A LAMPERTI TYPE REPRESENTATION OF CONTINUOUS-STATE BRANCHING PROCESSES WITH IMMIGRATION

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Abstract. Guided by the relationship between the breadth-first walk of a rooted tree and its sequence of generation sizes, we extend the Lamperti representation of continuous-state branching processes to allow immigration. The representation is obtained by solving a random ordinary differential equation defined by a pair of independent Lévy processes. Stability of the solutions is studied and gives, in particular, limit theorems (of a type previously studied by Grimvall, Kawazu and Watanabe, and Li) and a simulation scheme for continuous-state branching processes with immigration. We further apply our stability analysis to extend Pitman's limit theorem concerning Galton-Watson processes conditioned on total population size to more general offspring laws.

1. Introduction

1.1. **Motivation.** In this document, we extend the Lamperti representation of Continuous State Branching Processes so that it allows immigration. First, we will see how to find discrete (and simpler) counterparts to our results in terms of the familiar Galton-Watson process with immigration and its representation using two independent random walks. Consider a genealogical structure with immigration such as the one depicted in Figure 1. When ordering its elements in breadth-first order (with the accounting policy of numbering immigrants after the established population in each generation), let χ_i be the number of children of individual i. Define a first version of the breadth first walk $\tilde{x} = (\tilde{x}_i)$ by $\tilde{x}_0 = 0$ and $\tilde{x}_{i+1} = \tilde{x}_i + \chi_i$. Consider also the immigration process $y = (y_i)_{i \geq 0}$ where y_i is the quantity of immigrants arriving at generations less than or equal to i (not counting the initial members of the population as immigrants). Finally, suppose the initial population has k members. If c_n denotes the number of individuals of generations 0 to n, c_{n+1} is obtained from c_n by adding the quantity of sons of each member of the n-th generation plus the immigrants, leading to

$$c_{n+1} = c_n + \left(\chi_{c_{n-1}+1} + \cdots + \chi_{c_n}\right) + \left(y_{n+1} - y_n\right).$$

By induction we get

$$c_{n+1} = k + \tilde{x}_{c_n} + y_{n+1}.$$

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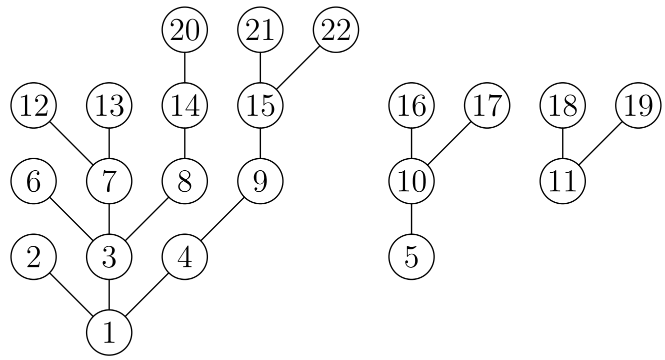

FIGURE 1. A genealogical structure allowing immigration

Let z_n denote the number of individuals of generation n so that $z_0 = c_0$ and for $n \geq 1$

$$z_n = c_n - c_{n-1};$$

if $\eta_i = \chi_i - 1$, we can define a second version of the breadth-first walk of the population by setting $x_0 = 0$ and $x_i = x_{i-1} + \eta_i$ (so that $x_i = \tilde{x}_i - i$). We then obtain

$$z_{n+1} = k + x_{c_n} + y_{n+1}. \tag{1}$$

This representation of the sequence of generation sizes z in terms of the breadth-first walk x and the immigration function y can be seen as a **discrete Lamperti transformation**. It is the discrete form of the result we aim at analyzing. However, we wish to consider a random genealogical structure which is not discrete. Randomness will be captured by making the quantity of sons of individuals an iid sequence independent of the iid sequence of immigrants per generation, so that the model corresponds to a Galton-Watson with immigration. Hence x and y would become two independent random walks, whose jumps take values in $\{-1, 0, 1, \ldots\}$ and $\{0, 1, \ldots\}$ respectively. Discussion of non-discreteness in the random genealogy model would take us far appart (we are motivated by Lévy trees with or without immigration, discussed for example in Duquesne and Le Gall (2002); Lambert (2002); Duquesne (2009); Abraham and Delmas (2009b)). We only mention that continuum trees are usually defined through a continuum analogue of the depth-first walk; our point of view is that generation sizes should be obtained in terms of the continuum analogue of the breadth-first walk. Indeed, in analogy with the discrete model, we just take X and Y as independent Lévy processes, the former without negative jumps (a spectrally positive Lévy process) and the latter with increasing sample paths (a subordinator). The discrete Lamperti transformation of (1) then takes the form

$$Z_t = x + X_{\int_0^t Z_s \, ds} + Y_t. \tag{2}$$

This should limit the continuum version of a Galton-Watson process with immigration, namely, the continuous-state branching processes with immigration introduced by Kawazu and Watanabe (1971).

1.2. **(Possibly killed) Lévy processes.** A spectrally positive Lévy process (spLp) is a stochastic process $X = (X_t)_{t\geq 0}$ with values on $(-\infty, \infty]$ with independent and stationary increments, càdlàg paths, and no negative jumps. Such a process is characterized by its Laplace exponent Ψ by means of the formula

$$\mathbb{E}\left(e^{-\lambda X_t}\right) = e^{t\Psi(\lambda)}$$

where

$$\Psi(\lambda) = -\kappa + a\lambda + \sigma^2\lambda^2/2 + \int_0^\infty \left(e^{\lambda x} - 1 - \lambda x \mathbf{1}_{x\leq 1}\right) \nu(dx)$$

where ν is the so called Lévy measure on $(0, \infty)$ and satisfies

$$\int 1 \wedge x^2 \, \nu(dx) < \infty.$$

The constant κ will be for us the killing rate; a Lévy process with killing rate k can be obtained by a Lévy process with zero killing rate by sending the latter to ∞ at an independent exponential time of parameter κ; σ^2 is called the Brownian component, while a is the drift. We shall also make use of subordinators, which are spLp with increasing trajectories. The Laplace exponent Φ of a subordinator X is defined as the negative of its Laplace exponent as a spLp:

$$\mathbb{E}\left(e^{-\lambda X_t}\right) = e^{-t\Phi(\lambda)}.$$

Since the Lévy measure ν of a subordinator actually satisfies

$$\int 1 \wedge x \, \nu(dx) < \infty,$$

and subordinators have no Brownian component, we can write

$$\Phi(\lambda) = -\kappa + d\lambda + \int \left(1 - e^{-\lambda x}\right) \nu(dx) \, .$$

Where, since $\sigma^2 = 0$, we have the relationship

$$d = a + \int_0^1 x \, \nu(dx)$$

between the parameters of X seen as a spLp and as a subordinator.

1.3. **Continuous-State Branching processes and the Lamperti representation.** Continuous-state branching (CB) processes are the continuous time and space version of Galton-Watson processes. They were introduced in different levels of generality by Jiřina (1958), Lamperti (1967b), and Silverstein (1968). They are Feller processes with state-space $[0, \infty]$ (use any metric that makes it homeomorphic to $[0, 1]$) satisfying the following branching property: the sum of two independent copies started at x and y has the law of the process started at $x + y$. The states 0 and ∞ are absorbing. The branching property can be recast by stating that the logarithm of the Laplace transform of the transition semigroup is given by a linear transformation of the initial state. As shown in Silverstein (1968), CB processes are in one to one correspondence with Laplace exponents of (killed) spectrally positive Lévy processes, which are called the branching mechanisms. In short, the logarithmic derivative of the semigroup of a CB process at zero applied to the function $x \mapsto e^{-\lambda x}$ exists and is equal to $x \mapsto x\Psi(\lambda)$. The function Ψ is the called the branching mechanism of the CB process and it is the Laplace exponent of a SPLP. A probabilistic form of this assertion is given by Lamperti (1967a) who states that if X is a SPLP with Laplace exponent Ψ, and for $x \geq 0$ we set τ for its hitting time of $-x$,

$$I_t = \int_0^t \frac{1}{x + X_{s\wedge\tau}}\, ds$$

and C equal to its right-continuous inverse, then

$$Z_t = x + X_{C_t}$$

is a CB process with branching mechanism Ψ, or CB(Ψ). This does not seem to be directly related to (2). The fact that it is related gives us what we think is the right perspective on the Lamperti transformation and the generalization considered in this work. Indeed, as previously shown in (Ethier and Kurtz, 1986, 6.1), Z is the only process satisfying

$$Z_t = x + X_{\int_0^t Z_s\, ds} \tag{3}$$

which is absorbed at zero . This is (2) in the absence of immigration. To see that a process satisfying (3) can be obtained as the Lamperti transform of X, note that if $C_t = \int_0^t Z_s\, ds$, then while Z has not reached zero C is strictly increasing so that it has an inverse, say I, whose right-hand derivative I'_+ is given by

$$I'_+(t) = \frac{1}{C'_+(I_t)} = \frac{1}{Z_{I_t}} = \frac{1}{x + X_{C\circ I(t)}} = \frac{1}{x + X_t}.$$

1.4. **Continuous-State Branching processes with Immigration.** Continuous-State Branching Processes with Immigration (or CBI processes) are the continuous time and space version of Galton-Watson processes with immigration and were introduced by Kawazu and Watanabe (1971). They are Feller processes with state-space $[0, \infty]$ such that the logarithm of the Laplace of the transition semigroup is given by an affine transformation of the initial state. (They thus form part of the affine processes studied in Dawson and Li (2006).) As shown in Kawazu and Watanabe (1971), they are characterized by the Laplace exponents of a SPLP and of a subordinator: the logarithmic derivative of the semigroup of a CB process at zero applied to the function $x \mapsto e^{-\lambda x}$ exists and is equal to the function

$$x \mapsto x\Psi(\lambda) - \Phi(\lambda)$$

where Ψ is the Laplace exponent of a SPLP and Φ is the Laplace exponent of a subordinator. We will give a probabilistic explanation of this characterization, similar to the Lamperti representation.

1.5. **A generalized Lamperti transformation and its consequences.** We propose to construct a CBI(Ψ, Φ) by solving the functional equation

$$Z_t = x + X_{\int_0^t Z_s \, ds} + Y_t. \tag{4}$$

We propose to call such a process Z the **Lamperti transform** of $(X, x+Y)$ and denote it by $Z = L(X, x+Y)$; however, the first thing to do is to show that there exists a unique process which satisfies (4). When Y is zero, a particular solution to (4) is the Lamperti transform of $X + x$ recalled above. Even in this case there can be many solutions to (4), in clear contrast to the discrete case where one can proceed recursively to construct the unique solution. Our stepping stone for the general analysis of (4) is the following partial result concerning existence and uniqueness proved in Section 2. A pair of càdlàg functions (f, g) such that f has no negative jumps, g is non-decreasing and $f(0)+g(0) \geq 0$ is termed an **admissible breadth-first pair**; f and g will be termed the **reproduction and immigration functions**. When g is constant, we say that $f + g$ is **absorbed at zero** if $f(x) + g = 0$ implies $f(y) + g = 0$ for all $y > x$.

