The component sizes of a critical random graph with pre-described degree sequence

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Summary. Consider a critical random multigraph $\mathcal{G}_n$ constructed by the configuration model such that its vertex degrees are independent random variables with the same distribution ν (criticality means that the second moment of ν is finite and equals twice its first moment). We specify the scaling limits of the ordered sequence of component sizes of $\mathcal{G}_n$ in different cases. When ν has finite third moment, the components sizes rescaled by $n^{-2/3}$ converge to the excursion lengths of a Brownian motion with parabolic drift, whereas when ν is a power law distribution with exponent $\gamma \in (3, 4)$, the components sizes rescaled by $n^{-(\gamma-2)/(\gamma-1)}$ converge to the excursion lengths of a drifted process with independent increments that will be characterized.

Key words. Critical random graph, random multigraph with given vertex degrees, power law, scaling limits, size-biased sampling, excursion.

A.M.S. Classification. 60C05, 05C80, 90B15.

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1 Introduction

The classical random graph model $G(n, p)$ has received a lot of attention since its introduction by Erdős and Rényi [12], especially because of the existence of a phase transition. In this model, a graph on n labeled vertices is constructed randomly by joining any pair of vertices by an edge with probability p, independently of the other pairs. For large n, the structure of this random graph depends on the value of np: for $p \sim c/n$ with $c < 1$, the largest connected component contains $O(\ln n)$ vertices, whereas when $p \sim c/n$ with $c > 1$, the largest component has $\Theta(n)$ vertices while the second largest component has $O(\ln n)$ vertices. The cases $c < 1$ and $c > 1$ are called subcritical and supercritical respectively. Much attention has been devoted to the critical case $p \sim 1/n$. When p is exactly equal to $1/n$, the largest components of have sizes of order $n^{2/3}$.

Molloy and Reed [19] showed that a random graph with a given degree sequence exhibits a similar phase transition. More precisely, let $\boldsymbol{d}^{(n)} = (d_i^{(n)})_{1\le i\le n}$ be a sequence of positive integers such that $\sum_{i=1}^n d_i^{(n)}$ is even. Let $G(n, \boldsymbol{d}^{(n)})$ be a random graph on n labeled vertices with degree sequence $\boldsymbol{d}^{(n)}$, uniformly chosen among all possibilities (tacitly assuming that there exists any such graph). Suppose that there exists a probability distribution $(\nu_k)_{k\ge1}$ such that $\#\{i : d_i^{(n)} = k\}/n \to \nu_k$ as $n \to \infty$. Let $\omega(n)$ be the highest degree in the graph. Under some further strong conditions on the sequences $\boldsymbol{d}^{(n)}$, Molloy and Reed proved that if $Q = \sum_{k=1}^\infty k(k-2)\nu_k < 0$ and $\omega(n) \le n^{1/8-\varepsilon}$ for some $\varepsilon > 0$, then with probability tending to 1, the size of the largest component of $G(n, \boldsymbol{d}^{(n)})$ is $O(\omega^2(n)\ln n)$, whereas if $Q > 0$ and $\omega(n) \le n^{1/4-\varepsilon}$ for some $\varepsilon > 0$, then with probability tending to 1, the size of the largest component is $\Theta(n)$, and if furthermore Q is finite, the size of the second largest component is $O(\ln n)$.

More recently, the near-critical behavior of such graphs has been studied. When $Q = 0$, the structure of $G(n, \boldsymbol{d}^{(n)})$ depends on how fast the quantity

$$\alpha_n = \sum_{k=1}^\infty k(k-2)\frac{\#\{i : d_i^{(n)} = k\}}{n} = \sum_{i=1}^n \frac{d_i^{(n)}(d_i^{(n)}-2)}{n}$$

converges to 0 (see Kang and Seierstad [18]). Requiring a fourth moment conditon, Janson and Luczak [17] proved that if $n^{1/3}\alpha_n \to \infty$, then the size of the largest component of $G(n, \boldsymbol{d}^{(n)})$ divided by $n\alpha_n$ converges in probability to $\frac{2\mu}{\beta}$, while the size of the second largest component of $G(n, \boldsymbol{d}^{(n)})$ divided by $n\alpha_n$ converges in probability to 0, where $\mu = \sum_{k=1}^\infty k\nu_k$ and $\beta = \sum_{k=3}^\infty k(k-1)(k-2)\nu_k \in (0,\infty)$. Furthermore, they noticed that their results can also be applied to some other random graph models by conditioning on the vertex degrees, provided that the random graph conditioned on the degree sequence has a uniform distribution over all possibilities. This is the case for $G(n, p)$ with $np \to 1$ and $n^{1/3}(np-1) \to \infty$. Note that if $n^{1/3}(np-1) = O(1)$, it is well-known that the largest component and the second largest component have are both sizes of the same order $n^{2/3}$, so that their results do not hold.

A major difficulty when dealing with the natural random graph $G(n, \boldsymbol{d}^{(n)})$ is that, despite its straightforward definition, it cannot be constructed via an easy algorithm (see Britton *et al.* [10]). To circumvent that obstacle, it is convenient to work with *multigraphs*, in which multiple edges and loops are allowed, using the explicit procedure provided by the *configuration model*, which was introduced by Bender and Canfield [4] and later studied by Bollobás [8] and Wormald [24]. See also Molloy and Reed [19, 20], Kang and Seierstad [18], Bertoin and Sidoravicius [6], van der Hofstad [23], Hatami and Reed [13]. Specifically, take a set of $d_i^{(n)}$ half-edges for the vertex with label i, $i \in \{1, \ldots, n\}$, and combine the half-edges into pairs by a uniformly random matching of the set of all half-edges. Observing that every graph $G(n, \boldsymbol{d}^{(n)})$ may be constructed through the same number, $d_1^{(n)}! \cdots d_n^{(n)}!$, of pairing of half-edges, we get that conditional on being a (simple) graph, the multigraph obtained by the configuration model has the same distribution as $G(n, \boldsymbol{d}^{(n)})$. That is why we shall deal with multigraphs.

The present work is devoted to the study inside the critical window. We suppose that we are given a probability distribution $\nu = (\nu_k)_{k\ge1}$ with finite second moment such that $\nu_2 < 1$ and $\sum_{k=1}^\infty k(k-2)\nu_k = 0$. Let D be a random variable with distribution ν. The multigraph $\mathcal{G}_n$ consisting of n vertices is defined by the configuration model as follows. Let $D_1, D_2, \ldots, D_n$ be n independent copies of D. Condition on $\sum_{i=1}^n D_i$ being even. Take a set of D_i half-edges for each vertex, and combine the half-edges into pairs by a uniformly random matching of the set of all half-edges. The random multigraph that this construction leads to is denoted by $\mathcal{G}_n$. We aim at specifying the asymptotics of the ordered sequence $\boldsymbol{\mathcal{C}}_n^\nu$ of component sizes of $\mathcal{G}_n$ in two different settings. First, we shall study the case when ν has finite third moment. We shall prove that $n^{-2/3}\boldsymbol{\mathcal{C}}_n^\nu$ then converges in distribution (with respect to a certain topology that will be detailed below) as $n \to \infty$ to the ordered sequence of the excursion lengths of a Brownian motion with parabolic drift. This should be viewed as an extension of Aldous' well-known result for the Erdős-Renyi model. Next the case when ν is a power law distribution with exponent $\gamma \in (3,4)$ will be studied. We shall show that $n^{-(\gamma-2)/(\gamma-1)}\boldsymbol{\mathcal{C}}_n^\nu$ converges in distribution as $n \to \infty$ to the ordered sequence of the excursion lengths of a certain drifted process with independent increments. Similar results have already been obtained, but for different random graph models (we refer to Bhamidi *et al.* [7] for inhomogeneous random graphs).

The paper is organized as follows. Sections 2, 3, 4 and 5 are devoted to the study of $\boldsymbol{\mathcal{C}}_n^\nu$ when ν has finite third moment; the main techniques developed there will be useful in the power law distribution case. The main results will be stated in Section 2. In Section 3, following the ideas of Aldous [1], we shall observe that the study may be reduced to the understanding of a walk defined via an algorithmic procedure related to the configuration model. Thank to [1], convergence of that walk turns out to be sufficient. It will be obtained in Section 4 using standard methodology from stochastic process theory (see, *e.g*, the CLT for continuous-time martingale). A key technique to obtain martingales is Poissonization. Basically, instead of considering mutigraphs with exactly n vertices, we shall deal with multigraphs with Poisson(n) vertices. Our approach also relies on size-biased ordering. Finally, in Section 5, we shall be interested in the number of cycles in the multigraph $\mathcal{G}_n$. To conclude, in Section 6, we shall study $\boldsymbol{\mathcal{C}}_n^\nu$ when ν is a power law distribution with exponent in $(3,4)$. We shall follow the same strategy, except we shall apply results of Aldous and Limic [2].

