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A geometric approach to integrability of Abel differential equations

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Abstract A geometric approach is used to study the Abel first order differential equation of the first kind. The approach is based on the recently developed theory of quasi-Lie systems which allows us to characterise some particular examples of integrable Abel equations. Second order Abel equations will be discussed and the inverse problem of the Lagrangian dynamics is analysed: the existence of two alternative Lagrangian formulations is proved, both Lagrangians being of a non-natural class. The study is carried out by means of the Darboux polynomials and Jacobi multipliers.

Keywords Abel equation · Lie systems · Jacobi multiplier

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1 Introduction: Abel differential equations

Nonlinear differential equations play a relevant role in the study of evolution in terms of an evolution parameter, what has motivated physicists' interest during the last forty years. The lack of a general procedure for determining their general solutions justifies the analysis of particular instances where solutions can be investigated via algebraic or geometric methods. An important example are Lie systems, which admit a superposition rule giving the general solution in terms of

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a finite set of particular solutions. These systems describe the integral curves of t-dependent vector fields which are t-dependent linear combinations of vector fields closing on a finite-dimensional real Lie algebra (see e.g. [1] and [2] for a modern presentation of the theory of Lie systems [3,4]).

An interesting instance of Lie system is the Riccati equation, a generalisation of the inhomogeneous linear equation. One of its main characteristics is the existence of an action of the group of curves in $SL(2,\mathbb{R})$ on the set of Riccati equations, what can be used for reduction of a given Riccati equation into a simpler one. This is for instance the case when a particular solution of a given equation is known [5]. Moreover, this property can be shown to be common to all Lie systems [6].

A generalisation of Lie systems has recently been proposed [7,8] allowing us to use some of the techniques applicable to Lie systems to this more general class of systems. This paper deals with a specific example, the Abel equation, with applications in many different branches of physics.

There are several differential equations called Abel equation. The first-order Abel equation of the first kind

$$\dot{x} = A_0(t) + A_1(t)x + A_2(t)x^2 + A_3(t)x^3, \tag{1}$$

is a generalisation of the Riccati equation (the particular case $A_3 = 0$) introduced by Abel within the theory of elliptic functions, while the first-order Abel equation of the second kind is of the form [9,10]:

$$(y + f(t))\,\dot{y} = B_0(t) + B_1(t)y + B_2(t)y^2 + B_3(t)y^3, \tag{2}$$

which reduces into one of the first kind by means of the transformation

$$x = (y + f(t))^{-1}, \tag{3}$$

with coefficients

$$\begin{aligned} &A_0 = -B_3, \quad A_1 = 3B_3\,f - B_2,\\ &A_2 = -f' - 3B_3\,f^2 + 2f\,B_2 - B_1,\\ &A_3 = f^3 B_3 - f^2\,B_2 + f\,B_1 - B_0. \end{aligned}$$

Therefore, we can restrict ourselves to study the Abel equations of the first kind.

We can also consider the generalised first-order Abel equation [11]

$$\dot{x} = \sum_{k=0}^{n} A_k(t)\,x^k, \quad n > 3, \quad n \in \mathbb{N}, \tag{4}$$

or the second-order Abel equation [12]:

$$\frac{d^2x}{dt^2} + 4x^2\Big(\frac{dx}{dt}\Big) + x^5 = 0\,. \tag{5}$$

The n-order Abel equation is a linear combination of the different members of a hierarchy of j-order Abel equations with functions $p_j = p_j(t)$ as coefficients [12]:

$$(p_0\mathbb{D}_A^n + p_1\mathbb{D}_A^{n-1} + \cdots + p_{n-1}\mathbb{D}_A + p_n)x + p_{n+1} = 0\,. \tag{6}$$

Here, $\mathbb{D}_A$ denotes the differential operator $\mathbb{D}_A = d/dt + x^2$, and the action of $\mathbb{D}_A$ leads to a family of differential equations whose first members are given by a sequence $\mathbb{D}_A^j x = 0$, $j = 0, 1, 2, \ldots$.