Theorem 1. *Let (f, g) be an admissible breadth-first pair. There exists a nonnegative h satisfying the equation*

$$h(t) = f\left(\int_0^t h(s) \, ds\right) + g(t)\,.$$

Furthermore, the solution is unique when g is strictly increasing, when $f + g(0)$ is a strictly positive function, or when g is constant and $f + g$ is absorbed at zero.

As a consequence of the analytic Theorem 1, we solve a probabilistic question raised by Lambert (1999, 2007).

Corollary 1. *Let X be a spectrally positive α-stable Lévy process. For any càdlàg and strictly increasing process Y independent of X, there is weak existence and uniqueness for the stochastic differential equation*

$$Z_t = x + \int_0^t |Z_s|^{1/\alpha} \, dX_s + Y_t. \tag{5}$$

When X is twice a Brownian motion and $Y_t = \delta t$ for some $\delta > 0$, this might be one of the simplest proofs available of weak existence and uniqueness of the SDE defining squared Bessel processes, since it makes no mention of the Tanaka formula or local times; it is based on Knight's theorem and Theorem 1. When X is a Brownian motion and $dY_t = b(t) \, dt$ for some Lipschitz and deterministic $b : [0, \infty) \to [0, \infty)$, Le Gall (1983) actually proves pathwise uniqueness through a local time argument. Our result further shows that if b is measurable and strictly positive then there is weak uniqueness. In the case Y is an $(\alpha - 1)$-stable subordinator independent of X, we quote Lambert (1999, 2007)

> ..whether or not uniqueness holds for (5) remains an open question.

Corollary 1 answers affirmatively. Note that when $Y = 0$ the stated result follows from Zanzotto (2002), and is handled by a time-change akin to the Lamperti transformation.

Fu and Li (2010) obtain strong existence and pathwise uniqueness for a different kind of SDE related to CBI processes with stable reproduction and immigration.

Regarding solutions to (4), Theorem 1 is enough to obtain the process Z when the subordinator Y is strictly increasing. When Y is compound Poisson, a solution to (4) can be obtained by pasting together Lamperti transforms. However, further analysis using the pathwise behavior of X when Y is zero or compound Poisson implies the following result.

Proposition 1. *Let $x \geq 0$. Then there is a unique càdlàg process Z which satisfies*

$$x + X_{-}\left(\int_0^t Z_s\, ds\right) + Y_t \leq Z_t \leq x + X\left(\int_0^t Z_s\, ds\right) + Y_t$$

and it coincides with the unique process which satisfies

$$Z_t = x + X\left(\int_0^t Z_s\, ds\right) + Y_t.$$

Our main result, a pathwise construction of a CBI, is the following.

Theorem 2. *Let X be a spectrally positive Lévy process with Laplace exponent Ψ and Y an independent subordinator with Laplace exponent Φ. The unique stochastic process Z which solves*

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_t$$

is a $\mathrm{CBI}(\Psi, \Phi)$ *that starts at x.*

We view Theorems 1 and 2 as a first step in the construction of branching processes with immigration where the immigration can depend on the current value of the population. One generalization would be to consider solutions to

$$Z_t = x + X_{\int_0^t a(s,Z_s)\, ds} + Y_{\int_0^t b(s,Z_s)\, ds},$$

where a is interpreted as the breeding rate and b as the rate at which the arriving immigration is incorporated into the population. For example Abraham and Delmas (2009a) consider a continuous branching process where immigration is proportional to the current state of the population. This could be modeled by the equation

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_{\int_0^t \alpha Z_s\, ds},$$

which, thanks to the particular case of Theorem 2 stated by Lamperti (1967a), has the law of a $\mathrm{CB}(\Psi - \alpha\Phi)$ started at x; this is the conclusion of Abraham and Delmas (2009a), where they rigorously define the model in terms of a Poissonian construction of a more general class of CBI processes which is inspired in previous work of Pitman and Yor (1982) for CBIs with continuous sample paths. Another representation of CBI processes, this time in terms of solutions to stochastic differential equations was given by Dawson and Li (2006) under moment conditions.

The usefulness of Theorem 2 is two-fold: firstly, we can use known sample path properties of X and Y to deduce sample-path properties of Z, and secondly, this representation gives a particular coupling with monotonicity properties which are useful in limit theorems involving Z, as seen in Corollary 6, Corollary 7, and Theorem 4. Simple applications of Theorem 2 include the following.

Corollary 2 (Kawazu and Watanabe (1971))**.** *If Ψ is the Laplace exponent of a spectrally positive Lévy process and Φ is the Laplace exponent of a subordinator, there exists a* CBI *process with branching mechanism Ψ and immigration mechanism Φ.*

Corollary 3. *A* $\mathrm{CBI}(\Psi,\Phi)$ *process does not jump downwards.*

Caballero et al. (2009) give a direct proof of this when $\Phi = 0$.

Corollary 4. *Let Z be a* $\mathrm{CBI}(\Psi,\Phi)$ *that starts at $x > 0$, let $\tilde{\Phi}$ be the right-continuous inverse of Ψ and define*

$$f(t) = \frac{\log|\log t|}{\tilde{\Phi}(t^{-1}\log|\log t|)}.$$

There exists $c \neq 0$ (which depends only on Ψ) such that

$$\liminf_{t\to 0}\frac{Z_t - x}{f(xt)} = c.$$

The case $x = 0$ in Corollary 4 is probably very different, as seen when $\Psi(\lambda) = 2\lambda^2$ and $\Phi(\lambda) = d\lambda$ which corresponds to the squared Bessel process of dimension d. Indeed, (Itô and McKean, 1974, p. 80) show that for a squared Bessel process Z of integer dimension that starts at 0, we have

$$\limsup_{t\to 0}\frac{Z_t}{2t\log\log 1/t} = 1.$$

We have not been able to obtain this result using the Lamperti transformation. However, note that starting from positive states we can obtain the lower growth rate, since it is the reproduction function X that determines it, while starting from 0, it is probably a combination of the local growth of X and Y that drives that of Z which might explain the change to an upper growth rate.

In the context of Theorem 1, much is gained by introducing the function c given by

$$c(t) = \int_0^t h(s)\,ds,$$

which has a right-hand derivative c'_+ equal to h. This is because the functional equation for h can then be recast as the initial value problem

$$\mathrm{IVP}(f,g) = \begin{cases} c'_+ = f\circ c + g \\ c(0) = 0 \end{cases}.$$

A solution c to $\mathrm{IVP}(f,g)$ is said to explode if there exists $t \in (0,\infty)$ such that $c(t) = \infty$. (Demographic) explosion is an unavoidable phenomena of $\mathrm{IVP}(f,g)$. When $f > 0$ and $g = 0$, it is known that explosion occurs if and only if

$$\int_0^\infty \frac{1}{f(x)}\,dx = \infty.$$

Actually, even when there is immigration, the main responsible for explosion is the reproduction function.

Proposition 2.
(1) If $\int^{\infty} 1/f^{+}(x)\,dx = \infty$ then no solution to $\mathrm{IVP}(f,g)$ explodes.
(2) If $\int^{\infty} 1/f^{+}(x)\,dx < \infty$ then a solution to $\mathrm{IVP}(f,g)$ explodes if and only if it is unbounded. In particular, if $g(\infty)$ exceeds the maximum of $-f$ then any solution to $\mathrm{IVP}(f,g)$ explodes.

We call f an **explosive reproduction function** if

$$\int^{\infty} \frac{1}{f^{+}(x)}\,dx < \infty.$$

Recall that ∞ is an absorbing state for CBI processes; Proposition 2 has immediate implications on how a CBI process might reach it. First of all, CBI processes might jump to ∞, which happens if and only if either the branching or the immigration correspond to killed Lévy processes. When there is no immigration and the branching mechanism Ψ has no killing rate, the criterion is due to Ogura (1970) and Grey (1974), who assert that the probability that a $\mathrm{CB}(\Psi)$ started from $x > 0$ is absorbed at infinity in finite time is positive if and only if

$$\int_{0+} \frac{1}{\Psi(\lambda)}\,d\lambda < \infty$$

and give a formula for its distribution in terms of the $\lim_{\lambda\to 0} u_t(\lambda)$; we call such Ψ an **explosive branching mechanism**. From Proposition 2 and Theorem 2 we get:

Corollary 5. *The probability that a $\mathrm{CBI}(\Psi,\Phi)$ jumps to ∞ is positive if and only if $\Psi(0)$ or $\Phi(0)$ are non-zero. The probability that it reaches ∞ continuously is positive if and only if $\Psi - \Psi(0)$ is an explosive branching mechanism and it is equal to if 1 $\Psi(0) = 0 = \Phi(0)$ and Ψ is explosive.*

When $\Psi(0) = \Phi(0) = 0$, the above explosion criterion was in fact obtained by Kawazu and Watanabe (1971).

We mainly use stochastic calculus in our proof of Theorem 2; however, a weak convergence type of proof, following the case $\Phi = 0$ presented in Caballero et al. (2009), could also be achieved in conjunction with a stability result. The following result deals with stability of $\mathrm{IVP}(f,g)$ under changes in f and g, but also with respect to discretization of the transformation itself. Indeed, consider the following approximation procedure: given $\sigma > 0$, called the span, consider the partition

$$t_i = i\sigma, \quad i = 0,1,2,\ldots,$$

and construct a function c^{σ} by the recursion

$$c^{\sigma}(0) = 0$$

and for $t \in [t_{i-1}, t_i)$:

$$c^\sigma(t) = c^\sigma(t_{i-1}) + (t - t_{i-1}) \left[f \circ c^\sigma(t_{i-1}) + g(t_{i-1})\right]^+ .$$

Equivalently, the function c^σ is the unique solution to the equation

$$\mathrm{IVP}_\sigma(f, g) = \left\{ c^\sigma(t) = \int_0^t \left[f \circ c^\sigma([s/\sigma]\sigma) + g([s/\sigma]\sigma)\right]^+ \, ds. \right.$$

We will write $\mathrm{IVP}_0(f, g)$ to mean $\mathrm{IVP}(f, g)$. Let D_+ denote the right-hand derivative.

The stability result is stated in terms of the usual **Skorohod J_1 topology** for càdlàg functions: a sequence f_n converges to f if there exist a sequence of homeomorphisms of $[0, \infty)$ into itself such that

$$f_n - f \circ \lambda_n, \lambda_n - \mathrm{Id} \to 0 \text{ uniformly on compact sets}$$

(where Id denotes the identity function on $[0, \infty)$). However, part of the theorem uses another topology on Skorohod space introduced in Caballero et al. (2009), which we propose to call the **uniform J_1 topology** and which is characterized by: a sequence f_n converges to f if there exist a sequence of homeomorphisms of $[0, \infty)$ into itself such that

$$f_n - f \circ \lambda_n, \lambda_n - \mathrm{Id} \to 0 \text{ uniformly on } [0, \infty).$$

Convergence in the uniform J_1 topology implies convergence in the Skorohod J_1 topology.