2 Formulation of the results in the finite third moment setting

In the first sections of the paper, we suppose that ν satisfies:

$$\sum_{k=1}^{\infty} k(k-2)\nu_k = 0, \qquad \sum_{k=1}^{\infty} k^3\nu_k < \infty \quad \text{and} \quad \nu_2 < 1. \tag{1}$$

The power law case will be studied in Section 6. Let

$$\mu = \sum_{k=1}^{\infty} k\nu_k \quad \text{and} \quad \beta = \sum_{k=3}^{\infty} k(k-1)(k-2)\nu_k.$$

Observe that $\beta > 0$. Define the Brownian motion with parabolic drift

$$W^\nu : t \geq 0 \mapsto \sqrt{\frac{\beta}{\mu}} W(t) - \frac{\beta}{2\mu^2} t^2,$$

where $(W(t), t \geq 0)$ is a standard Brownian motion. The reflected process valued the nonnegative half-line is

$$R^\nu : t \geq 0 \mapsto W^\nu(t) - \min_{0 \leq s \leq t} W^\nu(s).$$

Call excursion interval of R^ν every time interval $\gamma = [l(\gamma), r(\gamma)]$ such that $R^\nu(l(\gamma)) = R^\nu(r(\gamma)) = 0$ and $R^\nu(t) > 0$ on $l(\gamma) < t < r(\gamma)$. The excursion has length $|\gamma| = r(\gamma) - l(\gamma)$. Aldous [1] observed that we can order excursions by length, that is the set of excursions of R may be written $\{\gamma_j, j \geq 1\}$ so that the lengths $|\gamma_j|$ are decreasing. In the notation of [1], define $l^2_{\searrow}$ as the set of infinite sequences $x = (x_1, x_2, \dots)$ with $x_1 \geq x_2 \geq \dots \geq 0$ and $\sum_i x_i^2 < \infty$, endowed with the Euclidean metric. We may regard the finite sequence $\boldsymbol{\mathcal{C}}_n^\nu$ as a random element of $l^2_{\searrow}$ by appending zero entries.

Our main result describes the component sizes of $\mathcal{G}_n$ for large n; it mirrors that of Aldous [1] for the critical random graph.

Theorem 1. *Suppose ν satisfies (1). Let $\boldsymbol{\mathcal{C}}_n^\nu$ be the ordered sequence of component sizes of $\mathcal{G}_n$. Then*

$$n^{-2/3}\boldsymbol{\mathcal{C}}_n^\nu \xrightarrow[n\to\infty]{(d)} (|\gamma_j|, j \geq 1)$$

with respect to the $l^2_{\searrow}$ topology.

Remark 1. Suppose $\nu_2 = 1$, *i.e.*, $D \equiv 2$. Then the components of $\mathcal{G}_n$ are cycles. It is well-known that the distribution of cycle lengths is given by the Ewens's sampling formula ESF(1/2), and thus the size of the largest component divided by n converges in distribution to a non-degenerate distribution on $[0, 1]$ (see [3, Lemma 5.7]). This is also the case for the k-th largest component, where k is a fixed positive integer. That is why the assumption $\nu_2 < 1$ made in (1) is crucial in our setting.

Remark 2. Consider the case when ν is the Poisson distribution with parameter 1 (observe though that $\mathbb{P}(D = 0) > 0$, so strictly speaking, it is out of our setting, but our result still holds as vertices with degree 0 play no role). Then, for large integers n, $\mathcal{G}_n$ is an approximation of the Erdős–Rényi random graph $G(n, 1)$. Now, in that case, $\mu = \beta = 1$, so the process W^ν is the Brownian motion with drift $-t$ at time t, which describes the asymptotic component sizes of $G(n, 1)$ (see [1]).

Note that in our setting,

$$\liminf_{n\to\infty} \mathbb{P}(\mathcal{G}_n \text{ is a simple graph}) > 0$$

(see Bollobás [9], Janson [16]). Let $\mathcal{SG}_n$ be the multigraph $\mathcal{G}_n$ conditioned on being simple. As observed in the introduction, $\mathcal{SG}_n$ is also the random simple graph consisting of n vertices such that, conditioned on the degree sequence $(D_1, \dots, D_n)$, it is uniformly distributed over all grahs with this degree sequence. Authors usually use those two points to first focus on $\mathcal{G}_n$, and then derive the results for $\mathcal{SG}_n$ (see for instance Pittel [22], Janson [15], Janson and Luczak [17]). Nevertheless, because the convergence in Theorem 1 is in distribution and not in probability, we cannot deduce easily results for the simple graph $\mathcal{SG}_n$. We nonetheless believe that the following assertion should hold:

Conjecture 1. *Suppose ν satisfies (1). Let $\mathcal{C}_n^{\nu,s}$ be the ordered sequence of component sizes of $\mathcal{SG}_n$. Then*

$$n^{-2/3}\mathcal{C}_n^{\nu,s} \xrightarrow[n\to\infty]{(d)} (|\gamma_j|, j \geq 1)$$

with respect to the $l^2_{\searrow}$ topology.

3 The depth-first search

3.1 An algoritmic construction of $\mathcal{G}_n$

We start by describing a convenient algorithm to construct the multigraph $\mathcal{G}_n$. Suppose that $\sum_{i=1}^n D_i$ is even. We partition the set of half-edges into three subsets: the set $\mathcal{S}$ of sleeping half-edges, the set $\mathcal{A}$ of active half-edges and the set $\mathcal{D}$ of dead half-edges. $\mathcal{S} \cup \mathcal{A}$ is the set of living half-edges. Initially, all the half-edges are sleeping.

Pick a sleeping half-edge uniformly at random and let v_1 denote the vertex it is attached to. Declare all the half-edges attached to v_1 active. While $\mathcal{A} \neq \emptyset$, proceed as follows.

- Let i be the largest integer k such that there exists an active half-edge attached to v_k.
- Consider an active half-edge l attached to v_i.
- Kill l, *i.e.*, remove it from $\mathcal{A}$ and place it into $\mathcal{D}$.
- Choose uniformly at random a living half-edge r and pair l to it.
- If r is sleeping, let v_{j+1} denote the vertex it is attached to, where j is the number of vertices which were found before the discovery of the vertex attached to r. Declare all the half-edges attached to v_{j+1} except r active.
- Kill r.

Iterate until $\mathcal{A} = \emptyset$. At that step, the first component has been totally explored. If $\mathcal{S} \neq \emptyset$, proceed similarly with the remaining living vertices until all the half-edges have been killed; the multigraph $\mathcal{G}_n$ is then constructed, and its vertices have been ordered via a depth-first search. See Figure 3.2 below for a simple illustration.

Note also that, by construction, the order in which the components appear in the depth-first search is size-biased order.

3.2 The depth-first walk

We now explain how the information on the component sizes may be encoded in a walk constructed via the depth-first search, which, as we shall see, is related to the process W^ν. We shall also need the notion of *cycle half-edge*.

Definition. *A half-edge l is called a* cycle half-edge *if there exists a half-edge r such that:*

- *l was killed before r,*

- *l was paired to r,*

- *r was active when l was paired to it.*

Observe that there exists a bijection between the set of cycle half-edges and the set of cycles, loops and multiple edges in $\mathcal{G}_n$.

Let us now define the walk associated to the depth-first search which will encode all the information that we need to study the component sizes. Write $(\hat{D}_i, i \in \{1, 2, \dots n\})$ the sequence of the degrees of the vertices of $\mathcal{G}_n$ ordered by their appearances in the depth-first search: for every $i \in \{1, \dots, n\}$,

$$\hat{D}_i = \text{degree of } v_i.$$

Define the depth-first walk $(W_n(i), 0 \leq i \leq n)$ by letting for every $i \in \{0, \dots, n\}$,

$$W_n(i) \quad = \quad \sum_{j=1}^{i} \left(\hat{D}_j - 2 - 2\# \left\{ \text{cycle half-edges attached to } v_k \right\} \right). \tag{2}$$

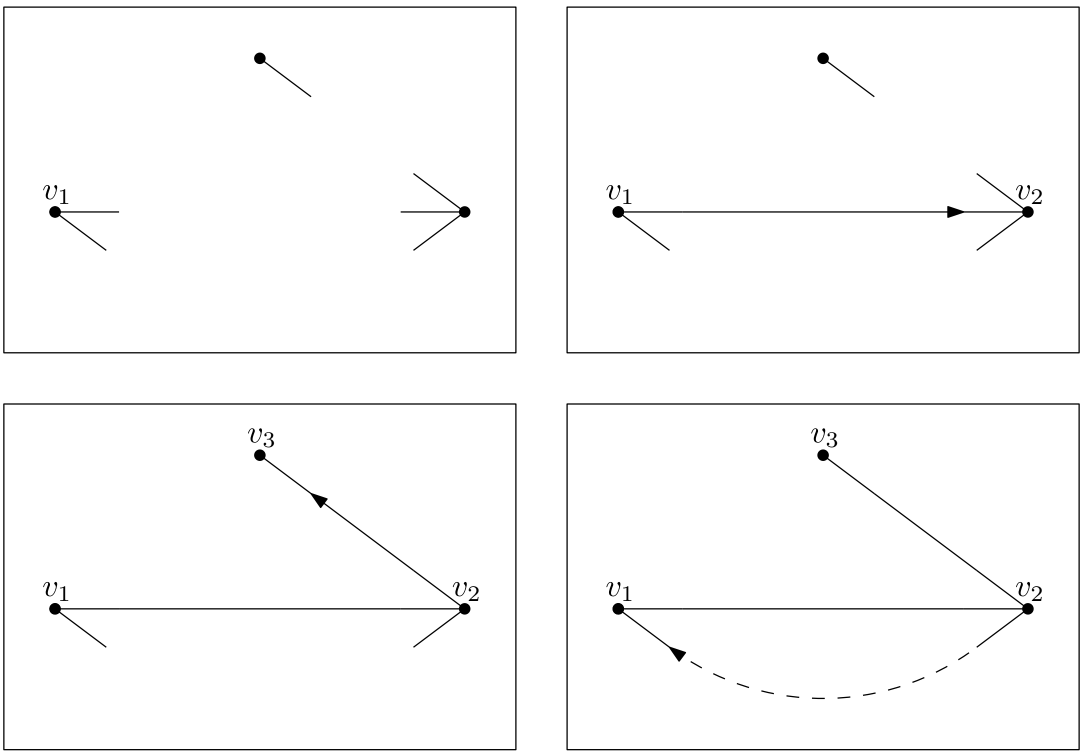

Figure 1: A realization of the algorithm constructing $\mathcal{G}_3$. The dashed oriented edge of the last picture contains a cycle half-edge at its origin; v_2 has a cycle half-edge. The sequence of the steps of the walk W_3 is $(0, -1, -2)$.

