NONASSOCIATIVE STRICT DEFORMATION QUANTIZATION OF C^*-ALGEBRAS AND NONASSOCIATIVE TORUS BUNDLES

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ABSTRACT. In this paper, we initiate the study of nonassociative strict deformation quantization of C^*-algebras with a torus action. We shall also present a definition of nonassociative principal torus bundles, and give a classification of these as nonassociative strict deformation quantization of ordinary principal torus bundles. We then relate this to T-duality of principal torus bundles with H-flux. We also show that the Octonions fit nicely into our theory.

INTRODUCTION

We shall present a definition of nonassociative principal torus bundles, inspired in part by the work on noncommutative principal torus bundles [9, 10, 11], and by our earlier work with Bouwknegt, [5, 6], on nonassociative T-duals of spacetimes that are principal torus bundles. In the previous work (with Bouwknegt), we were unable to give an independent description of the nonassociative T-duals, however here, we achieve such a description in many interesting special cases via nonassociative strict deformation quantization initiated in this paper.

Associative operator theoretic deformation quantization was studied in detail by Rieffel [22] (see the references therein), who called it *strict* deformation quantization, mainly to distinguish it from *formal* deformation quantization. Formal nonassociative deformation quantization has been previously studied in the mathematical physics literature, cf. [8, 12, 20, 21], but as far as we know, ours is the first paper to analyse *strict* nonassociative deformation quantization via nonassociative operator algebras or C^*-algebras in tensor categories [5, 6]. More precisely, we define and study nonassociative strict deformation quantization of C^*-algebras with a torus action, and apply it to show that some of the T-duals of spacetimes with H-flux that are principal torus bundles (of rank ≥ 3), are nonassociative strict deformation quantization of principal torus bundles (with basic H-flux), thereby giving a satisfactory description of the T-dual at least in these cases.

In [5, 6] we showed how a natural definition of T-duals in terms of crossed products could lead to nonassociative algebras. For a locally compact group G and a Moore 3-cocycle $\phi \in Z^3(G, \mathbf{T})$ (or more conveniently an antisymmetric tricharacter [5]), the twisted compact operators $\mathcal{K}_\phi(L^2(G))$ provide a simple example of such an algebra: one starts gives the integral kernels K_j defining compact operators a deformed product

$$(K_1 \star K_2)(x, z) = \int_G \phi(x, y, z) K_1(x, y) K_2(y, z) \, dy.$$

More generally, nonassociativity can occur for an algebra $\mathcal{A}$ whenever $C_0(G)$ acts on $\mathcal{A}$, so that $\phi \in C_0(G \times G \times G)$ acts on $(\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A}$. Using this to identify $(\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A}$ with $\mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A})$ leads to non-associativity in products of three elements in $\mathcal{A}$. (We shall refer to this as ϕ-nonassocitivity.) In the case of the kernels we can take the action of $f \in C_0(G)$ to be $(f.K)(z, w) = f(zw^{-1})K(z, w)$, [5,

Section 10]. An action β (of G) by automorphisms of $\mathcal{A}$ is consistent with this if $\beta_x[f.a] = \rho_x[f].\beta_x[a]$ where $\rho_x[f](y) = f(xy)$.

One of the aims of this paper is to show that the nonassociative bundles of [5] have a characterisation analogous to the noncommutative principal bundles of [9, 11]. In fact, we shall see that this automatically allows for noncommutativity as well.

The paper starts with a section showing that nonassociative torus algebras can be viewed as nonassociative strict deformation quantizations of their associative counterparts, and Section 2 shows the relation to T-duality, which is a fundamental symmetry in string theory. In Section 3 we introduce a definition of nonassociative principal torus bundles and show in the next Section that the bundles encountered in [5] are examples of such bundles. This involves an interesting new variant of the Takai duality Theorem, which combines duality with MacLane's notion of strictification to strictly associative categories. The nonassociative principal torus bundles are then classified in Section 5 by a minor modification of the classification of noncommutative principal torus bundles [9, 11]. Finally, in the appendix we show that the Octonions (stabilized) can also be recovered as a nonassociative strict deformation quantization.