Abel equations appear in the reduction of order of many second- and higher-order equations, and hence are frequently found in the modeling of real problems in many areas, for instance, Emden equation, $y''+(2/x)y'+y^n=0$, Emden–Fowler equation, $y''=A\,x^n\,y^m\,(y')^l$, the Van der Pol equation $y''-\epsilon(1-y^2)y'+\alpha\,y=0$, the Duffing equation, $y''+ay+b\,y^3=0$, and many other equations. They also play a relevant role in the study of quadratic systems in the plane [13] and the center-focus problem [14].

The Liénard equation $x''+f(x)\,x'+g(x)=0$ was studied in [15]. Defining $\xi(x)=x'$, it may be written $\xi\,\xi'+f(x)\,\xi+g(x)=0$, which is an Abel equation of the second kind, related by $u=1/y$ (see (3)) to the Abel equation of the first kind: $u'=f(u)\,u^2+g(u)\,u^3$. In particular, the one studied by Bougoffa [10], $\xi\,\xi'-\xi=\Phi(x)$ reduces to $u'+u^2+\Phi(x)\,u^3=0$. For instance, the autonomous dissipative Milne–Pinney equation can be transformed into a second kind Abel equation:

$$y\left(\frac{dy}{dt}+1\right)=\frac{1}{t^3},$$

which is related to the first kind Abel equation $\dot{x}=x^2-\frac{1}{t^3}x^3$.

The Abel equation was used by Majorana in the study of Thomas–Fermi equation:

$$y''=\frac{y^{3/2}}{x^{1/2}}, \tag{7}$$

which is a particular case of Emden–Fowler equation, and it is also used in the study of cosmological models [16]. The general solution of the cosmological Einstein-Friedmann equations for the universe filled with a scalar field for a given potential can be expressed via the general solution of the Abel equation of the first kind [17]. Another more recent application can be found in [18].

2 Lie systems and superposition rules

There are (systems of n) first-order differential equations (to be called Lie systems) that admit a superposition rule, i.e. a function $\phi:\mathbb{R}^{n(m+1)}\to\mathbb{R}^n$ such that the general solution can be written as $x=\phi(x_{(1)},\ldots,x_{(m)},k)$, the simplest example being the inhomogeneous linear equation

$$\dot{x}=c_0(t)+c_1(t)\,x, \tag{8}$$

for which there is a superposition rule involving two generic particular solutions:

$$x(t)=\phi(x_1(t),x_2(t);k)=x_1(t)+k\,(x_2(t)-x_1(t)). \tag{9}$$

Moreover, it admits a solution in terms of two quadratures:

$$x(t)=\exp\left(\int^t c_1(t')\,dt'\right)\left[C-\int^t\exp\left(\int^{t'}c_1(t'')\,dt''\right)c_2(t')\,dt'\right]$$

Another well known example is the Riccati equation, i.e. the following nonlinear differential equation:

$$\dot{x}=c_0(t)+c_1(t)\,x+c_2(t)\,x^2\,. \tag{10}$$

This type of equations does not admit integration by quadratures in the general case. Their solutions are the integral curves of the t-dependent vector field

$$X_t = c_0(t)X_0 + c_1(t)X_1 + c_2(t)X_2,$$

which is a linear combination with t-dependent coefficients of the vector fields

$$X_0 = \partial_x\,, \quad X_1 = x\,\partial_x\,, \quad X_2 = x^2\,\partial_x\,,$$

that close on a three-dimensional real Lie algebra, with defining relations

$$[X_0, X_1] = X_0\,, \quad [X_0, X_2] = 2X_1\,, \quad [X_1, X_2] = X_2\,,$$

isomorphic to the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra. Therefore it is a Lie system admitting a superposition rule which turns out to be [19]

$$\phi(x_1, x_2, x_3; k) = \frac{k\,x_1(x_3 - x_2) + x_2(x_1 - x_3)}{k\,(x_3 - x_2) + (x_1 - x_3)}\ . \tag{11}$$