Order the components $\mathcal{C}_1, \mathcal{C}_2, \ldots$ according to the depth-first search. Let

$$\begin{aligned} \zeta(k) &= \sum_{j=1}^{k} |\mathcal{C}_j|, \\ \zeta^{-1}(i) &= \min\{k : \zeta(k) \geq i\}, \end{aligned}$$

so that $\zeta^{-1}(i)$ is the index of the component containing v_i. It is easily seen that

$$W_n(\zeta(k)) = -2k \quad \text{and} \quad W_n(i) \geq -2k-1 \quad \text{for all } \zeta(k) \leq i < \zeta(k+1). \tag{3}$$

It follows that we can recover component sizes and indices from the walk via

$$\begin{aligned} \zeta(k) &= \min\{i : W_n(i) = -2k\}, \\ |\mathcal{C}_j| &= \zeta(j) - \zeta(j-1), \\ \zeta^{-1}(i) &= 1 - \left\lceil \min_{j<i} \frac{W_n(j)}{2} \right\rceil . \end{aligned}$$

Our main result relates the walk to the process W^ν:

Theorem 2. *Rescale the depth-first walk W_n by defining for every $t \in [0, n^{1/3}]$*

$$\overline{W}_n(t) = n^{-1/3} W_n\left(\lfloor t n^{2/3} \rfloor\right).$$

Then

$$\overline{W}_n \xrightarrow[n\to\infty]{(d)} W^\nu.$$

To see how Theorem 1 follows from Theorem 2, we refer to Section 3.4 of the remarkable paper [1] of Aldous. Intuitively, the result should be clear from property (3) of depth-first walk. Component sizes are indeed encoded as lengths of path segments above past even minima; these converge to lengths of excursions of W^ν above past minima, which are just lengths of excursions of the reflected process $(W^\nu(t) - \min_{0\leq s\leq t} W^\nu(s), t \geq 0)$ above 0. Similarly, Conjecture 1 would be proven as soon as the following result is shown:

Conjecture 2. *The law of the rescaled walk $\overline{W}_n$ conditioned on the event $\{\mathcal{G}_n$ is simple$\}$ converges in distribution to W^ν as $n \to \infty$.*

The next two sections are devoted to the proof of Theorem 2. In Section 4, we shall be interested in the depth-first walk $(\sum_{j=1}^{i}(\hat{D}_j - 2), 0 \leq i \leq n)$. It is easier to study the latter than the walk W_n since it ignores cycle half-edges and its law only depends on the sequence $(\hat{D}_j, 1 \leq j \leq n)$, which has the law of the size-biased ordering of n independent copies of D. Let

$$\overline{s}_n : t \in \left[0, n^{1/3}\right] \mapsto n^{-1/3} \sum_{1\leq j\leq tn^{2/3}} \left(\hat{D}_j - 2\right).$$

We shall show that the walk $\overline{s}_n$ converges in distribution to W^ν as $n \to \infty$. In Section 5, we shall see that the difference between the two rescaled depth-first walks $\overline{W}_n$ and $\overline{s}_n$ is so small that in the limit, these processes have the same behavior. The combination of the two remarks yields Theorem 2.

4 Convergence of the walk $\overline{s}_n$

As mentioned above, in this section, we forget the contribution of the cycle half-edges to the depth-first walk W_n (we shall see in Section 5 that there are indeed few cycle half-edges up to time $tn^{2/3}$ for every $t > 0$) and we only focus on the simpler walk $(\sum_{j=1}^{i}(\hat{D}_j - 2), 0 \leq i \leq n)$.

It is easily seen that the configuration model defining $\mathcal{G}_n$ induces a degree-biased ordering of its vertices: conditionally on the degree sequence $(D_1, \ldots, D_n)$, the sequence $(\hat{D}_1, \ldots, \hat{D}_n)$ has the law of a size-biased reordering of $(D_1, \ldots, D_n)$. Conditionally on $(D_1, \ldots, D_n) = (d_1, \ldots, d_n)$, a convenient way to degree-biased order the vertices of $\mathcal{G}_n$ is to assign an exponential clock with parameter d_i to the vertex i, $i \in \{1, 2, \ldots, n\}$, and to order the vertices according to the times the clocks they are attached to ring. Furthermore, since we want to discover vertices regularly as the walk $(\sum_{j=1}^{i}(\hat{D}_j - 2), 0 \leq i \leq n)$ does, time should elapse quicker and quicker, so we need to speed it up.

Specifically, let

$$\mathcal{L} : t \geq 0 \mapsto \sum_{k \in \mathbb{N}^*} e^{-kt} \nu_k$$

be the Laplace transform of ν. Let $\phi = 1 - \mathcal{L}$. Write ψ for the inverse of ϕ. Following [21], consider a Poisson point process Π_n on $\mathbb{N}^* \times (0, n)$ with intensity π_n, where

$$\pi_n(k, \mathrm{d}s) \quad = \quad k e^{-k\psi(s/n)} \psi'(s/n) \nu_k \mathrm{d}s.$$

The k-components of the atoms of Π_n should be viewed as degrees whereas the s-components should be seen as time. Observe that since ψ is a convex increasing function, time is indeed speeded up.

Let $(\hat{\mathcal{D}}_1, \hat{\mathcal{D}}_2, \ldots)$ be the sequence of the k-components of the atoms ordered according to the s-components increasing. Conditionally on Π_n having exactly n atoms, $(\hat{\mathcal{D}}_1, \ldots \hat{\mathcal{D}}_n)$ has the same law as the random vector $(\hat{D}_1, \ldots, \hat{D}_n)$. Nonetheless, since the number of atoms of Π_n is a Poisson variable with parameter n and, as we shall soon see, we are only interested in what happens up to time $O(n^{2/3})$, we shall study the process Π_n without the latter conditioning. We thus get a Markovian process.

Observe that for every $s \in (0, n)$, $\sum_{k \in \mathbb{N}^*} k e^{-k\psi(s/n)} \psi'(s/n) \nu_k = 1$. Hence the following result, which shows that the atoms of Π_n are discovered regularly with respect to time.

Lemma 1. *The point process*

$$\{s \in (0, n) : \textit{there exists } k \in \mathbb{N}^* \textit{ such that } (k, s) \in \Pi_n\}$$

is Poisson point process on $(0, n)$ *with intensity* $\mathrm{d}t$.

It should now be natural to introduce the process $(S_n(t))_{t \geq 0}$ defined as the sum of the k-components of the atoms of Π_n minus 2 with s-components less than or equal to t:

$$S_n(t) = \sum_{(k,s) \in \Pi_n} (k - 2) \mathbf{1}_{s \leq t}.$$

In other words, letting $N_n(t) = \#\{s \in (0, t] : \text{there exists } k \in \mathbb{N}^* \text{ such that } (k, s) \in \Pi_n\}$,

$$S_n(t) = \sum_{j=1}^{N_n(t)} \left(\hat{\mathcal{D}}_j - 2 \right).$$

We can now state the key result of the present work:

Proposition 1. *Rescale S_n by defining $\overline{S}_n : t \geq 0 \mapsto n^{-1/3} S_n(tn^{2/3})$. Then*

$$\overline{S}_n \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} W^\nu.$$

Proof. We follow the ideas of Aldous [1]. Let

$$A_n : t \mapsto \int \pi_n(k, \mathrm{d}s)(k-2)\mathbf{1}_{s\leq t}$$

be the continuous bounded variation process such that

$$M_n(t) = S_n(t) - A_n(t), \quad t \geq 0,$$

is a martingale. Observe that A_n is deterministic. Rescaling as usual to define $\overline{A}_n$ and $\overline{M}_n$, we shall see in Lemma 2 below that for every $t_0 > 0$,

$$\lim_{n\to\infty} \sup_{t\leq t_0} \left| \overline{A}_n(t) + \frac{\beta}{2\mu^2} t^2 \right| = 0$$

and in Lemma 3 below that

$$\overline{M}_n \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} \sqrt{\frac{\beta}{\mu}} B,$$

where B denotes a standard Brownian motion, which will imply Proposition 1. □

Lemma 2. *For every $t_0 > 0$,*

$$\lim_{n\to\infty} \sup_{t\leq t_0} \left| \overline{A}_n(t) + \frac{\beta}{2\mu^2} t^2 \right| = 0.$$

Proof. By definition,

$$\begin{aligned} A_n(t) &= \int_0^t \sum_{k\in\mathbb{N}^*} (k^2 - 2k) e^{-k\psi(s/n)} \psi'(s/n) \nu_k \mathrm{d}s \\ &= \int_0^t (a_n(s) - 2) \mathrm{d}s, \end{aligned}$$

where

$$a_n(s) = \frac{\mathcal{L}''(\psi(s/n))}{-\mathcal{L}'(\psi(s/n))}.$$

Since $\mathbb{E}[D^2] = 2\mathbb{E}[D]$, $a_n(s)$ tends to 2 as $n \to \infty$. Moreover, it is easily seen by approximating $\psi(s/n)$ by $\frac{s}{\mu n}$ that $a_n(s) - 2$ is approximatively $-\frac{\beta}{\mu^2}\frac{s}{n}$. Let us be more precise.