1. NONASSOCIATIVE TORUS AND NONASSOCIATIVE STRICT DEFORMATION QUANTIZATION

Let T be a torus, and $\widehat{T}$ denote the Pontryagin dual of T. Let $\phi \in Z^3(\widehat{T}, \mathbf{T})$ be a $\mathbf{T}$-valued 3-cocycle on $\widehat{T}$.

Following the constructions of [5, Section 4], consider the unitary operator $u(\beta, \gamma)$ acting on $L^2(\widehat{T})$ given by

$$(u(\beta, \gamma)\psi)(\alpha) = \phi(\alpha, \beta, \gamma)\psi(\alpha)$$

for all $\psi \in L^2(\widehat{T})$. We easily verify that

$$\phi(\alpha, \beta, \gamma)u(\alpha, \beta)u_t(\alpha\beta, \gamma) = \xi_\alpha[u(\beta, \gamma)]u(\alpha, \beta\gamma)$$

where $\xi_\alpha = \mathrm{ad}(\rho(\alpha))$ and $(\rho(\alpha)\psi)(g) = \psi(g\alpha)$ is the right regular representation. One can define, as was done in [5, 6], a twisted convolution product and adjoint on $C(\widehat{T}, \mathcal{K})$, where $\mathcal{K} = \mathcal{K}(L^2(\widehat{T}))$, by

$$(1) \qquad (f *_\phi g)(x) = \sum_{y \in \Gamma} f(y)\xi_y[g(y^{-1}x)]u(y, y^{-1}x),$$

and

$$(2) \qquad f^*(x) = u(x, x^{-1})^{-1}\xi_x[f(x^{-1})]^*.$$

The operator norm completion is the nonassociative twisted crossed product C*-algebra known as the *nonassociative torus*,

$$C^*(\mathcal{K}, \widehat{T}, \phi) = \mathcal{K}(L^2(\widehat{T})) \rtimes_{\xi, u} \widehat{T}.$$

When ϕ is trivial, the nonassociative torus specializes to the continuous functions on the torus T, stabilized.

Our next goal is to extend this construction to general C*-algebras with a T-action. Let $\mathcal{A}$ be a C*-algebra with a continuous action α of T and $\phi \in Z^3(\widehat{T}, \mathbf{T})$ be a $\mathbf{T}$-valued 3-cocycle on $\widehat{T}$. Then we define the *nonassociative strict deformation quantization* of $\mathcal{A}$, denoted $\mathcal{A}_\phi$ as follows.

We have the direct sum decomposition,

$$\begin{aligned} \mathcal{A} \otimes \mathcal{K} &\cong \widehat{\bigoplus}_{\chi \in \hat{T}} \mathcal{A}_\chi \otimes \mathcal{K} \\ a &= \sum_{\chi \in \hat{T}} a_\chi, \end{aligned}$$

where $\mathcal{K} = \mathcal{K}(L^2(\widehat{T}))$ and for $\chi \in \widehat{T}$,

$$\mathcal{A}_\chi := \{a \in \mathcal{A} \mid \alpha_t(a) = \chi(t) \cdot a \quad \forall t \in T\}.$$

Since T acts by $\star$-automorphisms, we have

$$\mathcal{A}_\chi \cdot \mathcal{A}_\eta \subseteq \mathcal{A}_{\chi\eta} \quad \text{and} \quad \mathcal{A}_\chi^* = \mathcal{A}_{\chi^{-1}} \qquad \forall \chi, \eta \in \widehat{T}.$$

The completion of the direct sum is explained as follows.