Theorem 3. *Let (f, g) be an admissible breadth-first pair and suppose there is a unique function c which satisfies,*

$$\int_s^t f_- \circ c(r) + g(r)\, dr \le c(t) - c(s) \le \int_s^t f \circ c(r) + g(r)\, dr, \quad \text{for } s \le t. \tag{6}$$

(In particular, $\mathrm{IVP}(f, g)$ *has c as its unique solution.)*

If $f_n \to f$ and $g_n \to g$ in the Skorohod J_1 topology, $\sigma_n \to 0$, and c_n is any solution to $\mathrm{IVP}_{\sigma_n}(f_n, g_n)$ *then $c_n \to c$ pointwise and uniformly on compact sets of $[0, \tau)$. Furthermore, if $f \circ c$ and g do not jump at the same time then $D_+ c_n \to D_+ c$*

(1) in the Skorohod J_1 topology if $\tau = \infty$.
(2) in the uniform J_1 topology if $\tau < \infty$ and we additionally assume that $f_n \to f$ in the uniform J_1 topology.

The hypothesis that (6) has a unique solution is related to the uniqueness of $\mathrm{IVP}(f, g)$.

Proposition 3. *Let (f, g) be an admissible breadth-first pair. If either g is strictly increasing, $f + g(0)$ is strictly positive, or g is constant and $f + g(0)$ is absorbed at zero, then (6) has a unique solution.*

On the other hand, it is not very hard to show that the jumping condition of Theorem 3 holds in a stochastic setting.

Proposition 4. *Let X be a spLp , Y an independent subordinator with Laplace exponents Ψ and Φ and, for $x \geq 0$, let Z the unique process such that*

$$Z_t = x + X_{C_t} + Y_t \text{ where } C_t = \int_0^t Z_s \, ds.$$

Almost surely, the processes $X \circ C$ and Y do not jump at the same time.

From Theorem 3 and Propositions 1 and 4, we deduce the following weak continuity result.

Corollary 6. *Let Ψ_n, Ψ be Laplace exponents of SPLPs and Φ_n, Φ be Laplace exponents of subordinators and suppose that $\Psi_n \to \Psi$ and $\Phi_n \to \Phi$ pointwise. If (x_n) is a sequence in $[0, \infty]$ converging to $x \in [0, \infty]$ and Z_n (resp. Z) are CBIs with branching and immigration mechanisms Ψ_n and Φ_n (resp. Ψ and Φ) then $Z_n \to Z$ in the Skorohod J_1 topology on càdlàg paths on $[0, \infty]$ if Ψ is non-explosive and in the uniform J_1 topology if Ψ is explosive.*

Theorem 3 also allows us to simulate CBI processes. Indeed, the approximation procedure of IVP_σ tells us that if we can simulate random variables with distribution X_t and Y_t for every $t > 0$, we can then approximately simulate the process Z as the right-hand derivative of the solution to $\mathrm{IVP}_\sigma(X, x + Y)$. The procedure $\mathrm{IVP}_\sigma(X, x + Y)$ actually corresponds to an Euler method of span σ to solve $\mathrm{IVP}(X, x + Y)$. Theorem 3 implies the convergence of the Euler method as the span goes to zero when applied to $\mathrm{IVP}(X, x + Y)$, even with the discontinuous driving functions X and Y! (The essential step in convergence of discretizations, recognized as far back as Viswanatham (1952); Coddington and Levinson (1952), is a uniqueness assertion, which in our case is given in Theorem 1.)

We also give an application of Theorem 3 to limits of Galton-Watson processes with immigration. If X^n and Y^n are independent random walks with step distributions μ_n and ν_n supported on $\{-1, 0, 1, \ldots\}$ and $\{0, 1, 2, \ldots\}$ and for any $k_n \geq 0$ we define recursively the sequences C^n and Z^n by setting

$$C^n_0 = k_n, \quad Z^n_{m+1} = X^n_{C^n_m} + Y^n_m, \quad \text{and} \quad C^n_{m+1} = C^n_m + Z^n_{m+1}.$$

As discussed in Subsection 1.1, the sequence Z^n is a Galton-Watson process with immigration with offspring and immigration distributions μ_n and ν_n. However, if X^n and Y^n are extended by constancy on $[m, m + 1)$ for $m \geq 0$ (keeping the same notation), then C^n is the approximation of the Lamperti transformation with span 1 applied to X^n and Y^n and Z^n is the right-hand derivative of C^n.

Corollary 7. *Suppose the existence of sequences a_n and b_n such*

$$X^n_n / a_n \quad \text{and} \quad Y^n_n / b_n$$

converge weakly to the infinitely divisible distributions μ and ν; denote by Ψ and Φ their Laplace exponents. Suppose that $k_n \sim x c_n$ where $x \geq 0$ and $c_n = b_n / a_n$. Then the sequence

$$S^{b_n / a_n}_{1/a_n} Z^n$$

converges in distribution to a $\mathrm{CBI}(\Psi, \Phi)$ that starts at x, in the Skorohod J_1 topology if Ψ is non-explosive and in the uniform J_1 topology otherwise.

When Ψ is non-explosive and $\Phi = 0$, the above theorem was proved by Grimvall (1974). He also proved the convergence of finite-dimensional distributions in the explosive case, which we complement with a limit theorem. For general Φ, but non-explosive Ψ, a similar result was proven by Li (2006).

Note that the scaling of Z^n is the same as that of Y^n, that is: immigration dominates in terms of scaling and this is natural since Y^n and Z^n are on the same time-scale. The sequence of initial states is constructed so as to be compatible with both the growth rate of X^n and of Y^n. In any case, since $x = 0$ is acceptable from the point of view of the limit theorem, it is not as delicate a balance as it might seem.

The stability result of Theorem 3 applies not only in the Markovian case of CBI processes. As an example, we generalize work of Pitman (1999) who considers the scaling limits of conditioned Galton-Watson processes in the case of the Poisson offspring distribution. Let μ be an offspring distribution with mean 1 and suppose that $Z^{k,n}$ is a Galton-Watson process started at k and conditioned on

$$\sum_{i=0}^{\infty} Z_i^{k,n} = n.$$

We shall consider the scaling limit of $Z^{k,n}$ as $k, n \to \infty$ whenever the shifted reproduction law $\tilde{\mu}_k = \mu_{k+1}$ is the domain of attraction of a stable law without the need of centering. The scaling limit of a random walk with step distribution $\tilde{\mu}$ is then a spectrally positive stable law of index $\alpha \in (1, 2]$ with which one can define the first passage bridge F^l between l and 0 of length 1 of the associated Lévy process. Informally this is the stable process conditioned to be above 0 on $[0, 1]$ and conditioned to end at 0 at time 1. This intuitive notion was formalized in Chaumont and Pardo (2009). The Lamperti transform of F^l will be the right hand derivative of the unique solution to $\mathrm{IVP}\big(F^l, 0\big)$.

Theorem 4. *Let $Z^{k,n}$ be a Galton-Watson process with critical offspring law μ such that $Z_0^{k,n} = k$ and is conditioned on $\sum_{i=1}^{\infty} Z_i^{k,n} = n$. Let S be a random walk with step distribution μ and suppose there exist constants $a_n \to \infty$ such that $(S_n - n)/a_n$ converges in law to a spectrally positive stable distribution with Laplace exponent Ψ. Let X be a Lévy process with Laplace exponent Ψ and F^l its first passage bridge from l to 0 of length 1. If $k_n/a_n \to l$ then the sequence $S_{a_n}^{n/a_n} Z^{k_n,n}$ converges in law to the Lamperti transform of F^l in the Skorohod J_1 topology.*

When $\alpha = 2$, the process F^l is a Bessel bridge of dimension 3 between l and 0 of length 1, up to a normalization factor. In this case, (Pitman, 1999, Lemma 14) tells us that the Lamperti transform Z^l of F^l satisfies the SDE

$$\begin{cases} dZ_v^l = 2\sqrt{Z_v^l}\, dB_v + \left[4 - \frac{\left(Z_v^l\right)^2}{t - \int_0^v Z_u^l\, du}\right] dv \\ Z_0^l = l \end{cases}$$

driven by a Brownian motion B, and it is through stability theory for SDEs that Pitman (1999) obtains Theorem 4 when μ is a Poisson distribution with mean 1. Theorem 4 is a complement to the convergence of Galton-Watson forests conditioned on their total size and number of trees given in Chaumont and Pardo (2009). When $l = 0$, our techniques cease to work. Indeed, the corresponding process F^0 would be a normalized Brownian excursion and the problem $\mathrm{IVP}\big(F^0, 0\big)$ does not have a unique solution. Hence, even if our techniques yield tightness in the corresponding limit theorem with $l = 0$, we would have to give further arguments to prove that any subsequential limit is the correct solution $\mathrm{IVP}\big(F^0, 0\big)$. The limit theorem when $l = 0$ and $\alpha = 2$ was conjectured by Aldous (1991), and proved in Drmota and Gittenberger (1997) by analytic methods. For any $\alpha \in (1, 2]$, the corresponding statement was stated and proved in Kersting (1998) by working with the usual Lamperti transformation, which chooses a particular solution to $\mathrm{IVP}\big(F^0, 0\big)$.

The paper is organized as follows. Theorem 1, Proposition 1, and Corollary 1 are proved in Section 2 which focuses on the analytic aspects of the Lamperti transformation and its basic probabilistic implications. The representation CBI processes of Theorem 2 is then proved in Section 3, together with Proposition 4 and Corollary 4. Finally, Section 4 is devoted to the stability of the Lamperti transformation with a proof of Theorem 3, Proposition 3, Corollary 6, Corollary 7, and Theorem 4. (Corollaries 2 and 3 are considered to follow immediately from Theorem 2, while Corollary 5 from Theorem 2 and Propostion 2; proofs have been omitted.)

2. The generalized Lamperti transformation as an initial value problem

Let f and g be càdlàg functions with g increasing and suppose that f has no negative jumps. We also impose that $f(0) + g(0) \geq 0$. We begin by studying the existence a nonnegative càdlàg function h which satisfies

$$h(t) = f\left(\int_0^t h(s)\, ds\right) + g(t)\,; \tag{7}$$

although there might be many solutions, only one of them will let us obtain CBI processes.

When g is identically equal to zero, a solution is found by the method of time-changes: let τ be the first hitting time of zero by f, let

$$i_t = \int_0^t \frac{1}{f(s \wedge \tau)}\, ds$$

and consider its right-continuous inverse c so that

$$h = f \circ c$$

satisfies (7) with $g = 0$ and it is the only solution for which zero is absorbing. This can be found in (Ethier and Kurtz, 1986, 6.1). In this case the transformation which takes f to h is called the Lamperti transformation, introduced in Lamperti (1967a). There is a slight catch: if f is never zero and goes to infinity, then h exists up to a given time (which might be infinite) when it also goes to infinity.

Solutions to (7) are not unique even when $g = 0$ as the next example shows: take $f(x) = \sqrt{|1-x|}$, $l > 0$, and consider

$$h_1(t) = \frac{(2-t)^+}{2} \text{ and } h_2(t) = \begin{cases} \frac{2-t}{2} & \text{if } t \leq 2 \\ 0 & \text{if } 2 \leq t \leq 2+l \\ \frac{t-2-l}{2} & \text{if } t \geq 2+l \end{cases}.$$

Then h_1 and h_2 are both solutions to (7).