Let $s \in [0, t_0 n^{2/3}]$. Observe that

$$\mathcal{L}''(x) = \mathbb{E}\left[D^2\right] - \mathbb{E}\left[D^3\right] x + o(x) \quad \text{and} \quad -\mathcal{L}'(x) = \mathbb{E}\left[D\right] - \mathbb{E}\left[D^2\right] x + o(x). \tag{4}$$

Because $\psi(x) = \frac{x}{\mu} + o(x)$, we deduce that there exists a function $\varepsilon(\cdot)$ tending to 0 at 0 such that:

$$a_n(s) - 2 = -\frac{\beta}{\mu^2}\frac{s}{n} + \frac{s}{n}\varepsilon\left(\frac{s}{n}\right).$$

We deduce that for every $t \in [0, t_0 n^{2/3}]$,

$$\left| A_n(t) + \frac{\beta}{\mu^2} \frac{t^2}{2n} \right| \leq \frac{1}{n} \int_0^t s \left| \varepsilon \left(\frac{s}{n} \right) \right| \mathrm{d}s \leq \frac{1}{n} \int_0^{t_0 n^{2/3}} s \left| \varepsilon \left(\frac{s}{n} \right) \right| \mathrm{d}s.$$

As a result, for every $\eta > 0$, there exists an integer $n_0(\eta)$ such that for every integer $n \geq n_0(\eta)$,

$$\sup_{t \leq t_0 n^{2/3}} \left| A_n(t) + \frac{\beta}{2\mu^2} \frac{t^2}{n} \right| \leq \frac{1}{n} \int_0^{t_0 n^{2/3}} s \eta \mathrm{d}s = \frac{t_0^2}{2} \eta n^{1/3},$$

which proves Lemma 2. □

Lemma 3. $\overline{M}_n \xrightarrow[n\to\infty]{(d)} \sqrt{\frac{\beta}{\mu}} B$, *where B denotes a standard Brownian motion.*

Proof. We want to apply the functional CLT for continuous-time martingales (see [11, Theorem 7.1.4(b)]). Since M_n is a purely discontinuous martingale, $[M_n]_t = \sum_{s \leq t} \Delta M_n(s)^2$, so that its predictable projection

$$\langle M_n \rangle : t \mapsto \int \pi_n(k, \mathrm{d}s)(k-2)^2 \mathbf{1}_{s \leq t}$$

is the continuous, increasing process such that $M_n^2 - \langle M_n \rangle$ is a martingale. Observe that $\langle M_n \rangle$ is deterministic. Defining $\overline{\langle M_n \rangle} : t \mapsto n^{-2/3} \langle M_n \rangle (t n^{2/3})$, all we have to prove is that for every $t_0 > 0$,

$$\overline{\langle M_n \rangle}(t_0) \xrightarrow[n\to\infty]{} \frac{\beta}{\mu} t_0 \tag{5}$$

and

$$\lim_{n\to\infty} \mathbb{E} \left[\sup_{t \leq t_0} \left| \overline{M}_n(t) - \overline{M}_n(t-) \right|^2 \right] = 0. \tag{6}$$

Let us establish (5). Elementary calculations yield:

$$\langle M_n \rangle (t) = \int_0^t b_n(s) \mathrm{d}s,$$

where

$$b_n(s) = \frac{\mathcal{L}^{(3)} + 4\mathcal{L}'' + 4\mathcal{L}'}{\mathcal{L}'} \circ \psi \left(\frac{s}{n} \right).$$

It is then easily seen that there exists a function $\varepsilon(\cdot)$ tending to 0 at 0 such that:

$$b_n(s) = \frac{\beta}{\mu} + \varepsilon \left(\frac{s}{n} \right).$$

We deduce that

$$\left| \langle M_n \rangle (t_0 n^{2/3}) - \frac{\beta}{\mu} t_0 n^{2/3} \right| \leq \int_0^{t_0 n^{2/3}} \left| \varepsilon \left(\frac{s}{n} \right) \right| \mathrm{d}s.$$

Hence, for every $\eta > 0$, there exists an integer $n_1(\eta)$ such that for every integer $n \geq n_1(\eta)$,

$$\left| \langle M_n \rangle (t_0 n^{2/3}) - \frac{\beta}{\mu} t_0 n^{2/3} \right| \leq \eta t_0 n^{2/3},$$

which proves (5).

We next turn our attention to (6). Note that $M_n(t) - M_n(t-) = S_n(t) - S_n(t-)$, so

$$\begin{aligned} \sup_{t \leq t_0 n^{2/3}} |M_n(t) - M_n(t-)|^2 &= \sup \left\{ (k-2)^2 : (k,s) \in \Pi_n \text{ and } s \leq t_0 n^{2/3} \right\} \\ &\leq \sup \left\{ k^2 : (k,s) \in \Pi_n \text{ and } s \leq t_0 n^{2/3} \right\}. \end{aligned}$$

Let S_n denote $\sup\{k : (k,s) \in \Pi_n \text{ and } s \leq t_0 n^{2/3}\}$ (we drop the dependency on t_0 in the notation). We have:

$$\mathbb{E}\left[S_n^2\right] = \sum_{k=1}^{\lfloor n^{1/3} \rfloor - 1} \mathbb{P}\left(S_n \geq \sqrt{k}\right) + \sum_{k \geq n^{1/3}} \mathbb{P}\left(S_n \geq \sqrt{k}\right) \leq n^{1/3} + \sum_{k \geq n^{1/3}} \mathbb{P}\left(S_n \geq \sqrt{k}\right).$$

Now, for every $m \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}(S_n \geq m) &= 1 - \mathbb{P}(S_n < m) \\ &= 1 - \mathbb{P}\left(\Pi_n\left(\{m, m+1, \dots\} \times [0, t_0 n^{2/3}]\right) = 0\right) \\ &= 1 - \exp\left(-\pi_n\left(\{m, m+1, \dots\} \times [0, t_0 n^{2/3}]\right)\right) \\ &\leq \pi_n\left(\{m, m+1, \dots\} \times [0, t_0 n^{2/3}]\right) \\ &= \sum_{l \geq m} \nu_l \int_0^{t_0 n^{2/3}} \mathrm{d}s l e^{-l\psi(s/n)} \psi'(s/n) \\ &= n \sum_{l \geq m} \nu_l \left(1 - e^{-l\psi(t_0 n^{-1/3})}\right) \\ &\leq n\psi(t_0 n^{-1/3}) \sum_{l \geq m} l \nu_l. \end{aligned}$$

As a result,

$$\begin{aligned} \mathbb{E}\left[S_n^2\right] &\leq n^{1/3} + \sum_{k \geq n^{1/3}} n\psi(t_0 n^{-1/3}) \sum_{l \geq \sqrt{k}} l \nu_l \\ &= n^{1/3} + n\psi(t_0 n^{-1/3}) \sum_{l \geq n^{1/6}} l \nu_l \sum_{k = \lfloor n^{1/3} \rfloor}^{l^2} 1. \end{aligned}$$

We deduce that for every integer n,

$$n^{-2/3} \mathbb{E}\left[\sup_{t \leq t_0 n^{2/3}} |M_n(t) - M_n(t-)|^2\right] \leq n^{-1/3} + n^{1/3} \psi(t_0 n^{-1/3}) \sum_{l \geq n^{1/6}} l^3 \nu_l.$$

Now, $n^{1/3}\psi(t_0 n^{-1/3})$ tends to $\frac{t_0}{\mu}$ and since $\mathbb{E}[D^3]$ is finite, $\sum_{l\geq n^{1/6}} l^3\nu_l$ tends to 0. Equation (6) is therefore proved. □

We now give a key consequence of Proposition 1 concerning the depth-first walk $\overline{s}_n$.

Corollary 1. *The rescaled depth-first walk $\overline{s}_n$ converges in distribution to W^ν as $n \to \infty$.*

Proof. Let T_n^j be the first time when j atoms of Π_n have been visited:

$$T_n^j = \inf\{t \geq 0 : N_n(t) \geq j\}.$$

Applying Lemma 1 and Proposition 1, the process

$$\overline{S}_n\left(n^{-2/3}T_n^{\lfloor tn^{2/3}\rfloor}\right), \quad t \geq 0,$$

converges in distribution to W^ν as $n \to \infty$. Now, for every $t \geq 0$,

$$\overline{S}_n\left(n^{-2/3}T_n^{\lfloor tn^{2/3}\rfloor}\right) = n^{-1/3}\sum_{1\leq j\leq tn^{2/3}}\left(\hat{\mathcal{D}}_j - 2\right).$$

A depoissonization completes the proof. □

5 Study of the cycle half-edges

In this section, we turn our attention to the cycle half-edges. In Section 5.1, we shall prove that there are few cycle half-edges in $\mathcal{G}_n$ (see Lemma 4). We shall then show in Section 5.2 how to derive Theorem 2 from Corollary 1 and Lemma 4.