The representation theory of T shows that $\bigoplus_{\chi \in \widehat{T}} \mathcal{A}_\chi$ is a T-equivariant dense subspace of $\mathcal{A}$, where T acts on $\mathcal{A}_\chi$ as follows: $\hat{\alpha}_t(a_\chi) = \chi(t) a_\chi$ for all $t \in T$. Then $\widehat{\bigoplus}_{\chi \in \hat{T}} \mathcal{A}_\chi$ is the completion in the C^*-norm of $\mathcal{A}$. Let

$$(\mathcal{A} \otimes \mathcal{K})^{alg} = \bigoplus_{\chi \in \hat{T}} \mathcal{A}_\chi \otimes \mathcal{K}.$$

The product on $\mathcal{A} \otimes \mathcal{K}$ then also decomposes as,

$$(ab)_\chi = \sum_{\chi_1 \chi_2 = \chi} a_{\chi_1} b_{\chi_2}$$

for $\chi_1, \chi_2, \chi \in \widehat{T}$. The product can be deformed to a nonassociative product $\star_\phi$ on $\mathcal{A} \otimes \mathcal{K}$ by setting

$$(a \star_\phi b)_\chi = \sum_{\chi_1 \chi_2 = \chi} a_{\chi_1} \xi_{\chi_1} [b_{\chi_2}] u(\chi_1, \chi_2).$$

We next describe the norm completion aspects in the deformed case. Let $\mathcal{H}_1$ denote the universal Hilbert space representation of the C^*-algebra $\mathcal{A} \otimes \mathcal{K}$ (with its usual product) which one obtains via the GNS theorem. By considering instead the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes L^2(T) \otimes \mathcal{H}_2$, where $\mathcal{H}_2$ is an infinite dimensional Hilbert space, where we note that every character of T occurs with infinite multiplicity in $L^2(T) \otimes \mathcal{H}_2$, we obtain a T-equivariant embedding $\varpi : \mathcal{A} \otimes \mathcal{K} \to B(\mathcal{H})$. The equivariance means that

$$\varpi(\varphi_\chi) = \varpi(\varphi)_\chi.$$

Now consider the action of $(\mathcal{A} \otimes \mathcal{K})^{alg}$ on $\mathcal{H}$ given by the deformed product $\star_\sigma$, that is, for $\varphi \in (\mathcal{A} \otimes \mathcal{K})^{alg}$ and $\Psi \in \mathcal{H}$,

$$(\varphi \star_\phi \Psi)_\chi = \sum_{\chi_1 \chi_2 = \chi} \varpi(\varphi_{\chi_1}) \xi_{\chi_1} [\Psi_{\chi_2}] u(\chi_1, \chi_2).$$

The operator norm completion of this action is the nonassociative strict deformation quantization $\mathcal{A}_\phi$ of $\mathcal{A}$, and is a C^*-algebra in a tensor category with associator equal to ϕ, see [6].

We might ask whether this can be understood using the Landstad–Kasprzak approach, as in [11]. However, if we followed precisely the route in [5], then a nonassociative algebra $\mathcal{A}^u$ would have a nonassociative crossed product $\mathcal{B}$ and we could not possibly have $\mathcal{A}^u \rtimes_{\alpha^u} G \cong \mathcal{B} \cong \mathcal{A} \rtimes_\alpha G$. Of course, the constructions of [5, Appendix A], which we discuss below, allow us to strictify the product, removing the nonassociativity. It does seem likely that a more sophisticated approach along those lines could work, but we leave that for future investigation.

We should also note that this notion of deformation is not quite the usual one since the condition that the restriction of ϕ to $\Lambda^\perp \times \Lambda^\perp \times \Lambda^\perp$ is trivial builds in a certain rigidity.
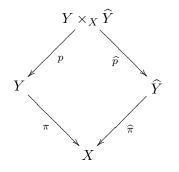
2. The relation to T-duality

We begin by summarizing T-duality for principal torus bundles with H-flux. T-duality, also known as target space duality, plays an important role in string theory and has been the subject of intense study for many years. In its most basic form, T-duality relates a string theory compactified on a circle of radius R, to a string theory compactified on the dual circle of radius $1/R$ by the interchange of the string momentum and winding numbers. T-duality can be generalized to locally defined circles (principal circle bundles, circle fibrations), higher rank torus bundles or fibrations,

and in the presence of a background H-flux which is represented by a closed, integral Čech 3-cocycle H on the spacetime manifold Y. It is closely related to mirror symmetry through the SYZ-conjecture, [23].