We propose to prove Theorem 1 by the following method: we first use the solution for the case $g = 0$ to establish the theorem when g is piecewise constant. When g is strictly increasing, we approximate it by a strictly decreasing sequence of piecewise constant functions $g_n > g$ and let h_n be the solution to (7) which uses g_n. In general, although h_n does not converge to a càdlàg function h in the usual Skorohod topology, its primitive (which starts at zero) does converge and this is enough to prove the existence of a function whose right-continuous derivative exists and solves (7). Actually, it is using primitives that one can compare the different solutions to (7) (and study uniqueness) and this is the point of view adopted in what follows. To this end, we generalize (7) into an initial value problem for the function c:

$$\mathrm{IVP}(f, g, x) = \begin{cases} c'_+(t) = f \circ c(t) + g(t) \\ c(0) = x \end{cases}.$$

(The most important case for us is $x = 0$ and we will write $\mathrm{IVP}(f, g)$ when referring to it.) We shall term

- f the **reproduction function**,
- g the **immigration function**,
- x the **initial cumulative population**, and
- c will be called the **cumulative population**.
- A solution c to $\mathrm{IVP}(f, g, x)$ is said to have **no spontaneous generation** if the condition $c'_+(t) = 0$ implies that $c(t+s) = c(t)$ as long as $g(t+s) = g(t)$.

In the setting of Theorem 1, spontaneous generation is only relevant when g is piecewise constant and it will be the guiding principle to chose solutions in this case.

A solution to $\mathrm{IVP}(f, g, x)$ without spontaneous generation when g is a constant κ is obtained by setting $f_x(s) = \kappa + f(x+s)$, calling ψ the Lamperti transform of f_x and setting $c = x + \psi$. We then have:

$$c'_+(t) = \psi'_+(t) = f_x(\psi(t)) = f(x + \psi(t)) + \kappa = f(c(t)) + g(t).$$

Let g be piecewise constant, say

$$g = \sum_{i=1}^{n} c_i \mathbf{1}_{[t_{i-1}, t_i)}$$

with $c_1 < c_2 < \cdots < c_n$ and $0 = t_0 < t_1 < \cdots < t_n$. Let us solve (7) by pasting the solutions on each interval: let ψ_1 solve $\mathrm{IVP}(f, c_1, 0)$ on $[0, t_1]$ without spontaneous generation. Let c

equal ψ_1 on $[0, t_1]$. Now, let ψ_2 solve IVP$(f, c_2, c(t_1))$ without spontaneous generation. (If $c(t_1) = \infty$, we set $\psi_2 = \infty$.) Set $c(t) = \psi_2(t - t_1)$ for $t \in [t_1, t_2]$ so that c is continuous. Also, for $t \in [t_1, t_2]$ we have

$$c'_+(t) = \psi'_{2+}(t - t_1) = f(\psi_2(t - t_1)) + c_2 = f(c(t)) + g(t) .$$

We continue in this manner. Note that if c'_+ reaches zero in $[t_{i-1}, t_i)$, say at t then c is constant on $[t, t_i)$ and that c'_+ solves (7) when g is piecewise constant. By uniqueness of solutions to (7) which are absorbing at zero when $g = 0$, we deduce the uniqueness of solutions to IVP$(f, g, 0)$ without spontaneous generation when the immigration is piecewise constant.

We first tackle the non-negativity assertion of Theorem 1. Since f is only defined on $[0, \infty)$, negative values of c do not make sense in equation (7). One possible solution is to extend f to $\mathbb{R}$ by setting $f(x) = f(0)$ for $x \leq 0$.

Lemma 1. *Any solution h to (7) is non-negative.*

Proof. Let h solve (7) where f is extended by constancy on $(-\infty, 0]$ and define

$$c(t) = \int_0^t h(s)\, ds,$$

so that c solves IVP(f, g). We prove that $h \geq 0$ by contradiction. Assume there exists $t \geq 0$ such that $h(t) < 0$. Note that since h has no negative jumps, h can only reach negative values continuously and, since h is right-continuous, if it is negative at a given t, then there exists $t' > t$ such that h is negative on $[t, t')$. Consider the non-empty set

$$\{t \geq 0 : h(t) = 0 \text{ and there exists } \varepsilon > 0 \text{ with } h < 0 \text{ on } (t, t + \varepsilon)\} ,$$

let τ be its infimum, and take $h > 0$ be such that h is negative on $(\tau, \tau + h)$. Note that $\tau > 0$ since $f(0) + g(0) \geq 0$. By definition of τ, c is non-decreasing and h is non-negative on $[0, \tau]$ and c is strictly decreasing and h negative on (τ, τ_h). So, there exist $t_1 \leq \tau < t_2 < \tau + h$ such that

$$c(t_1) = c(t_2)$$

(when $\tau = 0$ this follows since $f(0) + g(0) \geq 0$) which implies

$$0 \leq h(t_1) = f \circ c(t_1) + g(t_1) = f \circ c(t_2) + g(t_1) \leq f \circ c(t_2) + g(t_2) = h(t_2) < 0.$$

□

2.1. **Monotonicity and existence.** We now establish a basic comparison lemma for solutions to IVP$(f, g, 0)$ which will lead to the existence result when g is strictly increasing.

Lemma 2. *Let c and $\tilde{c}$ solve IVP(f, g) and IVP$\left(\tilde{f}, \tilde{g}\right)$ where $f \leq \tilde{f}$ and $g \leq \tilde{g}$. If $g(0) + f(0) < \tilde{g}(0) + \tilde{f}(0)$ and either $g_- < \tilde{g}_-$ or $f_- < \tilde{f}_-$ then $c < \tilde{c}$ on $(0, \infty)$.*

It is important to note that the inequality $c \leq \tilde{c}$ cannot be obtained from the hypothesis $g \leq \tilde{g}$ using the same reproduction function f. Indeed, we would otherwise have unicity for IVP(f, g, x) which, as we have seen, is not the case even when $g = 0$.

Proof. Let $\tau = \inf\{t > 0 : c(t) = \tilde{c}(t)\}$. Since

$$c'_+(0) = f(0) + g(0) < f(0) + \tilde{g}(0) = \tilde{c}'_+(0)\,,$$

and the right-hand derivatives of c and $\tilde{c}$ are right-continuous, then $\tau > 0$. We now argue by contradiction. If τ were finite, we know that

$$c(\tau) = \tilde{c}(\tau)$$

and then

$$c'_-(\tau) = f(c(\tau)-) + g(\tau-) = f(\tilde{c}(\tau)-) + g(\tau-) < f(\tilde{c}(\tau)-) + \tilde{g}(\tau-) = \tilde{c}'_-(\tau)\,.$$

It follows that $c'_- < \tilde{c}'_-$ in some interval $(\tau - \varepsilon, \tau)$. However, for $0 < t < \tau$ we have $c(t) < \tilde{c}(t)$ and this implies

$$c(\tau) < \tilde{c}(\tau)\,.$$

□

Proof of Theorem 1, Existence. Consider a sequence of piecewise constant càdlàg functions g_n satisfying $g \le g_{n+1} < g_{n-}$ and such that $g_n \to g$ pointwise. Let c_n solve $\mathrm{IVP}(f, g_n, 0)$ with no spontaneous generation. By Lemma 2, the sequence of functions c_n is decreasing, so that it converges to a limit c. Let

$$\tau = \inf\{t \ge 0 : c(t) = \infty\} = \lim_{n\to\infty} \inf\{t \ge 0 : c_n(t) = \infty\}\,.$$

Since f is right-continuous and $c < c_n$, $f \circ c_n + g_n$ converges pointwise to $f \circ c + g$ on $[0, \tau)$. By bounded convergence, for $t \in [0, \tau)$:

$$c(t) = \lim_{n\to\infty} c_n(t) = \lim_{n\to\infty} \int_0^t f \circ c_n(s) + g_n(s)\, ds = \int_0^t f \circ c(s) + g(s)\, ds.$$

Hence, $h = c'_+$ proves the existence part of Theorem 1. □

2.2. **Uniqueness.** To study uniqueness of $\mathrm{IVP}(f, g)$, we use the following lemma.

Lemma 3. *If g is strictly increasing and c solves $\mathrm{IVP}(f, g)$ then c is strictly increasing.*

Proof. By contradiction: if c had an interval of constancy $[s, t]$, with $t > s$, then

$$\begin{aligned} 0 &= c'_+\left(\frac{t+s}{2}\right) \\ &= f \circ c\left(\frac{t+s}{2}\right) + g\left(\frac{t+s}{2}\right) \\ &> f \circ c(s) + g(s) \\ &= 0. \end{aligned}$$

□

Proof of Theorem 1, Uniqueness. Let g be strictly increasing and suppose that c and $\tilde{c}$ solve $\mathrm{IVP}(f,g)$. To show that $c=\tilde{c}$, we argue by contradiction by studying their inverses i and $\tilde{i}$. Since c and $\tilde{c}$ are strictly increasing by Lemma 3, then i and $\tilde{i}$ are continuous. If $c\neq\tilde{c}$ then $i\neq\tilde{i}$ and we might without loss of generality suppose there is x_1 such that $i(x_1)<\tilde{i}(x_1)$. Let

$$x_0=\sup\left\{x\leq x_1: i(x)\geq \tilde{i}(x)\right\}$$

and note that, by continuity of i and $\tilde{i}$, $x_0<x_1$ and $i\leq\tilde{i}$ on $[x_0,x_1]$. Since i and $\tilde{i}$ are continuous, they satisfy

$$i_y=\int_0^y \frac{1}{f(x)+g\circ i(x)}\,dx.$$

There must exist $x\in[x_0,x_1]$ such that $i'(x)$ and $\tilde{i}'(x)$ both exist and the former is strictly smaller since otherwise the inequality $\tilde{i}\leq i$ would hold on $[x_0,x_1]$. For this value of x:

$$f(x)=\frac{1}{\tilde{i}'(x)}-g\circ\tilde{i}(x)<\frac{1}{i'(x)}-g\circ i(x)=f(x)\,,$$

which is a contradiction. □

Remark. The above proof shows that if all solutions to $\mathrm{IVP}(f,g)$ are strictly increasing then uniqueness holds. Another type of hypothesis leading to uniqueness is therefore: f is strictly positive. Even another one is: f is absorbed at 0, meaning that if $f(s)=0$ then $f(t)=0$ for all $t\geq s$.

2.3. **Uniqueness in the stochastic setting.** We now verify that solutions to (4) are unique even if Y is compound Poisson. The following lemma is used in the proof. We say that a solution to $\mathrm{IVP}(f,g)$ is minimal if it is a lower bound for all other solutions to the same initial value problem.

Lemma 4. *A solution c to* $\mathrm{IVP}(f,g)$ *is minimal if it has no spontaneous generation.*

Proof. Suppose that c has no spontaneous generation and let $\tilde{c}$ be another solution to $\mathrm{IVP}(f,g)$; let i and $\tilde{i}$ stand for their right-continuous inverses.

We proceed by contradiction. If c did not stay below $\tilde{c}$ then i would not stay above $\tilde{i}$. Let z be such that $i(z)<\tilde{i}(z)$ and define

$$x=\sup\left\{y<z: i(y)\geq\tilde{i}(y)\right\}.$$

Then $i(x)=\tilde{i}(x)$. Indeed, $\tilde{i}$ cannot go over i by a jump at x since this would imply a constancy interval of c (when it reaches level x) which is shorter than the corresponding constancy interval of $\tilde{c}$ which would mean that c leaves x before g grows.