The vector fields X_0, X_1 and X_2 are a basis of the fundamental vector fields relative to the action of the group $SL(2,\mathbb{R})$ on $\overline{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$:

$$\Phi(A,x) = \begin{cases} \dfrac{\alpha x+\beta}{\gamma x+\delta}\,, & \text{if } x \notin \{-\delta/\gamma, \infty\}, \\ \alpha/\gamma\,, & \text{if } x = \infty, \\ \infty\,, & \text{if } x = -\delta/\gamma, \end{cases} \quad \text{with } A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R}).$$

A Riccati equation can be seen as a curve in $\mathbb{R}^3$ and each element $A(t)$ of the group $\mathcal{G}$ of smooth $SL(2,\mathbb{R})$-valued curves, $\mathrm{Map}(\mathbb{R}, SL(2,\mathbb{R}))$, transforms every curve $x(t)$ in $\bar{\mathbb{R}}$ into a new one $\bar{x}(t) = \Phi(A(t), x(t))$ satisfying a new Riccati equation with coefficients $\bar{c}_2, \bar{c}_1, \bar{c}_0$:

$$\begin{aligned} \bar{c}_2 &= \delta^2\,c_2 - \delta\gamma\,c_1 + \gamma^2\,c_0 + \gamma\dot{\delta} - \delta\dot{\gamma}\ , \\ \bar{c}_1 &= -2\,\beta\delta\,c_2 + (\alpha\delta + \beta\gamma)\,c_1 - 2\,\alpha\gamma\,c_0 + \delta\dot{\alpha} - \alpha\dot{\delta} + \beta\dot{\gamma} - \gamma\dot{\beta}\ , \\ \bar{c}_0 &= \beta^2\,c_2 - \alpha\beta\,c_1 + \alpha^2\,c_0 + \alpha\dot{\beta} - \beta\dot{\alpha}\ . \end{aligned}$$

These expressions define an affine action of the group $\mathcal{G}$ on the set of Riccati equations. An appropriate choice for the curve, $A(t)$, transforms the original Riccati equation into a simpler one, for instance an integrable by quadratures one. The conditions for the existence of such a transformation provide us with integrability conditions [20]. Moreover, similar properties hold for any other Lie system.

3 Geometric approach to the Abel equation of the first kind

Each t-dependent vector field whose integral curves are the solutions of an Abel equation is a linear combination with t-dependent coefficients of the vector fields in the linear space $V_{\mathrm{Abel}}(\mathbb{R})$ defined by

$$V_{\mathrm{Abel}}(\mathbb{R}) = \left\langle \partial_x, x\partial_x, x^2\partial_x, x^3\partial_x \right\rangle .$$

Such a t-dependent vector field is not, in general, a Lie system. Indeed, consider a t-dependent vector field $X(t,x) = x^2\partial_x + h(t)x^3\partial_x$, with a non-constant function

h. If $X(t,x)$ determines a Lie system, there must exist a finite-dimensional Lie algebra $V_0(\mathbb{R})$ including the vector fields $x^2\partial_x$ and $x^3\partial_x$ such that $X \in V_0(C^\infty(\mathbb{R}))$. Nevertheless, this is impossible as such a Lie algebra should contain the successive Lie brackets of $x^2\partial_x$ and $x^3\partial_x$, which span an infinite family of linearly independent vector fields (note that $[x^2\partial_x, x^n\partial_x] = (n-2)x^{n+1}\partial_x$).

As pointed out in a recent paper [7], we can however look for a linear subspace $W(\mathbb{R}) \subset V_{\rm Abel}(\mathbb{R})$ such that $W(\mathbb{R})$ is a Lie algebra satisfying that $[W(\mathbb{R}), V_{\rm Abel}(\mathbb{R})] \subset V_{\rm Abel}(\mathbb{R})$. In such a case the flows of time-dependent vector fields with values in $W(\mathbb{R})$ leave invariant the linear space $V_{\rm Abel}(\mathbb{R})$ and starting with a given Abel equation we obtain a new Abel equation from each flow. Maybe, by an appropriate flow, the transformed Abel equation becomes a Lie system. In this case the initial Abel equations is called a quasi-Lie system.