A striking feature of T-duality is that it can relate topologically distinct spacetime manifolds by the interchange of topological characteristic classes with components of the H-flux. Specifically, denoting by $(Y, [H])$ the pair of an (isomorphism class of) principal circle bundle $\pi : Y \to X$, characterized by the first Chern class $[F] \in H^2(X, \mathbf{Z})$ of its associated line bundle, and an H-flux $[H] \in H^3(Y, \mathbf{Z})$, the T-dual again turns out to be a pair $(\widehat{Y}, [\widehat{H}])$, where the principal circle bundles

$$
\begin{array}{ccc}
\mathbf{T} \longrightarrow Y \; , & \qquad\qquad & \mathbf{T} \longrightarrow \widehat{Y} \\
\quad\downarrow{\scriptstyle \pi} & & \quad\downarrow{\scriptstyle \widehat{\pi}} \\
X & & X
\end{array}
$$

are related by $[\widehat{F}] = \pi_*[H]$, $[F] = \widehat{\pi}_*[\widehat{H}]$, such that on the correspondence space

$$
\begin{array}{ccc}
 & Y \times_X \widehat{Y} & \\
 & \swarrow{\scriptstyle p} \qquad \searrow{\scriptstyle \widehat{p}} & \\
Y & & \widehat{Y} \\
 \searrow{\scriptstyle \pi} & & \swarrow{\scriptstyle \widehat{\pi}} \\
 & X &
\end{array}
$$

we have $p^*[H] - \widehat{p}^*[\widehat{H}] = 0$ [3, 4].

In earlier papers we have argued that the twisted K-theory $K^\bullet(Y, [H])$ (see, e.g., [2]) classifies charges of D-branes on Y in the background of H-flux $[H]$ [7], and indeed, as a consistency check, one can prove that T-duality gives an isomorphism of twisted K-theory (and the closely related twisted cohomology $H^\bullet(Y, [H])$ by means of the twisted Chern character ch_H) [3]

$$
\begin{array}{ccc}
K^\bullet(Y, [H]) & \xrightarrow{\;T_!\;} & K^{\bullet+1}(\widehat{Y}, [\widehat{H}]) \\
\downarrow{\scriptstyle ch_H} & & \downarrow{\scriptstyle ch_{\widehat{H}}} \\
H^\bullet(Y, [H]) & \xrightarrow{\;T_*\;} & H^{\bullet+1}(\widehat{Y}, [\widehat{H}])
\end{array}
$$

The above considerations were generalized to principal torus bundles in [5, 6].

Since the projective unitary group of an infinite dimensional Hilbert space $\mathsf{PU}(\mathcal{H})$ is a model for $K(\mathbf{Z}, 2)$, we can 'geometrize' the H-flux in terms of an (isomorphism class of) principal $\mathsf{PU}(\mathcal{H})$-bundle P over Y. We can reformulate the discussion of T-duality above in terms of noncommutative geometry as follows. The space of continuous sections vanishing at infinity, $\mathcal{A} = C_0(Y, \mathcal{E})$, of the associated algebra bundle of compact operator $\mathcal{K}$ on the Hilbert space $\mathcal{E} = P \times_{\mathsf{PU}(\mathcal{H})} \mathcal{K}$. $\mathcal{A}$ is a stable, continuous-trace, C*-algebra with spectrum Y, and has the property that it is locally Morita equivalent to continuous functions on Y. Thus the H-flux has the effect of making spacetime noncommutative. The K-theory of $\mathcal{A}$ is just the twisted K-theory $K^\bullet(Y, [H])$. The $\mathbf{T}$-action on Y lifts essentially uniquely to an $\mathbf{R}$-action on $\mathcal{A}$. In this context T-duality is the operation of taking the crossed product $\mathcal{A} \rtimes \mathbf{R}$, which turns out to be another continuous trace algebra associated to $(\widehat{Y}, [\widehat{H}])$ as above. A fundamental property of T-duality is that when applied twice, yields the original algebra $\mathcal{A}$, and the reason that it works in this case is due to Takai duality.

