably be created.

□

Let now $P' \subseteq P$ be polytopes with $\mathrm{Area}(P') + 1 = \mathrm{Area}(P) = d$, $g(P) = g(P') = 0$ and such that $P \setminus P' = T$ is a triangle of normalized area 1. Let D and $D' = D \setminus T$ be regular triangulations of P and P' respectively. Assume moreover that $\dim(\mathrm{Sec}_2(X_P)) = \dim(\mathrm{Sec}_2(X_{P'})) = 5$. Notice that, under these hypotheses, if P belongs to the class $P^0(4, 2\delta + i, \delta + i)$, $0 \leq i \leq 3$, with $d = 2\delta + i$ for some δ, i, then P' has also the form $P^0(4, 2\delta' + i', \delta + i')$, $0 \leq i' \leq 3$, with $d - 1 = 2\delta' + i'$ for some δ', i'. Notice moreover that if D is lattice equivalent to one of the configurations in Figure 5, then D' is.

Define $D'' = \{T'' \in D' : T'' \cap T = \emptyset\} \subseteq D' \subseteq D$. D'' is given by those triangles of D which do not intersect T. Using these notations we can describe $N_k(D)$ as the set given by the skew k-sets contained in D' and by those involving T, namely $N_k(D) = N_k(D') \cup \{(T, (T''_1, \ldots, T''_{k-1})) : (T''_1, \ldots, T''_{k-1}) \in N_{k-1}(D'')\}$.

Lemma 5.3. *In the above notation, D is k-delightful if and only if D' is k-delightful and D'' is $(k-1)$-delightful.*

Proof. Since D'' contains at most $d-3$ triangles, then $\bar{\nu}_{k-1}(D'') \leq \binom{(d-3)-2(k-2)}{k-1}$. Hence $\bar{\nu}_k(D) = \bar{\nu}_k(D') + \bar{\nu}_{k-1}(D'') \leq \binom{(d-1)-2(k-1)}{k} + \binom{(d-3)-2(k-2)}{k-1} = \binom{d-2(k-1)}{k}$. Since the number on the right equals $\nu_k(X_P)$ the thesis follows. □

This argument allows to use induction on $d = 2\delta + i$ and k to prove that the degenerations depicted in Figure 5 of the trapezia $P^0(4, 2\delta + i, \delta + i)$, $0 \leq i \leq 3$, are k-delightful, for k such that $3k - 1 \leq d + 1$.

Proposition 5.4. *The triangulations in Figure 5 are delightful.*

Proof. Let D denote one of the triangulations of Figure 5 and let $d = 2\delta + i$ be the number of triangles of D.

Fix $k = 2$. We first prove that D is 2-delightful by induction on d and exploiting the fact that D is 2-delightful if and only if $\bar{\nu}_2(D') = \binom{d-3}{2}$ and $\bar{\nu}_1(D'') = \#(D'') = d - 3$ (see the proof of Lemma 5.3), for each D as in Figure 5. Then we consider the case $k \geq 3$.

For $d = 4$, the degenerations $D_{2,2}$, $D'_{2,2}$ of $S(2,2)$ and $D'_{1,3}$ of $S(1,3)$ are clearly 1-delightful indeed each of them contains exactly one pair of disjoint triangles and $\nu_2(S(2,2)) = \nu_2(S(1,3)) = 1$. The same holds in the case $d = 5$ for $D_{2,3}$, $D'_{2,3}$ and $D'_{1,4}$: one can easily check that each contains exactly three pairs of disjoint triangles and it is $\nu_2(S(2,3)) = \nu_2(S(1,4)) = 3$.

For $d \geq 6$, assume the thesis true for any degree $\leq d - 1$. If d is even, write $d = 2\delta$. The degeneration $D_{\delta,\delta}$ (or $D'_{\delta,\delta}$) of $P^0(4, 2\delta, \delta)$ is obtained from $D' = D_{\delta-1,\delta}$ ($D' = D'_{\delta-1,\delta}$ respectively) by adding a triangle on the right. Now $\bar{\nu}_2(D) \leq \bar{\nu}_2(D') + \bar{\nu}_1(D'') = \binom{d-2}{2} + d - 3 = \binom{d-3}{2} = \nu_2(S(\delta,\delta))$. In the same way, the degeneration $D = D'_{\delta-1,(\delta-1)+2}$ of $P^0(4, 2(\delta-1)+2, (\delta-1)+2)$ is obtained by adding a triangle to $D'_{\delta-1,(\delta-1)+1}$ and computing the number $\bar{\nu}_2(D)$ we get the same conclusion. If d is odd, write $d = 2\delta + 1$. The degenerations $D_{\delta,\delta+1}$ and $D'_{\delta,\delta+1}$ are obtained respectively from $D_{\delta,\delta}$ and $D'_{\delta-1,\delta+1}$ by adding a triangle on the right end and the computation done in the case d even also works. Similarly, $D = D'_{\delta-1,(\delta-1)+3}$, which is obtained from $D'_{\delta-1,(\delta-1)+2}$, turns out to be 2-delightful.