Our aim is, given $Y = f(x)\,\partial_x \in \mathfrak{X}(\mathbb{R})$, to determine f such that $[Y, V_{Abel}(\mathbb{R})] \subset V_{Abel}(\mathbb{R})$. In particular,

$$[Y, \partial_x] = -f'(x)\partial_x \in V_{\rm Abel}(\mathbb{R}).$$

Hence, $-f'(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ and $f(x) = -\frac{c_3}{4}x^4 - \frac{c_2}{3}x^3 - \frac{c_1}{2}x^2 + c_0x + c_{-1}$. Using now the condition $[Y, x\,\partial_x] = (f(x) - xf'(x))\partial_x \in V_{\rm Abel}(\mathbb{R})$, we obtain that $c_3 = 0$, and $\left[Y, x^2\partial_x\right] = (2xf(x) - x^2f'(x))\partial_x \in V_{\rm Abel}(\mathbb{R})$ yields that $c_2 = 0$. Moreover, as Y must satisfy that $\left[Y, x^3\,\partial_x\right] = (3x^2f(x) - x^3f'(x))\partial_x \in V_{\rm Abel}(\mathbb{R})$, we also see that $c_1 = 0$. Consequently, every symmetry group of $V_{\rm Abel}(\mathbb{R})$ admits a Lie algebra of fundamental vector fields W_0 contained in the Lie algebra

$$W_{\rm Abel} = \langle \partial_x, x\partial_x \rangle\,.$$

We find in this way the so-called structure invariance group [21], which turns out to be the affine group in one dimension.

Correspondingly, the set of first-order Abel equations of the first kind is invariant under all the following transformations [22]

$$\bar{x}(t) = \alpha(t)\,x(t) + \beta(t), \qquad \alpha(t) \neq 0, \tag{12}$$

Under such a transformation the given Abel equation (1) becomes a new Abel equation $\dot{\bar{x}} = \bar{A}_0(t) + \bar{A}_1(t)\,\bar{x} + \bar{A}_2(t)\,\bar{x}^2 + \bar{A}_3(t)\,\bar{x}^3$, with

$$\begin{aligned}
&\bar{A}_3(t) = A_3(t)\alpha^2(t),\\
&\bar{A}_2(t) = \alpha(t)(3A_3(t)\beta(t) + A_2(t)),\\
&\bar{A}_1(t) = 3A_3(t)\beta^2(t) + 2A_2(t)\beta(t) + A_1(t) - \dot{\alpha}(t)\alpha^{-1}(t),\\
&\bar{A}_0(t) = \alpha^{-1}(t)\left(A_3(t)\beta^3(t) + A_2(t)\beta^2(t) + A_1(t)\beta(t) + A_0(t) - \dot{\beta}(t)\right).
\end{aligned} \tag{13}$$

This is a very useful property: the action of the group of curves in the affine group produces orbits, i.e. equivalence classes of Abel equations, with the same properties of integrability or existence of superposition rules. For instance, Abel equations of an orbit containing an integrable by quadratures Riccati equation are also integrable by quadratures. The orbits are characterized by different values of invariant functions.

One must determine the Lie algebras contained in $V_{\rm Abel}$ reachable by the set of transformations (the coefficient of X_3 cannot be transformed to be zero, see first equation in (13)). It is possible to check that the only subalgebra of dimension three is the one of Riccati equation which is not reachable by the considered set of transformations. On the other hand, there is a one-parameter family of two-dimensional subalgebras of $V_{\rm Abel}$ reachable from a proper Abel equation, those generated by

$$(-2\mu^3+3\mu x^2+x^3)\partial_x, \qquad (\mu+x)\partial_x. \tag{14}$$

These are integrable by two quadratures. For instance the case $\mu=0$ corresponds to Bernoulli equation. Finally, the algebras generated by vector fields

$$(c_0+c_1x+c_2x^2+x^3)\partial_x, \tag{15}$$

lead to separable differential equations, therefore integrable by one quadrature.