The isomorphism of the D-brane charges in twisted K-theory is, in this context, due to the Connes-Thom isomorphism. These methods have been generalized to principal torus bundles in [17, 18, 19], however novel features arise. First of all the $\mathbf{T}^n$-action on the principal torus bundle Y need not always lift to an $\mathbf{R}^n$-action on $\mathcal{A}$. Even if it does, this lift need not be unique. Secondly, the crossed product $\mathcal{A} \rtimes \mathbf{R}^n$ need not be a continuous-trace algebra, but rather, it could be a continuous field of noncommutative tori [17], and necessary and sufficient conditions are given when these T-duals occur. In recent work [10, 11], we describe the T-duals of trivial torus bundles with H-flux that has no component H_0 as below, in terms of parametrised strict deformation quantization of principal torus bundles with basic H-flux.

More generally, as argued in [5], when the $\mathbf{T}^n$-action on the principal torus bundle Y does not lift to an $\mathbf{R}^n$-action on $\mathcal{A}$, one has to leave the category of C^*-algebras in order to be able to define a "twisted" lift. The associator in this case is the restriction of the H-flux H to the torus fibre of Y, and the "twisted" crossed product is defined to be the T-dual. The fibres of the T-dual are noncommutative, nonassociative tori. That this is a proper definition of T-duality is due to our results which show that the analogs of Takai duality and the Connes-Thom isomorphism hold in this context. Thus an appropriate context to describe nonassociative algebras that arise as T-duals of spacetimes with background flux, such as nonassociative tori, is that of C*-algebras in tensor categories.

Let $H_j \in H^j(X; H^{3-j}(T; \mathbf{Z})) \subset H^3(X \times T; \mathbf{Z})$ for $j = 0, 1, 2, 3$. Here we prove the result,

Theorem 2.1. *In the notation above, $(X \times T, H_0 + H_1 + H_2 + H_3)$ and the parametrised strict deformation quantization of $(Y, q^*(H_3))$ with deformation parameter ϕ_1, $[\phi_1] = H_1$, further deformed nonassociatively with deformation 3-cocycle ϕ, $[\phi] = H_0$, are T-dual pairs, where the 1st Chern class $c_1(Y) = H_2$.*

Proof. By [10, 11] we know that

$$\mathrm{CT}(Y, q^*(H_3))_{\phi_1} \cong \mathrm{CT}(X \times T, H_1 + H_2 + H_3) \rtimes V.$$

Now $\mathrm{CT}(X \times T, H_0 + H_1 + H_2 + H_3) \cong \mathrm{CT}(X \times T, H_1 + H_2 + H_3) \otimes \mathrm{CT}(T, H_0)$. Also by [5, 6], we have $\mathrm{CT}(T, H_0) \rtimes V \cong A_\phi$, where we recall that $\mathrm{CT}(T, H_0) \rtimes V$ is a twisted (nonassociative) crossed product and A_ϕ denotes the nonassociative torus. Also, by §5

$$\mathrm{CT}(Y, q^*(H_3))_{\phi_1} \otimes A_\phi \cong ((\mathrm{CT}(Y, q^*(H_3))_{\phi_1})_\phi.$$

Therefore

$$((\mathrm{CT}(Y, q^*(H_3))_{\phi_1})_\phi \cong \mathrm{CT}(X \times T, H_0 + H_1 + H_2 + H_3) \rtimes V,$$

proving the result. $\square$

3. Nonassociative principal bundles

In [5] the nonassociative bundles arose as duals of associative algebras, but in this section they are the main object of interest, so we shall take $G = \widehat{V}$, where V is a vector group. The quotient of V/Λ by a maximal rank lattice Λ is a torus T, whose Pontrjagin dual $\widehat{T} \cong \Lambda^\perp \subset \widehat{V}$ is the reciprocal lattice in the dual group which is trivial on Λ.