5.1 Upper bound of the number of cycle half-edges

In this section, we prove the following result:

Lemma 4. *Let $t > 0$ and $M > 0$. Introduce the event*

$$E_n(t,M) = \left\{\max_{i\leq t}\left\{\overline{s}_n(i) - \min_{k\leq i}\overline{s}_n(k)\right\} \leq M\right\}.$$

Then we have

$$\limsup_{n\to\infty}\mathbb{E}\left[\#\left\{\textit{cycle half-edges attached to } v_i,\ i=1,\ldots,\lfloor tn^{2/3}\rfloor,\ \textit{in } \mathcal{G}_n\right\}\mathbf{1}_{E_n(t,M)}\right] < \infty.$$

Proof. We first study the number of active half-edges, given they contribute to the appearance of cycle half-edges. It is easily seen that, during the depth-first search, when a component is explored, the number of active half-edges when k vertices have been partially or totally explored is less than or equal to the maximal height of the current excursion of the depth-first walk $(\sum_{j=1}^{i}(\hat{D}_j - 2), 0 \leq i \leq n)$ above its past minimum plus 1, up to time k. (Note that this would not be true if we had considered the breadth-first search.) Consequently, under the event $E_n(t,M)$, during the first $\lfloor tn^{2/3}\rfloor$ steps, $\#\mathcal{A}$ is always less than or equal to $Mn^{1/3}+1$.

For every deterministic sequence $(x_1, \ldots, x_n)$ of positive integers such that $\sum_{i=1}^n x_i$ is even, conditionally on the event $(\hat{D}_1, \ldots, \hat{D}_n) = (x_1, \ldots, x_n)$, one has:

$$\begin{aligned}
& \mathbb{E}\left[\# \left\{\text{cycle half-edges attached to } v_i,\ i = 1, \ldots, \lfloor tn^{2/3} \rfloor\right\} \mathbf{1}_{E_n(t,M)} \middle| \hat{D}_1 = x_1, \ldots, \hat{D}_n = x_n\right] \\
= \ & \mathbb{E}\left[\sum_{m=1}^{\lfloor tn^{2/3} \rfloor} \sum_{k=1}^{\hat{D}_m} \mathbf{1}_{\{\text{the } k\text{-th half-edge of } v_m \text{ is a cycle half-edge}\}} \mathbf{1}_{E_n(t,M)} \middle| \hat{D}_1 = x_1, \ldots, \hat{D}_n = x_n\right] \\
\leq \ & \sum_{m=1}^{\lfloor tn^{2/3} \rfloor} \sum_{k=1}^{x_m} \mathbb{P}\left(\text{the } k\text{-th half-edge of } v_m \text{ is a cycle half-edge} \middle| \hat{D}_1 = x_1, \ldots, \hat{D}_n = x_n \text{ and } E_n(t,M)\right) \\
\leq \ & \sum_{m=1}^{\lfloor tn^{2/3} \rfloor} x_m \frac{Mn^{1/3} + 1}{\sum_{i=1}^n x_i - \sum_{i=1}^{\lfloor tn^{2/3} \rfloor} x_i} \\
\leq \ & \sum_{m=1}^{\lfloor tn^{2/3} \rfloor} x_m \frac{Mn^{1/3} + 1}{n - tn^{2/3}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \mathbb{E}\left[\# \left\{\text{cycle half-edges attached to } v_i,\ i = 1, \ldots, \lfloor tn^{2/3} \rfloor, \text{ in } \mathcal{G}_n\right\} \mathbf{1}_{E_n(t,M)}\right] \\
\leq \ & \frac{Mn^{1/3} + 1}{n - tn^{2/3}} \mathbb{E}\left[\sum_{m=1}^{\lfloor tn^{2/3} \rfloor} \hat{D}_m\right] \\
\leq \ & \frac{Mn^{1/3} + 1}{n - tn^{2/3}} tn^{2/3} \mathbb{E}\left[\hat{D}_1\right].
\end{aligned}$$

Note that $\mathbb{E}[\hat{D}_1] \leq \sum_{k=1}^{\infty} k \frac{k\nu_k}{\mu}$. Hence

$$\limsup_{n \to \infty} \mathbb{E}\left[\# \left\{\text{cycle half-edges attached to } v_i,\ i = 1, \ldots, \lfloor tn^{2/3} \rfloor\right\} \mathbf{1}_{E_n(t,M)}\right] \leq 2Mt,$$

which completes the proof of Lemma 4. □

Remark 3. We can prove that in fact, for every $t > 0$,

$$\limsup_{n \to \infty} \mathbb{E}\left[\# \left\{\text{cycle half-edges attached to } v_i,\ i = 1, \ldots, \lfloor tn^{2/3} \rfloor, \text{ in } \mathcal{G}_n\right\}\right] < \infty.$$

Remark 4. We stress that a consequence of [6, Theorem 1] is that the expected number of cycle half-edges in $\mathcal{G}_n$ is $o(n)$.

5.2 End of the proof of Theorem 2

In this section, we prove Theorem 2. We keep the notation of Section 5.1. Let $t > 0$. Let f be a bounded, Lipschitz function defined on $(\mathcal{C}[0,t], \mathbb{R})$. Applying Corollary 1 and the Portmanteau theorem, it suffices to prove that $\mathbb{E}[f(\overline{W}_n)] - \mathbb{E}[f(\overline{s}_n)]$ tends to 0 as $n \to \infty$.

There exists $K > 0$ such that for every $w, w' \in (\mathcal{C}[0,t], \mathbb{R})$, $|f(w)| \leq K$ and $|f(w) - f(w')| \leq K\|w - w'\|$. Let $M > 0$. One has:

$$\begin{aligned} & \left|\mathbb{E}\left[f\left(\overline{W}_n\right)\right] - \mathbb{E}\left[f\left(\overline{s}_n\right)\right]\right| \\ = \; & \mathbb{E}\left[\left|f\left(\overline{W}_n\right) - f\left(\overline{s}_n\right)\right| \mathbf{1}_{E_n(t,M)}\right] + \mathbb{E}\left[\left|f\left(\overline{W}_n\right) - f\left(\overline{s}_n\right)\right| \left(1 - \mathbf{1}_{E_n(t,M)}\right)\right] \\ \leq \; & \mathbb{E}\left[K \left\|\overline{W}_n - \overline{s}_n\right\| \mathbf{1}_{E_n(t,M)}\right] + \mathbb{E}\left[2K\left(1 - \mathbf{1}_{E_n(t,M)}\right)\right] \\ \leq \; & K n^{-1/3} \mathbb{E}\left[\#\left\{\text{cycle half-edges attached to } v_i,\, i = 1, \ldots, \lfloor t n^{2/3} \rfloor\right\} \mathbf{1}_{E_n(t,M)}\right] \\ & + 2K\mathbb{P}\left(\max_{i \leq t}\left\{\overline{s}_n(i) - \min_{k \leq i} \overline{s}_n(k)\right\} \geq M\right). \end{aligned}$$

Lemma 4 ensures that

$$\lim_{n\to\infty} n^{-1/3} \mathbb{E}\left[\#\left\{\text{cycle half-edges attached to } v_i,\, i = 1, \ldots, \lfloor t n^{2/3} \rfloor\right\} \mathbf{1}_{E_n(t,M)}\right] = 0.$$

Moreover, applying Corollary 1 and the Portmanteau theorem,

$$\limsup_{n\to\infty} \mathbb{P}\left(\max_{i \leq t}\left\{\overline{s}_n(i) - \min_{k \leq i} \overline{s}_n(k)\right\} \geq M\right) \leq \mathbb{P}\left(\max_{s \leq t}\left\{W^\nu(s) - \min_{u \leq s} W^\nu(u)\right\} \geq M\right).$$

Therefore, for every $M > 0$,

$$\limsup_{n\to\infty} \left|\mathbb{E}\left[f\left(\overline{W}_n\right)\right] - \mathbb{E}\left[f\left(\overline{s}_n\right)\right]\right| \leq 2K\mathbb{P}\left(\max_{s \leq t}\left\{W^\nu(s) - \min_{u \leq s} W^\nu(u)\right\} \geq M\right).$$

Now, the continuity of W^ν implies that

$$\lim_{M\to\infty} \mathbb{P}\left(\max_{s \leq t}\left\{W^\nu(s) - \min_{u \leq s} W^\nu(u)\right\} \geq M\right) = 0.$$

Hence

$$\lim_{n\to\infty} \mathbb{E}\left[f\left(\overline{W}_n\right)\right] - \mathbb{E}\left[f\left(\overline{s}_n\right)\right] = 0.$$

Theorem 2 is therefore proved. □

6 The power law distribution setting

In this section, we do not suppose the finiteness of the moment of order 3 for distribution ν, and rather we replace assumption (1) by the following

$$\sum_{k=1}^{\infty} k(k-2)\nu_k = 0 \quad \text{and} \quad \nu_k \underset{k\to\infty}{\sim} c k^{-\gamma}, \tag{7}$$

where $c > 0$ and $\gamma \in (3, 4)$. This implies that (4) has to be replaced by

$$\mathcal{L}''(x) = 2\mu - \frac{c\,\Gamma(4-\gamma)}{\gamma - 3} x^{\gamma-3} + \underset{x\to 0}{o}(x^{\gamma-3}). \tag{8}$$

We are interested in the component sizes of the multigraph constructed the same way as before. To have a good idea of what the order of the component sizes should be, we adopt the same strategy. Specifically, taking the same notation as in Section 4, we shall consider the process $(S_n(t))_{t\geq 0}$:

$$S_n(t) \;=\; \sum_{(k,s)\in\Pi_n} (k-2)\mathbf{1}_{s\leq t}.$$

Recall that Π_n is a Poisson point process on $\mathbb{N}^* \times (0,n)$ with intensity π_n, where $\pi_n(k, \mathrm{d}s) = ke^{-k\psi(s/n)}\psi'(s/n)\nu_k \mathrm{d}s$. We intend to prove the following result:

Theorem 3. *Rescale S_n by defining $\overline{S}_n : t \geq 0 \mapsto n^{-1/(\gamma-1)} S_n\left(tn^{(\gamma-2)/(\gamma-1)}\right)$. Then*

$$\overline{S}_n \;\xrightarrow[n\to\infty]{(d)}\; X^\nu + A^\nu,$$

where

$$A^\nu : t \mapsto -\frac{c\,\Gamma(4-\gamma)}{(\gamma-3)(\gamma-2)\mu^{\gamma-2}} t^{\gamma-2}$$

and X^ν is the unique process with independent increments such that for every $t \geq 0$ and $u \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(iuX^\nu_t\right)\right] \;=\; \exp\left(\int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}x \left(e^{iux} - 1 - iux\right) \frac{c}{\mu}\frac{1}{x^{\gamma-1}} e^{-xs/\mu}\right).$$

Proof. As before, let

$$A_n : t \mapsto \int \pi_n(k, \mathrm{d}s)(k-2)\mathbf{1}_{s\leq t}$$

be the deterministic continuous bounded variation function such that

$$M_n(t) = S_n(t) - A_n(t), \quad t \geq 0,$$

is a martingale. Rescaling as in Theorem 3 to define $\overline{A}_n$ and $\overline{M}_n$, we can easily see as in Lemma 2 (recall (8)) that for every $t > 0$,

$$\lim_{n\to\infty} \sup_{s\leq t} \left|\overline{A}_n(s) - A^\nu(s)\right| = 0.$$

To complete the proof of Theorem 3, it thus suffices to show that

$$\overline{M}_n \xrightarrow[n\to\infty]{(d)} X^\nu.$$

This will be done in Lemmas 5 and 6 by applying general results on convergence to a process with independent increments borrowed from [14]. □

The rest of this section is organized as follows. We shall first study in Lemma 5 the martingale $\overline{M}^{(1)}_n$ related to the small jumps of $\overline{M}_n$. Then, in Lemma 6, we shall be interested in the martingale $\overline{M}^{(2)}_n$ which counts the big jumps. The fact that $\overline{M}_n = \overline{M}^{(1)}_n + \overline{M}^{(2)}_n$ converges to X^ν, which is the sum of the limits of $\overline{M}^{(1)}_n$ and $\overline{M}^{(2)}_n$, stems from the independence of $\overline{M}^{(1)}_n$ and $\overline{M}^{(2)}_n$ (since they never jump simultaneously). To ease notation, let

$$a = \frac{1}{\gamma - 1}.$$

Lemma 5. *The martingale* $\overline{M}_n^{(1)}$ *defined for every* $t \geq 0$ *by*

$$\overline{M}_n^{(1)}(t) = \sum_{(k,s)\in\Pi_n} \mathbf{1}_{k<n^a}(k-2)n^{-a}\mathbf{1}_{s\leq tn^{1-a}} - \int \pi_n(k,ds)\mathbf{1}_{k<n^a}(k-2)n^{-a}\mathbf{1}_{s\leq tn^{1-a}}$$

converges in distribution as $n \to \infty$ *to a process* $(X_t^{(1)})_{t\geq 0}$ *with independent increments characterized by: for every* $t \geq 0$ *and* $u \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(iuX_t^{(1)}\right)\right] = \exp\left(\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}x \left(e^{iux} - 1 - iux\right) \frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\right).$$

Proof. First observe that the process $X^{(1)}$ may be defined as the limit for the metric induced by the norm

$$\|Y\| = \mathbb{E}\left[\sup\{Y_s^2 : 0 \leq s \leq t\}\right]^{1/2}$$

of the Cauchy family

$$t \mapsto \sum_{s\leq t} \mathbf{1}_{\Delta_s > \varepsilon}\Delta_s - \int_0^t \mathrm{d}s \int_\varepsilon^1 \mathrm{d}x\ x\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}$$

as ε tends to 0, where Δ is a Poisson point process with intensity $\mathbf{1}_{x\in(0,1)}\nu(\mathrm{d}s, \mathrm{d}x)$ with

$$\nu(\mathrm{d}s, \mathrm{d}x) = \frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\mathrm{d}s\mathrm{d}x.$$

To prove Lemma 5, we rely on [14, Theorem VII.3.7]. Dealing with small jumps of the martingale $\overline{M}_n$ indeed enables us to work with "square-integrable" processes (note that $\int_0^t \int_{\mathbb{R}} x^2\mathbf{1}_{x\in(0,1)}\nu(\mathrm{d}s, \mathrm{d}x) < \infty$).

Taking the same notation as in [14], we first have to compute the characteristics (B^n, C^n, ν^n) of $\overline{M}_n^{(1)}$, which are defined via the equation : for every $t \geq 0$ and $u \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(iu\overline{M}_n^{(1)}(t)\right)\right] = \exp\left(iuB^n(t) - \frac{1}{2}u^2C^n(t) + \int_0^t \int_{-n^{-a}}^1 \left(e^{iux} - 1 - iux\right)\nu^n(\mathrm{d}s, \mathrm{d}x)\right).$$

The exponential formula for Poisson point processes yields

$$\mathbb{E}\left[\exp\left(iu\overline{M}_n^{(1)}(t)\right)\right] = \exp\left\{n\sum_{k<n^a}\nu_k\left(1 - e^{-k\psi(tn^{-a})}\right)\left(e^{iu(k-2)n^{-a}} - 1 - iu(k-2)n^{-a}\right)\right\}.$$

Consequently, $B^n = C^n = 0$ and

$$\nu^n(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s \sum_{k<n^a} \delta_{(k-2)n^{-a}}(\mathrm{d}x)\ n^{1-a}k\nu_k\psi'(sn^{-a})e^{-k\psi(sn^{-a})}.$$

According to [14, Theorem VII.3.7], Lemma 5 will be proved as soon as we have shown that for every $t \geq 0$:

$$\int_0^t \int_{-n^{-a}}^1 x^2\nu^n(\mathrm{d}s, \mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_0^t \int_0^1 x^2\nu(\mathrm{d}s, \mathrm{d}x), \tag{9}$$

and

$$\text{for every } g \in C_2(\mathbb{R}_+), \qquad \int_0^t \int_{-n^{-a}}^1 g(x)\nu^n(\mathrm{d}s, \mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_0^t \int_0^1 g(x)\nu(\mathrm{d}s, \mathrm{d}x), \qquad (10)$$

where $C_2(\mathbb{R}_+)$ is the set of all continuous bounded functions $\mathbb{R}_+ \to \mathbb{R}$ which are 0 on a neighborhood 0 and have a limit at infinity.

Let us establish (9). Elementary calculations yield:

$$\int_0^t \int_{-n^{-a}}^1 x^2\nu^n(\mathrm{d}s, \mathrm{d}x) \;=\; n^{1-2a} \sum_{k<n^a} (k-2)^2 \nu_k \left(1 - e^{-k\psi(tn^{-a})}\right).$$

A difficulty stems from the lack of good estimates for ν_k when k is small. That is why we write

$$\begin{aligned}\int_0^t \int_{-n^{-a}}^1 x^2\nu^n(\mathrm{d}s, \mathrm{d}x) \;&=\; n^{1-2a} \sum_{k\in\mathbb{N}^*} (k-2)^2 \nu_k \left(1 - e^{-k\psi(tn^{-a})}\right) \\ &\quad - n^{1-2a} \sum_{k\geq n^a} (k-2)^2 \nu_k \left(1 - e^{-k\psi(tn^{-a})}\right).\end{aligned}$$

It is easy to see that the first term in the difference tends to $\frac{c\Gamma(4-\gamma)}{(\gamma-3)\mu^{\gamma-3}} t^{\gamma-3}$. As for the second, recalling that $\nu_k \sim ck^{-\gamma}$,

$$n^{1-2a} \sum_{k\geq n^a} (k-2)^2 \nu_k \left(1 - e^{-k\psi(tn^{-a})}\right) \underset{n\to\infty}{\sim} n^{1-2a} \int_{n^a}^{\infty} \mathrm{d}x x^2 c x^{-\gamma} \left(1 - e^{-x\psi(tn^{-a})}\right).$$

A change of variables and an application of the dominated convergence theorem (recall that $\psi(x) = \frac{x}{\mu} + o(x)$) yield

$$n^{1-2a} \sum_{k\geq n^a} (k-2)^2 \nu_k \left(1 - e^{-k\psi(tn^{-a})}\right) \underset{n\to\infty}{\longrightarrow} \int_1^{\infty} \mathrm{d}x \; c\frac{1 - e^{-xt/\mu}}{x^{\gamma-2}}.$$

Noticing that

$$\frac{c\,\Gamma(4-\gamma)}{(\gamma-3)\mu^{\gamma-3}} t^{\gamma-3} = \int_0^{\infty} \mathrm{d}x \; c\frac{1 - e^{-xt/\mu}}{x^{\gamma-2}},$$

we finally get

$$\int_0^t \int_{-n^{-a}}^1 x^2\nu^n(\mathrm{d}s, \mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_0^1 \mathrm{d}x \; c\frac{1 - e^{-xt/\mu}}{x^{\gamma-2}},$$

which proves (9).