4 Canonical forms of Abel equations of the first kind

Starting from an Abel equation of the first kind with $A_3\neq 0$, we can define a transformation $\bar x = x+\frac{1}{3}A_2(t)/A_3(t)$, which reduces the original equation to a new one $\dot{\bar x}=\bar A_0(t)+\bar A_1(t)\,\bar x+\bar A_3(t)\,\bar x^3$, where

$$\bar A_0 = A_0-\frac{A_1A_2}{3A_3}+\frac{2}{27}\frac{A_2^3}{A_3^2}+\frac{1}{3}\frac{d}{dt}\left(\frac{A_2}{A_3}\right), \quad \bar A_1=A_1-\frac{1}{2}\frac{A_2^2}{A_3}, \quad \bar A_2=0, \quad \bar A_3=A_3.$$

In fact, a translation $\bar x = x+\beta$ transforms the given equation into:

$$\dot{\bar x}=\dot\beta+A_3(\bar x-a)^3+A_2(\bar x-a)^2+A_1(\bar x-a)+A_0,$$

and then

$$\dot{\bar x}=A_3\bar x^3+(A_2-3\beta A_3)\bar x^2+(3\beta^2A_3-2\beta A_2+A_1)\bar x+\dot\beta-\beta^3A_3+\beta^2A_2-\beta A_1+A_0.$$

Choosing $\beta=\dfrac{A_2(t)}{3\,A_3(t)}$ we find that $\bar A_3=A_3$, $\bar A_2=0$ and $\bar A_1$ and $\bar A_0$ are given by the above given expressions. When $\bar A_0=0$, we obtain a first possibility for the canonical form

$$\dot x = A_1(t)\,x+A_3(t)\,x^3,$$

which is a Bernoulli equation for $n=3$, and therefore solvable by the change of variable $u=1/x^2$, which leads to the inhomogeneous linear equation

$$\dot u+2A_1u+A_3=0\,.$$

On the contrary, if $\bar A_0\neq 0$, under the transformation $\bar x=\bar{\bar x}\,\bar A_0$, the equation becomes $\bar{\bar x}'=\bar{\bar A}_3(t)\bar{\bar x}^3+\bar{\bar A}_1(x)\,y+1$, and we obtain as a canonical form

$$\dot x = 1+A_1(t)\,x+A_3(t)\,x^3. \tag{16}$$

An important instance is the case $A_0=A_1=0$: the canonical forms are such that

$$\bar A_1=-\frac{1}{3}\frac{A_2^2}{A_3}, \qquad \bar A_0=\frac{2}{27}\frac{A_2^3}{A_3^2}+\frac{1}{3}\frac{d}{dt}\left(\frac{A_2}{A_3}\right).$$

Liouville proved that if Φ_3 and Φ_5 are given by

$$\begin{aligned}\Phi_3 &= A_2\,A_3'-A_2'\,A_3+3\,A_0\,A_3^2-A_1\,A_2\,A_3+\tfrac{2}{9}A_2^3\\ \Phi_5 &= A_3\,\Phi_3'-3\left(A_3'+\tfrac{1}{3}A_2^2-A_1\,A_3\right)\end{aligned}$$

then the quotient Φ_3^5/Φ_5^3 is an invariant under the structure invariance group. The first canonical form corresponds to $\Phi_3 = 0$. Any two equations with $\Phi_3 = 0$ are in the same orbit, i.e. are related by the structure invariance group.

For equations with the second canonical form there is a new invariant, and they are in the same orbit if and only if the value of the invariant is the same for both equations.