We recall that an algebra $\mathcal{A}(X)$ over a locally compact Hausdorff space X is an algebra with a homomorphism from $C_0(X)$ to the centre of the multiplier algebra $\mathcal{M}\mathcal{A}(X)$, and, for consistency in the nonassociative case, we require this action to commute with that of $C_0(T)$.

This suggests the following definition:

Definition 3.1. An nonassociative algebra $\mathcal{A}(X)$ over X, with nonassociativity defined by ϕ is called a *nonassociative principal (T, ϕ) bundle* (or $NAP(T, \phi)$-bundle), if there is an action γ of T as

automorphisms of $\mathcal{A}(X)$ (commuting with the $C_0(X)$-action) and an isomorphism of nonassociative algebras
$$\mathcal{A}(X) \rtimes_\alpha T \cong C_0(X, \mathcal{K}_\phi),$$
for twisted compact operators on some space.

Example 3.2. This is clearly modelled on the definition of a noncommutative principal bundle in [9], to which it reduces in the associative case when $\phi \equiv 1$. As a concrete example we may consider the nonassociative torus algebra generated by unitary elements $U(\xi)$ for $\xi \in \widehat{T}$ satisfying
$$U(\xi)(U(\eta)U(\zeta)) = \phi(\xi, \eta, \zeta)(U(\xi)U(\eta))U(\zeta)$$
for all $\xi, \eta, \zeta \in \widehat{T}$. There is an obvious action of $v \in T$ action given by $v : U(\xi) \mapsto \xi(v)U(\xi)$ which preserves the above defining relation.

The definition is also motivated in part by the following observation. Recalling that $\widehat{T} \cong \Lambda^\perp \subset \widehat{V}$, [5, Theorem 8.3] tells us that, in the case discussed there, the nonassociative torus bundle is isomorphic to
$$C_0(X, \mathcal{K}_\phi(L^2(\widehat{V})) \rtimes_{\gamma, u} \Lambda^\perp = C_0(X, \mathcal{K}_\phi(L^2(\widehat{V})) \rtimes_{\gamma, u} \widehat{T},$$
where the crossed product is a Leptin–Busby-Smith crossed product ([5, Section 3]) defined by an action γ and an algebra valued cocycle u satisfying
$$
\begin{aligned}
\gamma_x \circ \gamma_y &= \mathrm{ad}(u(x,y)) \circ \gamma_{xy}, \\
\phi(x,y,z)u(x,y)u(xy,z) &= \gamma_x[u(y,z)]u(x,yz).
\end{aligned}
$$
On the other hand, such a crossed product has a dual action $\widehat{\gamma}$ of T and if we knew a suitable version of Takai duality [5, Theorem 9.2] for nonassociative algebras we should have
$$C_0(X, \mathcal{K}_\phi(L^2(\widehat{V})) \rtimes_{\gamma, u} \widehat{T} \rtimes_{\widehat{\gamma}} T \cong C_0(X, \mathcal{K}_\phi(L^2(\widehat{V})) \otimes \mathcal{K}_{\overline{\phi}}(L^2(\widehat{V}),$$
showing that the definition does match what happens in the known nonassociative case, apart, possibly, from the nonassociativity.

To investigate this last point we shall now that the multiplier u does not compound the existing nonassociativity.