Now, fix $k \geq 3$, and consider d such that $3k - 1 \leq d + 1$. Let D be one of the degenerations of $P = P^0(4, 2\delta + i, \delta + i)$ in Figure 5. We prove the statement by induction on d and k using an argument similar to that of above. Let D' and D'' be as above and assume D' is k-delightful and D'' is $(k-1)$-delightful. Then $\bar{\nu}_k(D) = \bar{\nu}_k(D') + \bar{\nu}_{k-1}(D'') = \binom{d-2(k-1)}{k}$. □

This proves Theorem 5.1.

Our result fits with the ones obtained by Sturmfels and Sullivant. In [16, Prop. 5.8] they proved that if a delightful term order exists for a rational normal scroll $S(\delta_1, \ldots, \delta_n)$ of dimension n, then we must have $\delta_j \in \{m, m+1, m+2, m+3\}$ for some m. They also proved in [16, Prop. 5.11] that the converse holds in the $n = 2$ case. With our approach we have proved the same result in the case $n = 2$ and we have also constructed these delightful triangulations.

5.2. **The elliptic case.** Here we prove a classification result for the $g = 1$ case. The polytopes we are dealing with are depicted in Theorem 1.3.

Theorem 5.5. *All polytopes with $g = 1$ and $5 \le d \le 8$ admit delightful triangulations. They are lattice equivalent to the ones in Figure 6.*

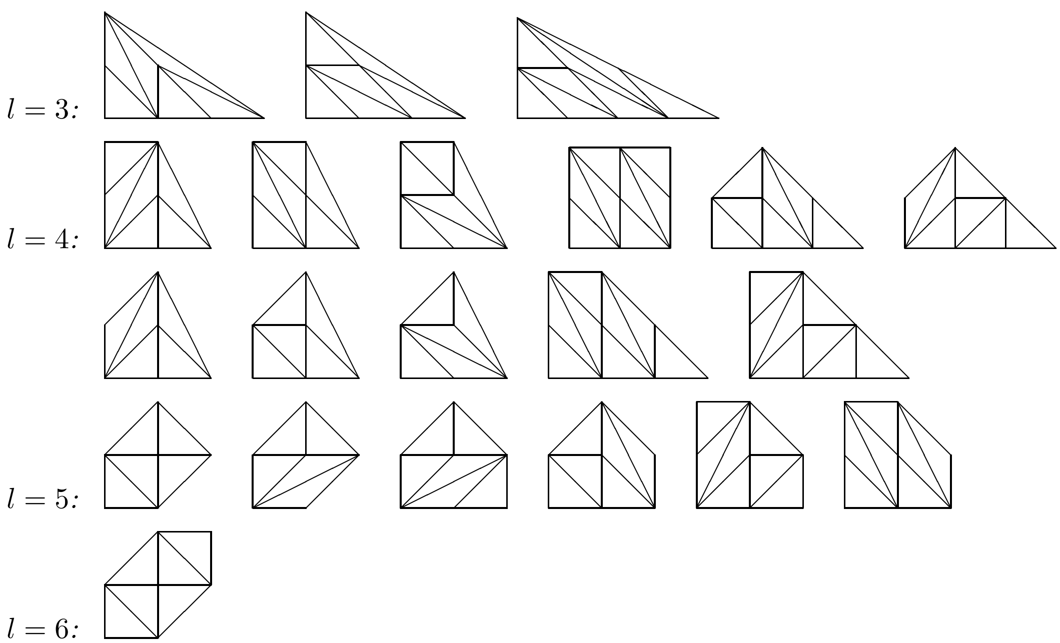


FIGURE 6. Delightful triangulations of $g = 1$ polytopes

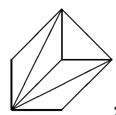
We need a preliminary remark. Notice that if D do not contain an elliptic singularity of multiplicity d, then D must contain at least a triangle T that do not have any vertex at the interior lattice point and $P \backslash T$ is convex. Define $P' := P \backslash T \subseteq P$: $\text{Area}(P') + 1 = \text{Area}(P) = d$, $g(P) = g(P') = 1$ and assume that that $\dim(\text{Sec}_2(X_P)) = \dim(\text{Sec}_2(X_{P'})) = 5$. Consider the triangulation $D' \subseteq D$ of P' obtained from D by deleting that T, D' is still regular. Set $D'' = \{T'' \in D' : T'' \cap T = \emptyset\} \subseteq D'$: we have that $\#(D'') \le d - 4$. From this follows that $\bar{\nu}_2(D) = \bar{\nu}_2(D') + \#(D'') \le \binom{d-4}{2} + (d-4) = \binom{d-3}{2} = \nu_2(X_P)$. We get the following lemma.