There exist a two-parameter structure invariance group for this type of equations in the case of first canonical form and a one-parameter one in the cases of the second form provided that an invariant has a constant value different from zero.

5 Second order Abel equation

Lie's recipe for order reduction of second-order linear differential equations leads to some Riccati equations. For instance $\ddot{y} = 0$ under the change of variable $y = e^z$ becomes $\mathbb{D}x = 0$ with $\mathbb{D} = d/dt + x$, where $x = \dot{z}$. Similarly $\dddot{y} = 0$ becomes under the same change of variable $\mathbb{D}^2 x = 0$. On the other hand, when comparing Abel equation with Riccati equation one sees that a possible generalization to second-order equation of Abel equation is $\mathbb{D}_A^2 x = 0$, where $\mathbb{D}_A = d/dt + x^2$ [12]. The second-order Abel equation (5) can be presented as a system of two first-order equations

$$\begin{cases} \dfrac{dx}{dt} = v, \\ \dfrac{dv}{dt} = -4x^2 v - x^5, \end{cases}$$

corresponding to the following vector field on the velocity phase space $\mathbb{R}^2$

$$\Gamma = v\,\partial_x + F_A\,\partial_v\,, \qquad F_A = -4x^2 v - x^5.$$

Such a nonlinear Abel equation of second-order can be derived from the Lagrangian

$$L_A(x,v) = \frac{1}{(v+x^3)^2}\,. \tag{17}$$

The corresponding conserved energy function is given by

$$E_{L_A} = -\,\frac{(3v+x^3)}{(v+x^3)^3}\,.$$

To be remarked that there exists an alternative Lagrangian given by:

$$\widetilde{L}_A(x,v) = (3v+x^3)^{2/3}. \tag{18}$$

Such Lagrangians can be obtained by means of the Jacobi last multiplier theory and Darboux polynomials for polynomial vector fields. More specifically, Darboux polynomials for a polynomial vector field X are polynomials $\mathcal{D}$ such that $X\mathcal{D} = f\mathcal{D}$. The function f is said to be the cofactor corresponding to such a Darboux polynomial and the pair $(f,\mathcal{D})$ is called a Darboux pair.

On the other side, given a vector field X in an oriented manifold (M,Ω), a function R such that $R\,i(X)\Omega$ is closed is said to be a Jacobi multiplier (JM) for X

[23]. Recall that the divergence of the vector field X (with respect to the volume form Ω) is defined by the relation

$$\mathcal{L}_X \Omega = (\operatorname{div} X)\, \Omega\,.$$

Then, R is a Jacobi multiplier if and only if $R\,X$ is a divergenceless vector field and therefore using

$$\mathcal{L}_{RX} \Omega = (\operatorname{div} RX)\, \Omega = [X(R) + R \operatorname{div} X]\, \Omega = 0\,,$$

we see that R is a last multiplier for X if and only if

$$X(R) + R \operatorname{div} X = 0\,. \tag{19}$$

because, for any function f,

$$X(f\,R) + f\,R \operatorname{div} X = (Xf)\,R + f(X(R) + R \operatorname{div} X).$$

The remarkable point is that if $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are Darboux polynomials with corresponding cofactors f_i, with $i = 1, \ldots, k$, one can look for multiplier factors of the form

$$R = \prod_{i=1}^{k} \mathcal{D}_i^{\nu_i}, \tag{20}$$

and then

$$\frac{X(R)}{R} = \sum_{i=1}^{k} \nu_i \frac{X(\mathcal{D}_i)}{\mathcal{D}_i} = \sum_{i=1}^{k} \nu_i\, f_i.$$

Consequently, if the coefficients ν_i can be chosen such that

$$\sum_{i=1}^{k} \nu_i\, f_i = -\operatorname{div} X, \tag{21}$$

we arrive to

$$\frac{X(R)}{R} = \sum_{i=1}^{k} \nu_i\, f_i = -\operatorname{div} X,$$

what implies that R is a Jacobi last multiplier for X.