Proposition 3.3. *The action γ of $\widehat{T} = \Lambda^\perp \subset \widehat{V}$ on $C_0(X, \mathcal{K}_\phi(L^2(\widehat{V})))$ defined (with the X argument suppressed) by $\gamma_\xi[K](\eta, \zeta) = \phi(\xi, \eta, \zeta)^{-1} K(\xi^{-1}\eta, \xi^{-1}\zeta)$ has a trivial associativity cocycle.*

Proof. We calculate that
$$
\begin{aligned}
(\gamma_\omega \gamma_\xi[K])(\eta, \zeta) &= \phi(\omega, \eta, \zeta)^{-1} \phi(\xi, \omega^{-1}\eta, \omega^{-1}\zeta)^{-1} K(\xi^{-1}\omega^{-1}\eta, \xi^{-1}\omega^{-1}\zeta) \\
&= \frac{\phi(\omega\xi, \eta, \zeta)}{\phi(\omega, \eta, \zeta)\phi(\xi, \omega^{-1}\eta, \omega^{-1}\zeta)} \gamma_{\omega\xi}[K](\eta, \zeta) \\
&= \phi(\xi, \omega, \zeta)\phi(\xi, \eta, \omega)\gamma_{\omega\xi}[K](\eta, \zeta) \\
&= \phi(\omega, \xi, \eta)\phi(\omega, \xi, \zeta)^{-1}\gamma_{\omega\xi}[K](\eta, \zeta) \\
&= \mathrm{ad}(u(\omega, \xi))\gamma_{\omega\xi}[K](\eta, \zeta),
\end{aligned}
$$
where $u(\omega, \xi)$ is the multiplication operator $\psi(\eta) \mapsto \phi(\omega, \xi, \eta)\psi(\eta)$ on $L^2(\widehat{V})$. Now the associativity cocycle is given by
$$
\begin{aligned}
u(\xi, \eta)u(\xi\eta, \zeta)u(\xi, \eta\zeta)^{-1}\gamma_\xi[u(\eta, \zeta)]^{-1} &= \phi(\xi, \eta, \cdot)\phi(\xi\eta, \zeta, \cdot)\phi(\xi, \eta\zeta\cdot)^{-1}\gamma_\xi[\phi(\eta, \zeta, \cdot)]^{-1} \\
&= \phi(\xi, \eta, \cdot)\phi(\xi\eta, \zeta, \cdot)\phi(\xi, \eta\zeta\cdot)^{-1}\phi(\eta, \zeta, \xi^{-1}\cdot)^{-1} \\
&= \phi(\eta, \zeta, \xi),
\end{aligned}
$$

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and for ξ, η, $\zeta \in \Lambda^{\perp}$, we have $\phi(\eta, \zeta, \xi) = 1$. $\qquad\square$

This result shows that in [5, Theorems 8.2, 8.3] the twisted crossed product by $\Lambda^{\perp} \cong \widehat{T}$ does not introduce any extra nonassociativity into
$$\mathcal{A}(X) \cong C_0(X, \mathcal{K}_\phi(L^2(\widehat{V})) \rtimes_{\gamma, u} \Lambda^{\perp}.$$

4. STRICTIFICATION AND DUALITY

In working directly from our definition our first task is to get a reverse Takai duality in which we start with $\alpha \sim \widehat{\gamma}$ and then do a twisted product so that we can get
$$\mathcal{A}(X) \cong C_0(X, \mathcal{K}_\phi(L^2(\widehat{V})) \rtimes_{\gamma, u} \widehat{T}.$$
We shall approach this by proving a new, particularly interesting duality Theorem which arises from combining the duality Theorem [5, Theorem 9.2] with the strictification procedure in [6, Appendix A].

Before stating the Theorem we recall that, according to MacLane's strictification Theorem, any monoidal category is equivalent to a strict monoidal category, in which the associativity map $A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ is the obvious identification by rebracketing: $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$. In [6] it was shown that, when the objects are $C_0(G)$-modules (or $C^*(\widehat{G})$-modules) and the associator map is given by the action of $\phi \in \mathcal{U}M(C_0(G) \times C_0(G) \times C_0(G))$, the equivalence is particularly simple, and is given by a functor taking an object A to $A \otimes C_0(G)$. This could be identified with the crossed product $A \rtimes \widehat{G}$ and it is useful to give its Fourier transformed version.