Lemma 5.6. *In the notation of above, D is 2-delightful if and only if there exists a triangle T such that $D' = D \setminus T$ is 2-delightful and such that there are exactly $d - 4$ triangles in D not intersecting T.*

Proof of Theorem 5.5. Assume $k = 2$. We start from the base case, $d = 5$ and then we increase the degree by adding a triangle. In this way we can exploit Lemma 5.6 and cover all cases $5 \le d \le 9$.

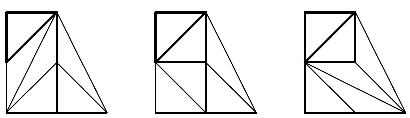
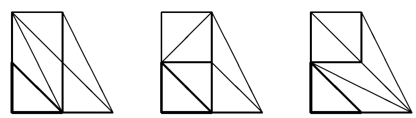
We first consider the sub-polytopes of the triangle $P^1(3, 9, 3)$, i.e. the ones corresponding to internal projections of the 3-ple Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$.

Fix $d = 5$. There are finitely many (regular) triangulations of each of these polytopes, up to equiaffinity. A part from the cases with five triangles covering the interior lattice point (see Table 8, first row), and from the case

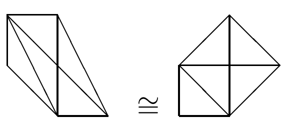
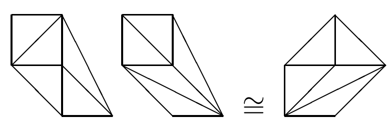

,

the remaining configurations contain a unique skew 2-set, so they are 2-delightful. It is easy and left to the reader.

Now fix $d = 6$. The only possible way to get 2-delightful triangulations of $P^1(4, 6, 2)$ is adding a triangle T to the 2-delightful triangulations of subpolytopes with $d = 5$ such that there are 2 triangles in D not intersecting T, by Lemma 5.6. The candidates have to be chosen among the following configurations

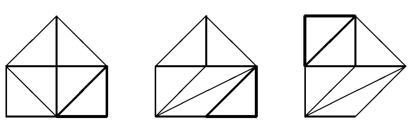
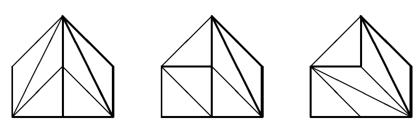
 and  .

The first set was obtained by adding a triangle to the 2-delightful triangulations of the quadrilateral $P^1(4, 5, 2)$ in Figure 6. Instead, to get the second set of configurations we first chose representatives of the equiaffinity class of the 2-delightful triangulations of $P^1(5, 5, 1)$ depicted in Figure 6, namely

 and  ,

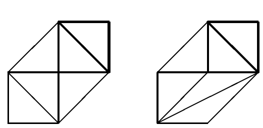
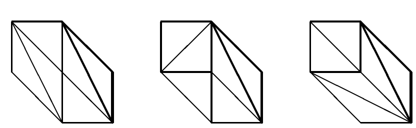
and then we added a triangle, as shown in the pictures. There are two triangles not intersecting T in the first, the forth and the sixth triangulation of $P(5, 6, 2)$ depicted above, so in these cases there are in all three skew 2-sets and we have 2-delightfulness.

Similarly, for $P(5, 6, 2)$ we may choose among the following configurations

 and 

The second triangulation is 2-delightful and the same holds for the third and the fifth which are lattice equivalent. While the remaining configurations contain less than 3 pairs of disjoint triangles.

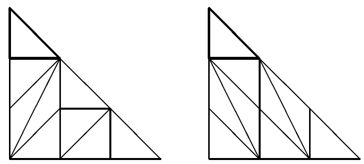
For the hexagon $P^1(6, 6, 1)$, the candidates are

 and 

Just the first and the third, which are lattice equivalent, are 2-delightful.

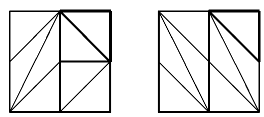
For $d = 7, 8$, namely for the polytopes $P^1(4,7,3)$, $P^1(5,7,2)$ and $P^1(4,8,3)$, the proof is similar and the details are left to the reader.