Also, on $(x,z]$, $\tilde{i}$ can have no jumps. Indeed, if $\Delta\tilde{i}_y>0$ for some $y\in(x,z]$ then it corresponds to a constancy interval of $\tilde{c}$. However, c has to reach y before $\tilde{c}$ (since $i(y)\leq\tilde{i}(y)$), and when it does, its derivative vanishes so that it has to stay at y until g grows. The first instant that g grows after $i(y)$, say

$$t=\inf\left\{r\geq i(y): g(r)>g(i(y))\right\},$$

would have to satisfy $i(y) \leq t$ or otherwise $\tilde{c}$ would not have a constancy interval at y (since $t < \tilde{i}(y)$ implies $\tilde{c}'_+(i(y)) = f(y) + g\big(\tilde{i}(y)\big) > f(y) + g(t) = c'_+(i(y)) \geq 0$) so that $\tilde{c}$ would reach $\tilde{c}$ at level y contradicting the definition of y. Hence, for $y \in (x, z]$ we get:

$$i(x) + \sum_{y' \in (x,y]} \Delta i_{y'} + \int_x^y \frac{1}{f(y') + g \circ i(y')}\, dy' = i(y) < \tilde{i}(y) = i(y) + \int_x^y \frac{1}{f(y') + g \circ \tilde{i}(y')}\, dx'.$$

It follows that at some point $y \in [x, z]$, we have

$$g \circ \tilde{i}(y) < g \circ i(y)$$

which is impossible as g is increasing and $i \leq \tilde{i}$ on $[x, z]$. □

Proof of Proposition 1. We first prove that there is an unique process Z which satisfies

$$Z_t = x + X\left(\int_0^t Z_s\, ds\right) + Y_t.$$

Let X be a SPLP and Y and independent subordinator. When Y is an infinite activity subordinator (its Lévy measure is infinite or equivalently it has jumps in any nonempty open interval) or it has positive drift, then its trajectories are strictly increasing and so uniqueness holds thanks to Theorem 1.

It then suffices to consider the case when Y is a compound Poisson process. There is a simple case we can establish: if X is also a subordinator and $x > 0$ then all solutions to $\mathrm{IVP}(X, x + Y)$ are strictly increasing and the previous argument still works. It remains to consider two cases when Y is compound Poisson: when X is a subordinator and $x = 0$ and when X is not a subordinator. In the first, note that the minimal solution to $\mathrm{IVP}(X, 0)$ is zero. To prove uniqueness, let C^x be the (unique) solution to $\mathrm{IVP}(X, x)$, so that C^x is greater than any solution to $\mathrm{IVP}(X, 0)$. If we prove that as $x \to 0$, $C^x \to 0$, then all solutions to $\mathrm{IVP}(X, 0)$ are zero and so uniqueness holds. For this, use the fact that as $t \to 0$, X_t/t converges almost to the drift coefficient of X, say $d \in [0, \infty)$ (cf. (Bertoin, 1996, Ch. III, Proposition 8, p.84)) so that

$$\int_{0+} \frac{1}{X_s}\, ds = \infty.$$

Let I^x be the (continuous) inverse of C^x (note that C^x is strictly increasing). Since

$$I^x(t) = \int_0^t \frac{1}{x + X_s}\, ds,$$

we see, by Fatou's lemma, that $I^x \to \infty$ as $x \to 0$, so that $C^x \to 0$. Now with X still a subordinator and Y compound Poisson, the preceding case implies that the solution to $\mathrm{IVP}(X, Y)$ is unique until the first jump time of Y; after this jump time, all solutions are strictly increasing, hence uniqueness follows. The only remaining case is when Y is compound Poisson and X is not a subordinator. The last hypothesis implies that 0 is regular for $(-\infty, 0)$, meaning that on every interval $[0, \varepsilon)$, X visits $(-\infty, 0)$ (cf. (Bertoin, 1996, Ch. VII, Thm. 1, p.189)); from this, it follows that if T is any stopping time with respect to the filtration $(\sigma(X_s, Y_r : s \leq t, r \geq 0), t \geq 0)$, then X visits $(-\infty, X_T)$ on any interval to the right of T. Let C be any solution to IVP$(X, x+Y)$; we will show that it has no spontaneous generation and hence that it is equal to the minimal solution, proving uniqueness. Indeed, let

$$[T_{i-1}, T_i),\ i = 1, 2, \ldots$$

be the intervals of constancy of Y; if C has spontaneous generation on one of these, say $[T_{i-1}, T_i)$, then X reaches the level $Y_{T_{i-1}}$ and then increases, which we know does not happen since the hitting time of $\{Y_{T_{i-1}}\}$ is a stopping time with respect to the filtration $\sigma(X_s, s \leq t) \vee \sigma(Y), t \geq 0$.

We end the proof by showing that any càdlàg process Z satisfying

$$x + X_-\left(\int_0^t Z_s\, ds\right) + Y_t \leq Z_t \leq x + X\left(\int_0^t Z_s\, ds\right) + Y_t \tag{8}$$

actually satisfies

$$Z_t = x + X\left(\int_0^t Z_s\, ds\right) + Y_t.$$

When Y is strictly increasing, an argument similar to the proof of the Monotonicity Lemma (Lemma 2) tells us that Z is strictly increasing, so that the càdlàg character of Z tells us that Z actually satisfies IVP$(X, x+Y)$. When $Y = 0$, the previous argument shows that, as long as Z has not reached 0, it coincides with the solution to IVP(X, x). If Z is such that

$$\inf\{t \geq 0 : Z_t = 0\} = \inf\{t \geq 0 : Z_{t-} = 0\},$$

then Z solves IVP(X, x), which has an unique solution, so that (8) has an unique solution. We then see that the only way in which Z can cease to solve IVP(X, x) is if X is such that

$$T_{0+} = \inf\{t \geq 0 : X_{t-} = 0\} < \inf\{t \geq 0 : X_t = 0\} = T_0,$$

which is ruled out almost surely by quasi-continuity of X. Indeed, T_{0+} is the increasing limit of the stopping times

$$T_\varepsilon = \inf\{t \geq 0 : X_t < \varepsilon\}$$

which satisfy $T_\varepsilon < T_{\varepsilon'}$ if $\varepsilon < \varepsilon'$ since X has no negative jumps. Hence X is almost surely continuous at T_{0+} which says that $X_{T_{0+}} = 0$ almost surely. In the remaining case when Y is a (non-zero) compound Poisson process, we condition on Y and argue similarly on constancy intervals of Y. □

2.4. **Explosion.** We now turn to the explosion criteria of solutions of IVP(f, g) of Proposition 2.

Proof.

(1) If $\int^\infty 1/f(x) = \infty$, let c be any solution to IVP(f, g). We show that it is finite at every $t > 0$. Indeed, using the arguments of Lemma 2, we see that c is bounded

by any solution to $\mathrm{IVP}(f, 1+g(t))$ on the interval $[0,t]$. A particular solution to $\mathrm{IVP}(f, 1+g(t))$ is obtained by taking the right-continuous inverse of

$$y \mapsto \int_0^y \frac{1}{f(x)+1+g(t)}\, dx.$$

Since

$$\int_0^\infty \frac{1}{f(x)+1+g(t)}\, dx = \infty,$$

the particular solution we have considered is everywhere finite.

(2) Only the converse assertion needs verification. Let c be a solution to $\mathrm{IVP}(f,g)$ which is unbounded and suppose f is an exploding reproduction function. To prove that c explodes, we note that since $\lim_{x\to\infty} f(x)$ exists and equals ∞, we can chose M so that $f>0$ on $[M,\infty)$. We then consider the right-continuous inverse i of c and note that for $y>M$

$$i(y)-i(M) = \int_M^y \frac{1}{f(x)+g\circ i(x)}\, dx \leq \int_M^y \frac{1}{f(x)}\, dx.$$

Hence, $i(y)$ converges to a finite limit as $y\to\infty$ so that c explodes.

□

2.5. **Application of the analytic theory.** We now pass to a probabilistic application of Theorem 1.

Proof of Corollary 1. We consider first the case where Y is deterministic. Since Y is assumed to be strictly increasing, we can consider the unique non-negative stochastic process Z which satisfies

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_t.$$

(The reader might find any qualms regarding measurability issues reassured by Lemma 6.) Since Z is non-negative, Theorems. 4.1 and 4.2 of Kallenberg (1992) imply the existence of a stochastic process $\tilde X$ with the same law as X such that

$$Z_t = x + \int_0^t Z_s^{1/\alpha}\, d\tilde X_s + Y_t.$$

Hence Z is a weak solution to (5).

Conversely, if Z is a solution to (5), we apply Theorems 4.1 and 4.2 of Kallenberg (1992) to deduce the existence of a stochastic process $\tilde X$ with the same law as X such that

$$Z_t = x + \tilde X_{\int_0^t Z_s\, ds} + Y_t.$$

Considering the mapping $(f,g)\mapsto F(f,g)$ that associates to every admissible breadth-first pair the solution h to (7), we see that Z has the law of $F\left(\tilde X, x+Y\right)$. Hence, weak uniqueness holds for (5).

When Y is not deterministic but independent of X, we just reduce to the previous case by conditioning on Y (or by augmenting the filtration with the σ-field $\sigma(Y_t : t\geq 0)$). □

3. The representation of CBI processes

We now move on to the analysis of Theorem 2. Let X and Y be independent Lévy processes such that X is spectrally positive and Y is a subordinator under the probability measure $\mathbb{P}$. Call Ψ and Φ their Laplace exponents (taking care to have $\Phi \geq 0$ as for subordinators). Note that the trajectories of Y are either zero, piecewise constant (in the compound Poisson case), or strictly increasing.

Let Z be the stochastic process that solves

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_t$$

and has no spontaneous generation (when Y is compound Poisson). To prove that Z is a CBI(Ψ, Φ) we should see that it is a càdlàg and homogeneous Markov process and that there exist functions $u_t : (0, \infty) \to (0, \infty)$ and $v_t : (0, \infty) \to (0, \infty)$ satisfying

$$\begin{cases} \frac{\partial}{\partial t} u_t(\lambda) = -\Psi \circ u_t(\lambda) \\ u_0(\lambda) = \lambda \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial}{\partial t} v_t(\lambda) = \Phi(u_t(\lambda)) \\ v_0(\lambda) = 0 \end{cases} \tag{9}$$

and such that for all $\lambda, t \geq 0$:

$$\mathbb{E}\left(e^{-\lambda Z_t}\right) = e^{-x u_t(\lambda) - v_t(\lambda)}$$

(At this point it should be clear that the equation for u characterizes it and that, actually, for fixed $\lambda > 0$, $t \mapsto u_t(\lambda)$ is the inverse function to

$$x \mapsto \int_x^\lambda \frac{1}{\Psi(y)}\, dy.)$$

3.1. **A characterization Lemma and a short proof of Lamperti's Theorem.** The way to compute the Laplace transform of Z is by showing, with martingale arguments to be discussed promptly, that

$$\mathbb{E}\left(e^{-\lambda Z_t}\right) = \int_0^t \mathbb{E}\left(\left[\Psi(\lambda)\, Z_s - \Phi(\lambda)\right] e^{-\lambda Z_s}\right)\, ds. \tag{10}$$

We are then in a position to apply the following result.