We now turn our attention to (10). Let $\varepsilon \in (0, 1)$ and $g : [\varepsilon, 1] \to \mathbb{R}$ be a continuous function. Then

$$\int_0^t \int_{-n^{-a}}^1 g(x)\nu^n(\mathrm{d}s, \mathrm{d}x) \;=\; n \sum_{\varepsilon n^a < k < n^a} g\left(\frac{k-2}{n^a}\right) \nu_k \left(1 - e^{-k\psi(tn^{-a})}\right).$$

Proceeding as before, we obtain

$$\int_0^t \int_{-n^{-a}}^1 g(x)\nu^n(\mathrm{d}s, \mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_\varepsilon^1 \mathrm{d}x g(x) c \frac{1-e^{-xt/\mu}}{x^\gamma},$$

completing the proof of Lemma 5. □

In order to finish of the proof Theorem 3, we now show the convergence of the martingale related to the big jumps.

Lemma 6. *The martingale $\overline{M}_n^{(2)}$ defined for every $t \geq 0$ by*

$$\overline{M}_n^{(2)}(t) = \sum_{(k,s)\in\Pi_n} \mathbf{1}_{k\geq n^a}(k-2)n^{-a}\mathbf{1}_{s\leq tn^{1-a}} - \int \pi_n(k, ds)\mathbf{1}_{k\geq n^a}(k-2)n^{-a}\mathbf{1}_{s\leq tn^{1-a}}$$

converges in distribution as $n \to \infty$ to a process $(X_t^{(2)})_{t\geq 0}$ with independent increments characterized by: for every $s, t \geq 0$, $u \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(iuX_t^{(2)}\right)\right] = \exp\left(\int_0^t \mathrm{d}s \int_1^\infty \mathrm{d}x \left(e^{iux} - 1 - iux\right) \frac{c}{\mu}\frac{1}{x^{\gamma-1}} e^{-xs/\mu}\right).$$

Proof. The existence of $X^{(2)}$ is easily obtained as the sum of

$$B^\nu : t \mapsto -\int_0^t \mathrm{d}s \int_1^\infty \mathrm{d}x\, x \frac{c}{\mu}\frac{1}{x^{\gamma-1}} e^{-xs/\mu}$$

and the partial sum of the jumps of a Poisson point process with intensity $\mathbf{1}_{x\geq 1}\nu(\mathrm{d}s, \mathrm{d}x)$ (recall that $\nu(\mathrm{d}s, \mathrm{d}x) = \frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\mathrm{d}s\mathrm{d}x$). Let us see how Lemma 6 derives from [14, Theorem VII.3.4].

As before, we first have to compute the characterics (B^n, C^n, ν^n) of $\overline{M}_n^{(2)}$, which are now defined via the equation : for every $s, t \geq 0$, $u \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(iu\overline{M}_n^{(2)}(t)\right)\right] = \exp\left(iuB^n(t) - \frac{1}{2}u^2C^n(t) + \int_0^t \int_{1-2n^{-a}}^\infty \left(e^{iux} - 1\right)\nu^n(\mathrm{d}s, \mathrm{d}x)\right).$$

The exponential formula for Poisson point processes yields

$$\mathbb{E}\left[\exp\left(iu\overline{M}_n^{(2)}(t)\right)\right] = \exp\left\{-iun^{1-a}\sum_{k\geq n^a}(k-2)\nu_k\left(1-e^{-k\psi(tn^{-a})}\right) + n\sum_{k\geq n^a}\nu_k\left(1-e^{-k\psi(tn^{-a})}\right)\left(e^{iu(k-2)n^{-a}}-1\right)\right\}.$$

Consequently, $C^n = 0$,

$$B^n(t) = -n^{1-a}\sum_{k\geq n^a}(k-2)\nu_k\left(1-e^{-k\psi(tn^{-a})}\right)$$

and

$$\nu^n(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s \sum_{k \geq n^a} \delta_{(k-2)n^{-a}}(\mathrm{d}x)\ k\nu_k n^{1-a}\psi'(sn^{-a})e^{-k\psi(sn^{-a})}.$$

According to [14, Theorem VII.3.4], Lemma 6 will be proved as soon as we have shown that for every $t \geq 0$:

$$\sup_{s \leq t} |B^n(t) - B^\nu_t| \underset{n\to\infty}{\longrightarrow} 0, \tag{11}$$

and

$$\text{for every } g \in C_2(\mathbb{R}_+), \quad \int_0^t \int_{1-2n^{-a}}^\infty g(x)\nu^n(\mathrm{d}s, \mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_0^t \int_1^\infty g(x)\nu(\mathrm{d}s, \mathrm{d}x). \tag{12}$$

Equation (12) can be shown exactly the same way as (10), and to prove (11), it suffices to compare the series to the corresponding integrals as we did above. □

Repeating what we did in Section 5, we deduce from Theorem 3 the following key result. As before, the walk defined via (2) is denoted by W_n.

Corollary 2. *Rescale the depth-first walk W_n by defining for every $t \in [0, n^{1/(\gamma-1)}]$*

$$\overline{W}_n(t) = n^{-1/(\gamma-1)} W_n\left(\lfloor tn^{(\gamma-2)/(\gamma-1)} \rfloor\right).$$

Then

$$\overline{W}_n \xrightarrow[n\to\infty]{(d)} X^\nu + A^\nu.$$

We now give an analogous result of Theorem 1 in the present setting. Let R^ν be the reflected process defined by

$$R^\nu : t \geq 0 \mapsto X^\nu_t + A^\nu_t - \inf_{0\leq s\leq t} \{X^\nu(s) + A^\nu(s)\}.$$

We define excursion intervals and excursion lengths of R^ν as in Section 2.

Theorem 4. *Suppose ν satisfies (7). Then the set of excursions of R^ν may be written $\{\gamma_j, j \geq 1\}$ so that the lengths $|\gamma_j|$ are decreasing. Moreover*

$$\sum_{j\geq 1} |\gamma_j|^2 < \infty$$

and letting $\boldsymbol{\mathcal{C}}^\nu_n$ be the ordered sequence of component sizes of $\mathcal{G}_n$,

$$n^{-(\gamma-2)/(\gamma-1)} \boldsymbol{\mathcal{C}}^\nu_n \xrightarrow[n\to\infty]{(d)} (|\gamma_j|, j \geq 1)$$

with respect to the $l^2_\searrow$ topology.

We borrow the technique for showing how Theorem 4 can be deduced from Corollary 2 from the deep paper of Aldous and Limic [2]. Indeed, observe that the component sizes of the multigraph $\mathcal{G}_n$, in the order of appearance in depth-first walk, are size-biased ordered. Applying [2, Proposition 17] (see also [1, Proposition 15 and Lemma 25]), Theorem 4 thus derives from Corollary 2 and the following lemma (we refer to [2, Proposition 14]).

Lemma 7. *The following four assertions hold.*

1. $X_t^\nu + A_t^\nu \xrightarrow{p} -\infty$ *as* $t \to \infty$.

2. $\sup\{|\gamma| : \gamma$ *is an excursion of* R^ν *s.t.* $l(\gamma) \geq t\} \xrightarrow{p} 0$ *as* $t \to \infty$.

3. *The set* $\{t : R_t^\nu = 0\}$ *contains no isolated points a.s.*

4. *For every* $t > 0$, $\mathbb{P}(R_t^\nu = 0) = 0$.

Proof of 1. By Lemma 5,

$$\mathbb{E}\left[\left(X_t^{(1)}\right)^2\right] = \frac{c}{\mu}\int_0^t \mathrm{d}s \int_0^1 \mathrm{d}x \frac{1}{x^{\gamma-3}} e^{-xs/\mu} \leq ct \int_0^{1/t} \mathrm{d}x \frac{1}{x^{\gamma-3}} + c\int_{1/t}^\infty \mathrm{d}x \frac{1}{x^{\gamma-2}},$$

so that

$$\mathbb{E}\left[\left(X_t^{(1)}\right)^2\right] \leq \frac{c}{(\gamma-3)(4-\gamma)} t^{\gamma-3}.$$

Applying Markov's inequality, we deduce that

$$t^{-(\gamma-3)} X_t^{(1)} \quad \xrightarrow[t\to\infty]{p} \quad 0. \tag{13}$$

Letting $\eta = (\gamma-3)/2$, this implies that $t^{-(1+\eta)} X_t^{(1)} \xrightarrow{p} 0$ as $t \to \infty$. Then notice that $X_t^{(2)}$ is less than $\sum_{s\leq t} \Delta_s$, where Δ is a Poisson point process with intensity $\mathbf{1}_{x\geq 1}\nu(\mathrm{d}s, \mathrm{d}x)$ (recall that $\nu(\mathrm{d}s, \mathrm{d}x) = \frac{c}{\mu}\frac{1}{x^{\gamma-1}} e^{-xs/\mu}\mathrm{d}s\mathrm{d}x$). Now $\mathbb{E}[\sum_{s\leq t}\Delta_s] = \frac{c}{\mu}\int_0^t \mathrm{d}s \int_1^\infty \mathrm{d}x \frac{1}{x^{\gamma-2}} e^{-xs/\mu} \leq \frac{c}{\mu(\gamma-3)} t$. Consequently, by Markov's inequality, $t^{-(1+\eta)}\sum_{s\leq t}\Delta_s \xrightarrow{p} 0$ as $t \to \infty$. Since $t^{-(1+\eta)} A_t^\nu \to -\infty$ as $t \to \infty$, property 1 is proved. □

Proof of 2. Restate 2 as follows : for every $\varepsilon > 0$,

$$\text{number of (excursion of } R^\nu \text{ with length} > 2\varepsilon) < \infty \text{ a.s.}$$