Consider finally the triangle $P^1(3,9,3)$. If there was any 2-delightful triangulation, it would be obtained by adding a triangle to some 2-delightful triangulation of $P^1(4,8,3)$, i.e.,



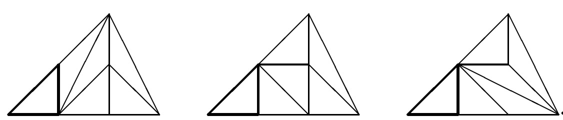
but they both are not 2-delightful.

Now we analyze the remaining polytopes. To get possible 2-delightful triangulations of $P^1(4,8,2)$ we add a triangle to the 2-delightful triangulations of $P^1(5,7,2)$ in Figure 6:



The second is the unique that is 2-delightful since T is disjoint from four distinct triangles and we have six more skew 2-sets from the subdivision of $P^1(5,7,2)$. So $\bar{\nu}_2(D) = 10$.

Consider $P^1(3,6,3)$. The candidates are the configurations obtained by adding a triangle to the 2-delightful triangulations of $P^1(4,5,2)$. We get:



The first and the second are 2-delightful, since we have two skew 2-sets coming from the triangulation of $P^1(4,5,2)$ and one more pair involving T.

Finally, for $P^1(3,8,4)$ the proof is similar and left to the reader.

Now assume $k \geq 3$. If $P = P^1(4,8,3)$, then $\mathrm{Sec}_3(X_P)$ fills up the space $\mathbb{P}^8$. The subdivisions of P depicted in Figure 6, that are 2-delightful, are also 3-delightful since a (unique) skew 3-set exists in both of them. In all other cases, namely for P not belonging to the class $P = P^1(4,8,3)$ and D as in Figure 6, we have $\dim(\mathrm{Sec}_3(X)) < 8$, see Section 2.1. □

6. Tables

In the following tables, we collect the triangulations of polytopes with $g \leq 1$, in which all the triangles have a common vertex p. They are non-delightful and in particular correspond to the singularities that cause k-delightfulness defect, for $k = 2, 3$, see Theorem 4.1.

In the first column we draw the subdivision of the polytope $Q = Q_p$; the degree of $Z = Z_p$, which corresponds to the number of triangles, is written in the second column, while the numbers $\nu_2(Z)$ and $\nu_3(Z)$ are collected respectively in the third and in the fourth column.

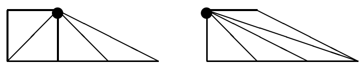
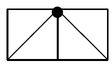
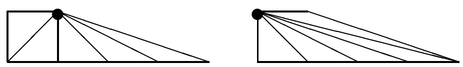
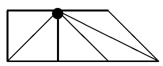
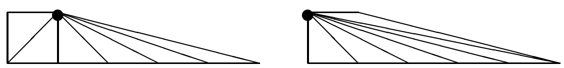
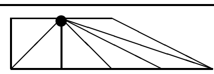
	triangulation of Q	$\deg(Z)$	$\nu_2(Z)$	$\nu_3(Z)$
1.		4	1	/
2.		4	1	/
3.		5	3	/
4.		5	3	/
5.		6	6	/
6.		6	6	/
7.	$S(1, \delta - 1)$	$\delta \geq 7$	$\binom{\delta-2}{2}$	/
8.	$S(2, \delta - 2)$	$\delta \geq 7$	$\binom{\delta-2}{2}$	$\binom{\delta-4}{3}$

FIGURE 7. Rational singularities

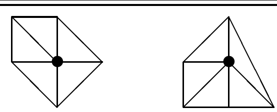
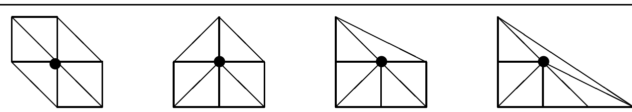
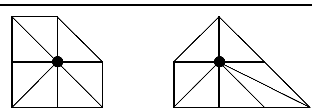
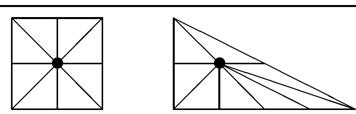
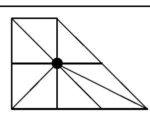
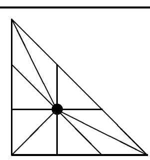
	triangulation of Q	$\deg(Z)$	$\nu_2(Z)$	$\nu_3(Z)$
1.		5	1	/
2.		6	3	/
3.		7	6	/
4.		8	10	/
5.		8	10	1
6.		9	15	4

FIGURE 8. Elliptic singularities

Acknowledgements

I would like to thank C. Ciliberto for introducing me to the problem of studying secant varieties and suggesting to me the idea of using toric degenerations as a tool to compute the secant degrees, while I was a PhD student of him. I also want to thank B. Sturmfels for suggesting me to classify the delightful triangulations. Finally, I am deeply grateful to R. Piene for many helpful and stimulating discussions during the preparation of this paper.

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