Lemma 5 (Characterization Lemma)**.** *If Z is a non-negative homogeneous Markov process with càdlàg paths starting at x and satisfying (10) for all $\lambda > 0$ then Z is a $CBI(\Psi, \Phi)$ that starts at x.*

Proof. Let us prove that the function

$$G(s) = \mathbb{E}\left(e^{-u_{t-s}(\lambda) Z_s - v_{t-s}(\lambda)}\right)$$

satisfies $G'(s) = 0$ for $s \in [0, t]$, so that it is constant on $[0, t]$ implying the equality

$$\mathbb{E}\left(e^{-\lambda Z_t}\right) = G(t) = G(0) = e^{-x u_t(\lambda) - v_t(\lambda)}.$$

We then see that Z_t has the same one-dimensional distributions as a CBI(Ψ, Φ) that starts at x, so that by the Markov property, Z is actually a CBI(Ψ, Φ).

To see that $G' = 0$, we first write

$$G(s+h) - G(s) = \Big(G(s+h) - \mathbb{E}\Big(e^{-u_{t-s-h}(\lambda)Z_s - v_{t-s-h}(\lambda)}\Big)\Big) + \Big(\mathbb{E}\Big(e^{-u_{t-s-h}(\lambda)Z_s - v_{t-s-h}(\lambda)}\Big) - G(s)\Big). \tag{11}$$

We now analyze both summands to later divide by h and let $h \to 0$.

For the first summand, use (10) to get:

$$\begin{aligned} &G(s+h) - \mathbb{E}\Big(e^{-Z_s u_{t-s-h}(\lambda) - v_{t-s-h}(\lambda)}\Big) \\ &= e^{-v_{t-s-h}(\lambda)} \int_s^{s+h} \mathbb{E}\Big(e^{-Z_r u_{t-s-h}(\lambda)} \left[Z_r \Psi \circ u_{t-s-h}(\lambda) - \Phi \circ u_{t-s-h}(\lambda)\right]\Big)\, dr, \end{aligned}$$

so that, since Z has càdlàg paths, we get

$$\begin{aligned} &\lim_{h\to 0} \frac{1}{h}\left[G(s+h) - \mathbb{E}\Big(e^{-Z_s u_{t-s-h}(\lambda) - v_{t-s-h}(\lambda)}\Big)\right] = \\ &\mathbb{E}\Big(e^{-u_{t-s}(\lambda)Z_{t-s} - v_{t-s}(\lambda)} \left[Z_s \Psi \circ u_{t-s}(\lambda) - \Phi \circ u_{t-s}(\lambda)\right]\Big). \end{aligned}$$

For the second summand in the right-hand side of (11), we differentiate under the expectation to obtain:

$$\begin{aligned} &\lim_{h\to 0} \frac{1}{h} \mathbb{E}\Big(e^{-u_{t-s-h}(\lambda Z_s - v_{t-s-h}(\lambda))} - e^{-u_{t-s}(\lambda Z_s - v_{t-s}(\lambda))}\Big) \\ &= \mathbb{E}\left(e^{-u_{t-s}(\lambda)Z_s - v_{t-s}(\lambda)} \left[Z_s \frac{\partial u_{t-s}(\lambda)}{\partial s} + \frac{\partial v_{t-s}(\lambda)}{\partial s}\right]\right). \end{aligned}$$

□

A simple case of our proof of Theorem 2 arises when $Y = 0$.

Proof of Theorem 2 when $\Phi = 0$. This is exactly the setting of Lamperti's theorem stated in Lamperti (1967a).

When $\Phi = 0$ (or equivalently, Y is zero), then C_t is a stopping time for X (since the inverse of C can be obtained by integrating $1/(x+X)$). Since Z is the time-change of X using the inverse of an additive functional, Z is a homogeneous Markov process. (Another proof of the Markov property of Z, based on properties of $\mathrm{IVP}(X, x+Y)$ is given in (3) of Lemma 6.) Also, we can transform the martingale

$$e^{-\lambda X_t} - \Psi(\lambda) \int_0^t e^{-\lambda X_s}\, ds$$

by optional sampling into the martingale

$$e^{-\lambda Z_t} - \Psi(\lambda) \int_0^t e^{-\lambda Z_s} Z_s\, ds.$$

We then take expectations and apply Lemma 5. □

3.2. **The general case.** For all other cases, we need the following measurability details. Consider the mapping F_t which takes a càdlàg function f with nonnegative jumps and starting at zero, a piecewise constant nondecreasing g starting at zero, and a nonnegative real x to $c'_+(t)$ where c solves $\mathrm{IVP}(f, x+g, 0)$ and has no spontaneous generation (if g is piecewise constant). Then

$$Z_{t+s} = F_t(X_{C_s+\cdot} - X_{C_s}, Y_{s+\cdot} - Y_s, Z_s)\,. \tag{12}$$

The assumed lack of spontaneous generation of Z will only be relevant to the proof of Theorem 2 to allow the following measurability details. We suppose that our probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is complete and let $\mathscr{T}$ stand for the sets in $\mathscr{F}$ of probability zero. For fixed $y, t \in [0, \infty]$, let $\mathscr{G}^t_y = \mathscr{F}^X_y \vee \mathscr{F}^Y_t \vee \mathscr{T}$.

Lemma 6 (Measurability details)**.**

(1) The filtration $\left(\mathscr{G}^t_y, y \geq 0\right)$ satisfies the usual hypotheses.

(2) C_t is a stopping time for the filtration $\left(\mathscr{G}^t_y, y \geq 0\right)$ and we can therefore define the σ-field

$$\mathscr{G}^t_{C_t} = \left\{A \in \mathscr{F} : A \cap \{C_t \leq y\} \in \mathscr{G}^t_y\right\}.$$

(3) Z is a homogeneous Markov process with respect to the filtration $\left(\mathscr{G}^t_{C_t}, y \geq 0\right)$.

Proof.

(1) We just need to be careful to avoid *one of the worst traps involving σ-fields* by using independence (cf. (Chaumont and Yor, 2003, Ex. 2.5, p. 29)).

(2) We are reduced to verifying

$$\{C_t < y\} \in \mathscr{G}^t_y. \tag{13}$$

We prove (13) in two steps, first when Y is piecewise constant, then when Y is strictly increasing.

If Y is piecewise constant, jumping at the stopping times $T_1 < T_2 < \ldots$ and set $T_0 = 0$. We first prove that

$$\{C_{T_n} < y\} \in \mathscr{F}^X_y \vee \mathscr{F}^Y_{T_n} \tag{14}$$

and this result and a similar argument will yield (13). The membership in (14) is proved by induction using the fact that C can be written down as a Lamperti transform on each interval of constancy of Y. Let I_t be the functional on Skorohod space that aids in defining the Lamperti transformation: when applied to a given function, it stops it upon reaching zero and then integrates its reciprocal from 0 to t. We then have

$$\{C_{T_1} < y\} = \{I_y(X + Y_0) > T_1\} \in \mathscr{F}^X_y \vee \mathscr{F}^Y_{T_1}$$

and if we suppose that

$$\{C_{T_n} < y\} \in \mathscr{F}^X_y \vee \mathscr{F}^Y_{T_n} \subset \mathscr{F}^X_y \vee \mathscr{F}^Y_{T_{n+1}}$$

then the decomposition

(15)

$$
\begin{aligned}
&\{C_{T_{n+1}} < y\}\\
&= \bigcup_{q\in(0,y)\cap\mathbb{Q}} \bigcup_{m=1}^{\infty} \{C_{T_n} < y - q - 1/2^m\} \cap \left\{I_q\big(X_{C_{T_n}+\cdot} + Y_{T_n}\big) > T_{n+1} - T_n\right\}\\
&= \bigcup_{q\in(0,y)\cap\mathbb{Q}} \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{2^m[y-q-1/2^m]} \left\{\frac{k}{2^m} \le C_{T_n} < \frac{k+1}{2^m}\right\} \cap \left\{I_{q-1/2^n}\big(X_{(k+1)/2^n+\cdot} + Y_{T_n}\big) > T_{n+1} - T_n\right\}
\end{aligned}
$$

allows us to obtain (14). We then write

$$
\begin{aligned}
&\{C_t < y\}\\
&= \bigcup_{n=0}^{\infty} \bigcup_{q\in(0,y)\cap\mathbb{Q}} \{T_n \le t < T_{n+1}\} \cap \{C_{T_n} < y - q\} \cap \left\{I_q\big(X_{C_{T_n}+\cdot} + Y_{T_n}\big) > t - T_n\right\}
\end{aligned}
$$

and decompose the right-hand side as in (15) to obtain when Y is piecewise constant.

When Y is strictly increasing, consider a sequence ε_n decreasing strictly to zero and a decreasing sequence (π_n) of partitions of $[0,t]$ whose norms tend to zero, with

$$\pi_n = \left\{t_0^n = 0 < t_1^n < \cdots < t_{k_n}^n = t\right\}.$$

Consider the process $Y^n = (Y_s^n)_{s\in[0,t]}$ defined by

$$Y_s^n = \varepsilon_n + \sum_{i=1}^{k_n} Y_{t_i^n} \mathbf{1}_{[t_{i-1}^n, t_i^n)}(s) + Y_t \mathbf{1}_{s=t}.$$

Since π^n is contained in π^{n+1} and $\varepsilon_n > \varepsilon_{n+1}$, $Y^n > Y^{n+1}$. If C^n is the solution to IVP$(X, x + Y^n, 0)$ with no spontaneous generation (defined only on $[0,t]$), then Lemma 2 gives $C^n > C^{n+1}$. Hence, (C^n) converges as $n \to \infty$, and since the limit is easily seen to be a solution to IVP$(X, x + Y, 0)$; the limit must equal C by the unicity statement in Theorem 1. To obtain (13), we note that

$$\{C_t^n < y\} \in \mathscr{F}_y^X \vee \mathscr{F}_t^{Y^n} \subset \mathscr{F}_y^X \vee \mathscr{F}_t^Y$$

and

$$\{C_t < y\} = \bigcup_n \{C_t^n < y\}.$$

(3) Mimicking the proof of the Strong Markov Property for Lévy processes (as in (Kallenberg, 2002, 13.11)) and using (13), one proves that the process

$$(X_{C_t+s} - X_{C_t}, Y_{t+s} - Y_t)_{s\ge 0}$$

has the same law as (X, Y) and is independent of $\mathscr{G}_{C_t}^t$ so that the strong Markov property for X and Y can be translated into

$$(X_{C_t+s} - X_{C_t}, Y_{t+s} - Y_t)_{s\geq 0} \text{ has the same law as } (X, Y) \text{ and is independent of } \left(X^{C_t}, Y^t\right).$$

Equation (12) implies that the conditional law of Z_{t+s} given $\mathscr{G}^s_{C_s}$ is actually Z_s measurable, implying the Markov property. The transition semigroup is homogeneous and in t units of time is given by the law $P_t(x,\cdot)$ of $F_t(X, Y, x)$ under $\mathbb{P}$. Note that this semigroup is conservative on $[0, \infty]$.