Finally, one can prove (see e.g. [23]) that if R is a Jacobi multiplier for a vector field which corresponds to a second-order differential equation, there is an essentially unique Lagrangian L (up to addition of a gauge term) such that $R = \partial^2 L/\partial v^2$.

Actually, Helmholtz analysed the set of conditions that a multiplier matrix $g_{ij}(x, \dot{x})$ must satisfy in order for a given system of second-order equations

$$\ddot{x}^j = F^j(x, \dot{x})\,, \quad j = 1, 2, \ldots, n,$$

or the corresponding system

$$\begin{cases} \dfrac{dx^i}{dt} = v^i, \\ \dfrac{dv^i}{dt} = F^i(x, v), \end{cases}$$

which provides us the integral curves for the vector field

$$X = v^i \partial_{x^i} + F^i(x, v)\partial_{v^i} \,, \tag{22}$$

when written of the form

$$g_{ij}(\ddot{x}^j - F^j(x, \dot{x})) = 0 \,, \quad i, j = 1, 2, \ldots, n,$$

to be the set of Euler–Lagrange equations for a certain Lagrangian L.

Each matrix solution g_{ij} can be identified with the Hessian matrix of L, $g_{ij} = \partial L/\partial v^i \partial v^j$, and then L can be obtained by direct integration of the g_{ij} functions.

The two first conditions just impose regularity and symmetry of the matrix g_{ij}; the two other equations introduce relations between the derivatives of g_{ij} and the derivatives of the functions F^i. The fourth set of conditions for g_{ij} is

$$X(g_{ij}) = g_{ik} A^k{}_j + g_{jk} A^k{}_i \,, \quad A^i{}_j = -\frac{1}{2}\partial_{v^j} F^i \,.$$

When the system is one-dimensional we have $i = j = k = 1$ and then the three first set of conditions become trivial and the fourth one reduces to one single P.D.E.

$$X(g) + g\,\frac{\partial F}{\partial v} \equiv v\,\frac{\partial g}{\partial x} + F\,\frac{\partial g}{\partial v} + g\,\frac{\partial F}{\partial v} = 0 \,,$$

which is the equation defining the Jacobi multipliers, because $\operatorname{div} X = \partial_v F$.

Then, the inverse problem reduces to find the function g which is a Jacobi multiplier and L is obtained by integrating the function g two times with respect to velocities. The function L so obtained is unique up to addition of a gauge term.

Coming back to the second-order Abel equation case, one can easily check that $\mathcal{D}_1(x, v) = v + x^3$ is a Darboux polynomial for Γ with cofactor $-x^2$ since

$$\left(v\,\frac{\partial}{\partial x} + F_A\,\frac{\partial}{\partial v}\right)(v + x^3) = -x^2(v + x^3) \,.$$

The divergence of the vector field Γ is $-4x^2$, and then we see that there is a Jacobi multiplier of the form $R = \mathcal{D}_1^{-4}$. Consequently, the Abel equation admits a Lagrangian description by means of a function L such that

$$\frac{\partial^2 L}{\partial v^2} = (v + x^3)^{-4} \,,$$

from where we obtain the Lagrangian $L = L_A$ given by (17).

But $\mathcal{D}_2(x, v) = 3v + x^3$ is also a Darboux polynomial for Γ with cofactor $-3x^2$,

$$\left(v\,\frac{\partial}{\partial x} + F_A\,\frac{\partial}{\partial v}\right)(3v + x^3) = 3x^2 v - 3(4x^2 v + x^5) = -3x^2(3v + x^3) \,,$$

and then we can find another Jacobi multiplier of the form $\mathcal{D}_2^{\nu_2}$ with $\nu_2 = -4/3$. The Abel equation admits a Lagrangian description by means of a function L such that

$$\frac{\partial^2 L}{\partial v^2} = (3v + x^3)^{-4/3} \,,$$

from where we obtain the Lagrangian $L = \widetilde{L}_A$ given by (18).

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