Fix $\varepsilon > 0$ and define events $C_n = \{\sup_{s\in[(n-1)\varepsilon, n\varepsilon]}(X_{(n+1)\varepsilon}^\nu + A_{(n+1)\varepsilon}^\nu - X_s^\nu - A_s^\nu) > 0\}$. It is easily seen that it suffices to show that $\mathbb{P}(C_n$ infinitely often$) = 0$. By (13), it is enough to prove that

$$\sum_{n\geq 1+s_0/\varepsilon} \mathbb{P}(C_n \cap C^{s_0}) < \infty, \quad \text{for every large } s_0, \tag{14}$$

where $C^{s_0} = \{\sup_{t\geq s_0} t^{-(\gamma-3)}|X_t^{(1)}| \leq \delta\}$ for some positive (small) constant $\delta > 0$ to be chosen later. Now

$$\begin{aligned} C_n \quad \subset \Bigg\{ & \sup_{s\in[(n-1)\varepsilon, n\varepsilon]} \left(X_{(n+1)\varepsilon}^{(2)} - X_s^{(2)}\right) \geq \frac{c\,\Gamma(4-\gamma)}{(\gamma-3)(\gamma-2)\mu^{\gamma-2}} \varepsilon^{\gamma-2}\left((n+1)^{\gamma-2} - n^{\gamma-2}\right) \\ & - \sup_{s\in[(n-1)\varepsilon, n\varepsilon]} \left(X_{(n+1)\varepsilon}^{(1)} - X_s^{(1)}\right) \Bigg\}. \end{aligned}$$

For every n larger than $1+s_0/\varepsilon$, on C^{s_0}, we have :

$$\sup_{s\in[(n-1)\varepsilon,n\varepsilon]}\left(X^{(1)}_{(n+1)\varepsilon}-X^{(1)}_s\right) \;\leq\; 2\delta\varepsilon^{\gamma-3}(n+1)^{\gamma-3}\leq 2\delta\varepsilon^{\gamma-3}2^{\gamma-3}n^{\gamma-3}.$$

Consequently, for every n larger than $1+s_0/\varepsilon$,

$$C_n\cap C^{s_0}\subset\left\{\sup_{s\in[(n-1)\varepsilon,n\varepsilon]}\left(X^{(2)}_{(n+1)\varepsilon}-X^{(2)}_s\right)\geq\left(\frac{c\,\Gamma(4-\gamma)}{(\gamma-3)\mu^{\gamma-2}}\varepsilon^{\gamma-2}-\delta\varepsilon^{\gamma-3}2^{\gamma-2}\right)n^{\gamma-3}\right\}.$$

Taking $\delta=\varepsilon\frac{c\,\Gamma(4-\gamma)}{(\gamma-3)\mu^{\gamma-2}2^{\gamma-1}}$, and denoting $\frac{c\,\Gamma(4-\gamma)}{2(\gamma-3)\mu^{\gamma-2}}\varepsilon^{\gamma-2}$ by ρ, we thus have for every n large enough :

$$C_n\cap C^{s_0}\subset\left\{\sup_{s\in[(n-1)\varepsilon,n\varepsilon]}\left(X^{(2)}_{(n+1)\varepsilon}-X^{(2)}_s\right)\geq\rho n^{\gamma-3}\right\}.$$

Now, considering a Poisson point process Δ with intensity $\mathbf{1}_{x\geq 1}\nu(\mathrm{d}s,\mathrm{d}x)$, where $\nu(\mathrm{d}s,\mathrm{d}x)=\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\mathrm{d}s\mathrm{d}x$, observe that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\in[(n-1)\varepsilon,n\varepsilon]}\left(X^{(2)}_{(n+1)\varepsilon}-X^{(2)}_s\right)\geq\rho n^{\gamma-3}\right)\\
&\leq\;\mathbb{P}\left(\sum_{s\in[(n-1)\varepsilon,(n+1)\varepsilon]}\Delta_s\geq\rho n^{\gamma-3}\right)\\
&\leq\;\rho^{-1}n^{-\gamma+3}\mathbb{E}\left[\sum_{s\in[(n-1)\varepsilon,(n+1)\varepsilon]}\Delta_s\right]\\
&=\;\rho^{-1}n^{-\gamma+3}\frac{c}{\mu}\int_{(n-1)\varepsilon}^{(n+1)\varepsilon}\mathrm{d}s\int_1^\infty\mathrm{d}x\;x\frac{1}{x^{\gamma-1}}e^{-xs/\mu}.
\end{aligned}$$

We deduce that for every n larger than $2+s_0/\varepsilon$,

$$\mathbb{P}\left(C_n\cap C^{s_0}\right)\;\leq\;\frac{2\varepsilon c}{\rho\mu}n^{-\gamma+3}\int_1^\infty\mathrm{d}x\;x^{2-\gamma}e^{-nx\varepsilon/(2\mu)}\leq\frac{4c}{\rho}n^{-\gamma+2}e^{-n\varepsilon/(2\mu)},$$

which proves (14) and completes the proof of assertion 2. □

Proof of 3. To show property 3, we first consider the case $t=0$. We aim at showing that $\inf\{s>0:X^\nu_s+A^\nu_s<0\}=0$ a.s. Since for every $s\in[0,\infty)$ and $x\in(0,\infty)$, $\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu}\leq\frac{c}{\mu}\frac{1}{x^{\gamma-1}}$, we can couple the process X^ν and construct a stable process L with index $\gamma-2$ with no negative jumps such that

$$\forall s\geq 0\quad\forall u\in\mathbb{R},\quad\mathbb{E}\left[\exp\left(iuL_s\right)\right]\;=\;\exp\left(s\int_0^\infty\mathrm{d}x\left(e^{iux}-1-iux\right)\frac{c}{\mu}\frac{1}{x^{\gamma-1}}\right)$$

satisfying

$$\forall s\geq 0,\quad X^\nu_s\;\leq\;L_s+\frac{c}{\mu}\int_0^s\mathrm{d}r\int_0^\infty\mathrm{d}x\frac{1}{x^{\gamma-2}}\left(1-e^{-xr/\mu}\right),\tag{15}$$

i.e.,

$$\forall s \geq 0, \quad X_s^\nu \ \leq \ L_s + \frac{c\,\Gamma(4-\gamma)}{(\gamma-3)(\gamma-2)\mu^{\gamma-2}} s^{\gamma-2}.$$

Consequently $X^\nu + A^\nu \leq L$. Since $\inf\{s > 0 : L_s < 0\} = 0$ a.s., with probability 1, 0 is not an isolated point of the set $\{t : R_t^\nu = 0\}$. This implies assertion 3 by routine arguments (see [2] for details). □

Proof of 4. Here again, we shall use a coupling argument. We need an inequality opposite to (15). Specifically, we have to bound the increments of $X^\nu + A^\nu$ from below. We first focus on $X^{(1)}$. Let $t \in (0,\infty)$. Arguing as before, since for every $s \in [0,t]$ and $x \in (0,1)$, $\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xs/\mu} \geq \frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xt/\mu}$, we can construct a Lévy process $(Q_s^{(1)})_{s\in[0,t]}$ such that

$$\forall s \in [0,t] \quad \forall u \in \mathbb{R}, \quad \mathbb{E}\left[\exp\left(iuQ_s^{(1)}\right)\right] \ = \ \exp\left(s\int_0^1 \mathrm{d}x\,\left(e^{iux} - 1 - iux\right)\frac{c}{\mu}\frac{1}{x^{\gamma-1}}e^{-xt/\mu}\right)$$

satisfying

$$\forall s \in [0,t], \quad X_t^{(1)} - X_s^{(1)} \ \geq \ Q_t^{(1)} - Q_s^{(1)} + \frac{c}{\mu}\int_s^t \mathrm{d}r \int_0^1 \mathrm{d}x \frac{1}{x^{\gamma-2}}\left(e^{-xt/\mu} - e^{-xr/\mu}\right).$$

Since for every $a, b \in (0,\infty)$ such that $a < b$, $e^{-a} - e^{-b} \leq b - a$, we have for every $s \in [0,t]$

$$X_t^{(1)} - X_s^{(1)} \ \geq \ Q_t^{(1)} - Q_s^{(1)} - \frac{c}{2(4-\gamma)\mu^2}(t-s)^2.$$

As a result, for every $s \in [0,t]$,

$$X_t^\nu - X_s^\nu \ \geq \ Q_t^{(1)} - Q_s^{(1)} - \frac{c}{2(4-\gamma)\mu^2}(t-s)^2 + B_t^\nu - B_s^\nu.$$

We easily deduce that there exists $C > 0$ (only depending on t) such that for every $s \in [0,t]$,

$$X_t^\nu + A_t^\nu - (X_s^\nu + A_s^\nu) \ \geq \ Q_t^{(1)} - Q_s^{(1)} - C(t-s).$$

Consequently

$$\sup\left\{X_t^\nu + A_t^\nu - (X_s^\nu + A_s^\nu) : s \in [0,t]\right\} \ \geq \ \sup\left\{Q_t^{(1)} - Ct - \left(Q_s^{(1)} - Cs\right) : s \in [0,t]\right\}.$$

Now, applying [5, Theorem VII. 2 and page 158] to the Lévy process $(Q_s^{(1)} - Cs)_{s\in[0,t]}$, we have

$$\mathbb{P}\left(Q_t^{(1)} - Ct = \inf\left\{Q_s^{(1)} - Cs : s \in [0,t]\right\}\right) = 0.$$

We deduce that

$$\mathbb{P}\left(X_t^\nu + A_t^\nu = \inf\left\{X_s^\nu + A_s^\nu : s \in [0,t]\right\}\right) = 0,$$

which is assertion 4. □

Acknowledgement. I am grateful to my thesis advisor Jean Bertoin who suggested this subject to me. I thank him for fruitful discussions and for his careful reading of this work.

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