□

We will need Proposition 4 for our proof of Theorem 2.

Proof of Proposition 4. Consider the filtration $(\mathscr{G}_y)$ given by

$$\mathscr{G}_t = \sigma(X_s : s \leq t) \vee \sigma(Y_s : s \geq 0) \vee \mathscr{T},$$

which satisfies the usual conditions. For fixed $\varepsilon > 0$, let $T_1 < T_2 < \dots$ be the jumps of Y of magnitude greater than ε. Arguing as in Lemma 6, we see that C_{T_i} is a $\mathscr{G}$-stopping time. Since X is a $\mathscr{G}$-Lévy process and C has continuous sample paths, quasi-continuity of X implies that X does not jump at C_{T_i} almost surely. □

Proof of Theorem 2. Since

$$\left(e^{-\lambda X_y} - \Psi(\lambda) \int_0^y e^{-\lambda X_s}\, ds\right)_{t\geq 0}$$

is a $\left(\mathscr{G}^t_y\right)_{y\geq 0}$-martingale, it follows that $M = (M_t)_{t\geq 0}$, given by

$$M_t = e^{-\lambda X_{C_t}} - \Psi(\lambda) \int_0^t e^{-\lambda X_{C_s}} Z_s\, ds,$$

is a $\left(\mathscr{G}^t_{C_t}\right)_{t\geq 0}$-local martingale. With respect to the latter filtration, the stochastic process $N = (N_t)_{t\geq 0}$ given by

$$N_t = e^{-\lambda Y_t} + \Phi(\lambda) \int_0^t e^{-\lambda X_s}\, ds$$

is a martingale. Applying integration by parts to $e^{-\lambda X\circ C}$ and $e^{-\lambda x + Y}$, we get

$$e^{-\lambda Z_t} = \text{Local Martingale} + \int_0^t e^{-\lambda Z_s} \left[\Psi(\lambda)\, Z_s + \Phi(\lambda)\right] ds + \left[e^{-\lambda X\circ C}, e^{-\lambda x - \lambda Y}\right]_t.$$

We now prove that the covariation in the preceding display is zero; hence

$$e^{-\lambda Z_t} - \int_0^t e^{-\lambda Z_s} \left[\Psi(\lambda)\, Z_s - \Phi(\lambda)\right] ds$$

is a martingale, since its sample paths are uniformly bounded on compacts thanks to the non-negativity of Z.

Given that $e^{-\lambda Y_t}$ is of bounded variation since Y is, (Kallenberg, 2002, Theorem 26.6.(vii)) implies that

$$\left[e^{-\lambda X\circ C}, e^{-\lambda x - \lambda Y}\right]_t = \sum_{s\leq t} \Delta e^{-\lambda X\circ C_s} \Delta e^{-\lambda x - \lambda Y_s}.$$

Since, by Propostion 4, $X \circ C$ and Y do not jump at the same time, we see that

$$\left[e^{-\lambda X\circ C}, e^{-\lambda x-\lambda Y}\right] = 0.$$

Taking expectations, we get (10) and we conclude by applying Lemma 5 since Z is a Markov process by Lemma 6. □

3.3. **Translating a law of the iterated logarithm.**

Proof of Corollary 4. Let $\tilde{\Phi}$ be the right-continuous inverse of Ψ and

$$f(t) = \frac{\log|\log|\, t}{\tilde{\Phi}(t)}.$$

As noted by Bertoin (1995), Fristedt and Pruitt (1971) prove the existence of a constant $c \neq 0$ such that

$$\liminf_{t\to 0} \frac{X_t}{f(t)} = c.$$

Recall that $\tilde{\Phi}$ is the Laplace exponent of a subordinator (cf. (Bertoin, 1996, Ch. VII, Thm. 1)); if $\tilde{d}$ is the drift coefficient of $\tilde{\Phi}$ then

$$\lim_{\lambda\to\infty} \frac{\Phi(\lambda)}{\lambda} = \tilde{d}.$$

If $\tilde{d} = 0$ then

$$\frac{1}{f(t)} = o\left(\frac{1}{t}\right),$$

and if $\tilde{d} > 0$ then

$$\frac{1}{f(t)} \sim \frac{\tilde{d}}{t}.$$

Let Z be the unique solution to

$$Z_t = x + X_{\int_0^t Z_s\, ds} + Y_t$$

with $x > 0$, where X and Y are independent Lévy processes, with X spectrally positive of Laplace exponent Ψ and Y a subordinator with Laplace exponent Φ. Since $Z_0 = x$, and Z is right-continuous, then

$$\lim_{t\to 0+} \frac{1}{t}\int_0^t Z_s\, ds = x$$

almost surely. Hence

$$\liminf_{t\to 0+} \frac{X_{\int_0^t Z_s\, ds}}{f(xt)} = c.$$

On the other hand, if d is the drift of Φ then

$$\lim_{t\to 0} \frac{Y_t}{t} = d,$$

(cf. (Bertoin, 1996, Ch. III, Prop. 8)). Hence,

$$\lim_{t\to 0+} \frac{Z_t - x}{f(xt)} = c.$$

If $\tilde{d} > 0$ then, actually,

$$\liminf_{t\to 0} \frac{X_t}{t} = -\frac{1}{\tilde{d}}$$

so that

$$\liminf_{t\to 0+} \frac{Z_t - x}{t} = -\frac{x}{\tilde{d}} + d.$$

□

4. Stability of the generalized Lamperti transformation

We now turn to the proof of Theorem 3, and to Corollaries 6 and 7, which summarize the stability theory for IVP(f, g).

4.1. **Proof of the analytic assertions.** The basic step in the proof of Theorem 3 is accomplished by the following lemma:

Lemma 7. *Under the assumptions of Theorem 3, if $(c_n(t)\,, n \geq 1)$ is bounded for some $t > 0$ then $c_n \to c$ uniformly on $[0, t]$.*

Proof. Let M be a bound for $c_n(t)$ and let K be an upper bound for $(f_n, n \geq 1)$ on $[0, M]$ and $(g_n, n \geq 1)$ on $[0, t]$ (which exists since $f_n \to f$ and $g_n \to g$). For any $s \in [0, t]$

$$D_+ c_n(s) = [f_n \circ c_n([\sigma_n s]/\sigma_n) + g_n(s)]^+ \leq 2K$$

implying that the family of functions $\{c_n : \sigma \in (0, 1]\}$ has uniformly bounded right-hand derivatives (on $[0, t]$) and starting points. (If $\sigma_n = 0$, we get the same upper bound for $D_+ c_n$ using the equality $D_+ c_n = f \circ c_n + g$.) Therefore, $\{c_n : n \geq 1\}$ is uniformly bounded and equicontinuous on $[0, t]$. By the Arzelà-Ascoli theorem, every sequence $(c_{n_k}, k \geq 1)$ has a further subsequence that converges to a function $\tilde{c}$ (which depends on the subsequence). We now prove that $\tilde{c} = c$, which implies that $c_n \to c$ as $n \to \infty$ uniformly on $[0, t]$.

Suppose that n_k is such that c_{n_k} has an limit $\tilde{c}$ as $k \to \infty$ uniformly on $[0, t]$. Since f has no negative jumps, we get

$$\liminf_{x\to y} f(x) = f_-(y) \quad \text{and} \quad \limsup_{x\to y} f(x) = f(y)$$

so that

$$f_- \circ \tilde{c} \leq \liminf_{k\to\infty} f \circ c_{n_k} \quad \text{and} \quad \limsup_{k\to\infty} f \circ c_{n_k} \leq f \circ \tilde{c}$$

Using Fatou's lemma we get

$$\int_s^t [f_- \circ \tilde{c}(r) + g_-(r)]^+ \, dr \leq \tilde{c}(t) - \tilde{c}(s) \leq \int_s^t [f \circ \tilde{c}(r) + g(r)]^+ \, dr.$$

□

Although it is not used in the proof of Theorem 3, we continue with a proof of Proposition 3.

Proof of Proposition 3. Let c be any solution to

$$\int_s^t f_- \circ c(r) + g(r)\, ds \leq c(t) - c(r) \leq \int_s^t f \circ c(r) + g(r)\, ds, \quad \text{for } s \leq t.$$

Note that c is strictly increasing if $f + g(0)$ is positive or g is strictly increasing, so that the lower and upper bounds for c are equal in these cases, implying that c solves $\text{IVP}(f, g)$ which has a unique solutions with these hypotheses. Indeed, if $f + g(0)$ is a positive function, then the lower bound integrand is strictly positive and so c cannot have a constancy interval. If on the other hand g is strictly increasing, note that initially $f \circ c + g$ is non-negative and since it has no negative jumps, it can only reach zero continuously. If c had a constancy interval $[s, t]$ with $s < t$, we then let

$$s' = \sup \{ r \leq s : f_- \circ c(r) + g(r) > 0 \},$$

so that

$$f_- \circ c\big(s'\big) + g\big(s'\big) = 0.$$

But then, since c is constant on $[s', t]$, there exists $r \in (s', t)$ such that

$$f_- \circ c\big(s'\big) + g\big(s'\big) = f_- \circ c\big(s'\big) + g(r) = 0$$

which implies that g has a constancy interval on $[0, t]$, a contradiction. Hence, c has no constancy intervals.

When g is a constant and $f + g$ is absorbed at zero, then c is strictly increasing until it is absorbed, so that again both bounds for c are equal. □

Proof of Theorem 3. For the convergence of cumulative population c_n towards c, we argue by cases along sequences $n_k \to \infty$, reducing by further subsequences to the alternatives: $(c_{n_k}(t))$ is bounded or goes to ∞. The latter possibility is handled by Lemma 7; we now prove that in the former, $c_{n_k} \to c$ pointwise on $[0, t]$ as $k \to \infty$. The conclusion is that $c_n \to c$ pointwise on $[0, \infty)$ and hence, by Lemma 7, uniformly on compact subsets of $[0, \tau)$.

Suppose that $n_k \to \infty$ is such that $c_{n_k}(t) \to \infty$. For any $x > 0$, consider the sequence $c_{n_k} \wedge x$. Note that it is uniformly bounded and eventually equal to x. Indeed, note that $c_{n_k}(s) \leq x$ implies $s \leq t$ for large enough n. For any $s \in [0, t]$ we have

$$D_+ c_{n_k} \wedge x(s) = [f \circ c_{n_k}([\sigma_{n_k} s]/\sigma_{n_k}) + g(s)]^+ \mathbf{1}_{c_{n_k}(s) \leq x} \leq \overline{f}(x) + g(t)$$

for large enough n, so that the sequence $c_{n_k} \wedge x$ is uniformly bounded and equicontinuous on $[0, t]$. Let $\tilde{c}$ be its uniform limit on $[0, t]$. If $\tilde{c}(s) < x$, we can argue as in the proof of Lemma 7 to see that $\tilde{c} = c$ on $[0, s]$. If $\tilde{c}(s) \geq x$ and we see that $\tilde{c}$ and c both reach x at the same point $s' \leq s$ and hence $\tilde{c}(s) = c(s) \wedge x$. Hence $c_{n_k} \wedge x \to c \wedge x$. Since x is arbitrary, we see that $c_{n_k} \to c$ pointwise on $[0, t]$, even if $c(t) = \infty$.

Let $h_n = D_+ c_n$ and $h = D_+ c$. We now prove that $h_n \to h$ in the Skorohod J_1 topology if the explosion time τ is infinite. Recall that $h = f \circ c + g$ and that when $\sigma_n = 0$ then $h_n = f_n \circ c_n + g_n$ while if $\sigma_n > 0$ then $h_n(t) = [f_n \circ c_n([t/\sigma_n]\sigma_n) + g_n([t/\sigma_n]\sigma_n)]^+$. Assume that $\sigma_n = 0$ for all n, the case $\sigma_n > 0$ being analogous. Then the assertion $h_n \to h$ is reduced to proving that: $f_n \circ c_n \to f \circ c$, which is related to the continuity of the composition mapping on Skorohod space, and then deducing that $f_n \circ c_n + g_n \to f \circ c + g$, which is related to continuity of addition on Skorohod space. Both continuity assertions require conditions to hold: the convergence $f_n \circ c_n \to f \circ c$ can be deduced from (Wu, 2008, Thm. 1.2) if we prove that f is continuous at every point at which c^{-1} is discontinuous, and then the convergence of $f_n \circ c_n + g_n$ will hold because of (Whitt, 1980, Thm 4.1) since we assumed that $f \circ c$ and g do not jump at the same time. Hence, the convergence $h_n \to h$ is reduced to proving that f is continuous at discontinuities of c^{-1}. If c is strictly increasing (which happens when g is strictly increasing or $f + g(0) > 0$), then c^{-1} is continuous. When c is not strictly increasing, we will use the assumed uniqueness of (6) to prove that f is continuous at discontinuities of c. The proof reduces to checking that, if c has an interval of constancy $[s, t]$, assumed to be maximal (which corresponds to a jump of c^{-1} at $c(s)$), then g is constant on $[s, t]$ and f reaches $-g(s)$ for the first time at $c(s)$. Since f has no negative jumps, then f is continuous at $c(s)$.

We now prove that f is continuous at discontinuities of c^{-1}. Suppose c^{-1} is discontinuous at x, let $s = c^{-1}(x-)$ and $t = c^{-1}(x)$, so that $c = x$ on $[s, t]$ while $c < x$ on $[0, s)$ and $c > x$ on (t, ∞). Since $D_+c = f \circ c + g = 0$ on $[s, t)$, we see that g is constant on $[s, t)$. We assert that

$$\inf \{y \geq 0 : f(y) = -g(s)\} = x.$$

Indeed, if f reached $-g(s)$ at $x' < x$, there would exist $s' < s$ such that

$$f \circ c(s') + g(s) = 0 \geq f \circ c(s') + g(s') \geq 0$$

so that actually g is constant on $[s', t)$. Hence, c has spontaneous generation which implies there are at least two solutions to IVP(f, g): one that is constant on (s', s), and c. This contradicts the assumed uniqueness to (6).

Finally, we assume that the explosion time τ is finite but that $f_n \to f$ in the uniform J_1 topology and prove that $h_n \to h$ in the uniform J_1 topology. Let $\varepsilon > 0$, d be any metric on $[0, \infty]$ that makes it homeomorphic to $[0, 1]$, and consider $M > 0$ such that $d(x, y) < \varepsilon$ if $x, y \geq M$. Since f is an explosive reproduction function, then $\lim_{t\to\infty} f(t) = \infty$. Then, because $f_n \to f$ in the uniform J_1 topology, we can then infer the existence of $K > 0$ such that $f(x), f_n(x) > M$ if $x > K$. Let $T < \tau$ be such that f is continuous at $c(T)$ and $K < c(T)$. Then, $f_n \to f$ in the usual J_1 topology on $[0, c(T)]$ and, arguing as in the non-explosive case, we see that $h_n = f_n \circ c_n + g_n \to f \circ c + g = h$ in the usual J_1 topology on $[0, T]$. Hence, there exists a sequence (λ_n) of increasing homeomorphisms of $[0, T]$ into itself such that $h_n - h \circ \tilde{\lambda}_n \to 0$ uniformly on $[0, T]$. Define now λ_n to equal $\tilde{\lambda}_n$ on $[0, T]$ and the identity on $[T, \infty)$. Then (λ_n) is a sequence of homeomorphisms of $[0, \infty)$ into itself which converges uniformly to the identity, and since $K < c(T)$, then $K < c_n(T)$ eventually and so $M < h_n, h$ eventually, so that $d(h_n(t), h(t)) < \varepsilon$ on $[T, \infty)$ eventually. Since ε is arbitrary, we see that $h_n \to h$ in the uniform J_1 topology.

□

In order to apply Theorem 3 to Galton-Watson type processes, we need a lemma relating the discretization of the Lamperti transformation and scaling. Define the scaling operators S_a^b by

$$S_a^b f(t) = \frac{1}{b} f(at)\,.$$

Let also c^σ be the approximation of span σ to $\mathrm{IVP}(f, g)$, which is the unique function satisfying

$$c^\sigma(t) = \int_0^t f \circ c^\sigma(\sigma[s/\sigma]) + g(\sigma[s/\sigma])\ ds.$$

We shall denote $c^\sigma(f, g)$ to make the explicit on f and g explicit in the following theorem.

Lemma 8. *We have:*

$$S_a^b c^\sigma(f, g) = c^{\sigma/a}\Big(S_b^{b/a} f,\ S_a^{b/a} g\Big)\,,$$

and if h^σ is the right-hand derivative of c^σ then

$$S_a^{b/a} h^\sigma(f, g) = h^{\sigma/a}\Big(S_b^{b/a} f,\ S_a^{b/a} g\Big)\,.$$

Proof. For the first assertion, we should prove that

$$S_a^b c^\sigma(t) = \int_0^t S_b^{b/a} f \circ S_a^b c^\sigma(\sigma/a[as/\sigma]) + S_a^{b/a} g(\sigma/a[as/\sigma])\ ds.$$

This follows from a change of variables and a definition chase:

$$\begin{aligned}
S_a^b c^\sigma(t) = \frac{1}{b} c^\sigma(at) &= \frac{1}{b} \int_0^{at} f \circ\ c^\sigma(\sigma[s/\sigma]) + g(\sigma[s/\sigma])\ ds \\
&= \frac{a}{b} \int_0^t f \circ c^\sigma(\sigma[as/\sigma]) + g(\sigma[as/\sigma])\ ds \\
&= \frac{a}{b} \int_0^t f\Big(b S_a^b c^\sigma(\sigma/a[as/\sigma])\Big) + S_a^1 g(\sigma/a[as/\sigma])\ ds \\
&= \int_0^t S_b^{b/a} f \circ S_a^b c^\sigma(\sigma/a[as/\sigma]) + S_a^{b/a} g(\sigma/a[as/\sigma])\ ds.
\end{aligned}$$

The second assertion now follows noting that

$$\int_0^t S_a^{b/a} h^\sigma(s)\ ds = S_a^b c^\sigma(t) = c^{\sigma/a}\Big(S_b^{b/a} f, S_a^{b/a} g\Big)(t) = \int_0^t h^{\sigma/a}\Big(S_b^{b/a} f, S_a^{b/a} g\Big)(t)\,.$$

□

4.2. **Weak continuity of CBI laws.**

Proof of Corollary 6. Let X_n and X be spLps with Laplace exponents Ψ_n and Ψ and Y_n and Y be subordinators with Laplace exponents Φ_n and Φ such that X_n (resp. X) is independent of Y_n (resp. Y). The hypotheses $\Psi_n \to \Psi$ and $\Phi_n \to \Phi$ imply that (X_n, Y_n) converges weakly (in the Skorohod J_1 topology) to (X, Y) and by Skorohod's representation theorem, we can assume that the convergence takes place almost surely on an adequate probability space.

Let Z_n (resp. Z) be the Lamperti transform of $(X_n, x_n + Y_n)$ (resp. $(X, x + Y)$). Proposition 1 and Theorem 3 then imply that Z_n converges almost surely to Z, which is a CBI(Ψ, Φ) thanks to Theorem 2.

□

4.3. **A limit theorem for Galton-Watson processes with immigration.**

Proof of Corollary 7. The weak convergence of X^n_n/a_n to μ where $a_n \to \infty$ implies that μ is infinitely divisible and that its Lévy measure has support in $(0, \infty)$ since X^n has jumps bounded below by -1. Let Ψ be the Laplace exponent of μ. Likewise, the weak convergence of Y^n_n/b_n to ν, where $b_n \to \infty$, implies, since Y^n has non-decreasing sample paths, that ν is the distribution at time 1 of a subordinator; call Φ the Laplace exponent of the latter. By Skorohod's theorem, if X and Y are Lévy processes whose distributions at time 1 are μ and ν then:

$$S^{a_n}_n X^n \to X \quad \text{and} \quad S^{b_n}_n Y^n \to Y,$$

where the convergence is in the J_1 topology if X does not drift to infinity, but can be strengthened to the uniform J_1 topology if X drifts to infinity.

Since $k_n \sim c_n x$ where $x \geq 0$ and $c_n = b_n/na_n$, then

$$S^{b_n/a_n}_{n/a_n} Z^n_0 = k_n/c_n \to x.$$

So that if $S^{b_n/a_n}_{n/a_n} Z^n$ converges in law, its limit starts at x. Furthermore, we can apply Lemma 8 to get:

$$S^{b_n/a_n}_{n/a_n} Z^n = L^{1/n}\left(S^{b_n/a_n}_{nb_n/a_n^2} X^n, S^{b_n/a_n}_{n/a_n} Y^n\right).$$

Considering subsequences, we see that

$$S^{b_n/a_n}_{nb_n/a_n^2} X^n \to X \quad \text{and} \quad S^{b_n/a_n}_{n/a_n} Y^n \to Y,$$

so that

$$S^{b_n/a_n}_{n/a_n} Z^n \to L(X, Y)\,,$$

thanks to Proposition 1 and Theorems 2 and 3. □

4.4. **A limit theorem for conditioned Galton-Watson processes.**

Proof of Theorem 4. Let $Z^{k,n}$ be a Galton-Watson process with critical offspring law μ such that $Z^{k,n}_0 = k$ and is conditioned on $\sum_{i=1}^{\infty} Z^{k,n}_i = n$. Then, $Z^{k,n}$ has the law of the discrete Lamperti transformation of of the n steps of a random walk with jump distribution $\tilde{\mu}$ (the shifted reproduction law) which starts at 0 and is conditioned to reach $-k$ in n steps; call the latter process $X^{k,n}$, so that

$$Z^n_k = L^1\left(k + X^{k,n}\right).$$

Thanks to Chaumont and Pardo (2009), if $k/\sigma\sqrt{n} \to l$ then

$$S_n^{a_n} X^{k,n} \to F^l.$$

Thanks to Lemma 8, we see that

$$S_n^{n/a_n} Z^{n,k} = L^{1/a_n}\left(S_n^{a_n} X^{k,n}, 0\right).$$

Since F^l is absorbed at zero (as easily seen by the pathwise construction of F^l in (Chaumont and Pardo, 2009, Thm. 4.3)), then Theorem 3 and Proposition 3 imply that

$$S_n^{n/a_n} Z^{n,k} \to L\left(F^l, 0\right).$$

□

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