Theorem 4.1. *The multiplication for the crossed product $\mathcal{A} \rtimes \widehat{G}$ can be written in terms of $\widehat{a}(\xi) = \int a(x)\xi(x)\,dx$ as*
$$(a \star b)(x) = a_x[b(x)].$$
where the action of $a_x \in \mathcal{A} \otimes C_0(G)$ is a combination of the multiplication in $\mathcal{A}$ and the action of $C_0(G)$.

Proof.
$$
\begin{aligned}
\widehat{(a \star b)}(\xi) &= \int \widehat{a}(\eta)\alpha_\eta[\widehat{b}(\xi - \eta)]\,d\eta \\
&= \int a(y)\eta(y)\alpha_\eta[b(x)](\xi\eta^{-1})(x)\,dxdyd\eta \\
&= \int a(z + x)\eta(z)\alpha_\eta[b(x)](\xi)(x)\,dxdzd\eta \\
&= \int a_x(z)\eta(z)\alpha_\eta[b(x)](\xi)(x)\,dxdzd\eta
\end{aligned}
$$
where $a_x(z) = a(z + x)$. Now
$$(f.b)(x) = f(z)\eta(z)\alpha_\eta[b(x)]$$
links the action of $f \in C_0(G)$ with that of $\widehat{G}$, so
$$\widehat{(a \star b)}(\xi) = \int a_x(z)\eta(z)\alpha_\eta[b(x)](\xi)(x)\,dxdzd\eta = \int a_x[b(x)](\xi)(x)\,dx,$$
showing that
$$(a \star b)(x) = a_x[b(x)]. \qquad\square$$

This can be identified with the crossed product $A \rtimes \widehat{G}$, but we wish to modify the tensor product $\otimes_{C_0(G)}$ to $\circ$ so that
$$(A \circ B) \otimes_{C_0(G)} C \equiv A \otimes_{C_0(G)} (B \otimes_{C_0(G)} C).$$
(An explicit construction shows that this is possible: we decompose $\phi^{-1} \in \mathcal{UM}(C_0(G)) \otimes \mathcal{UM}(C_0(G)) \otimes \mathcal{UM}(C_0(G))$ as $\phi' \otimes \phi'' \otimes \phi'''$ and then act by $a \otimes b \mapsto [(\phi'.a) \otimes (\phi''.b)].\phi'''$. This enables us to get a functor sending $\mathcal{A} \circ \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.

Definition 4.2. Changing the multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ an algebra $\mathcal{A}$ to the multiplication $\mathcal{A} \circ \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ yields the modified crossed product $\mathcal{A} \rtimes^\phi \widehat{G}$.

(We note that this crossed product is defined for any antisymmetric tricharacter ψ, and not just for ϕ. In general it changes the nonassociativity from that defined for one tricharacter to that defined by another.

Theorem 4.3. *Let $\mathcal{A} = \mathcal{B} \rtimes_{\beta,u} G$ be a nonassociative Leptin–Busby–Smith generalised crossed product, with associator defined by ϕ. Then for the dual action $\widehat{\beta}$ of $\widehat{G}$, we have*
$$\mathcal{A} \times^{\psi\phi}_{\widehat{\beta}} \widehat{G} \cong \mathcal{B} \otimes \mathcal{K}_\psi(L^2(G)),$$
and, in particular,
$$\mathcal{A} \times^\phi_{\widehat{\beta}} \widehat{G} \cong \mathcal{B} \otimes \mathcal{K}(L^2(G)),$$
and
$$\mathcal{A} \times_{\widehat{\beta}} \widehat{G} \cong \mathcal{B} \otimes \mathcal{K}_{\overline{\phi}}(L^2(G)).$$
An ordinary crossed product $\mathcal{A} \rtimes_\alpha G$ has a natural $\widehat{G}$ action $\widehat{\alpha}$ and $(\mathcal{A} \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ with the double dual action $\widehat{\widehat{\alpha}}$ of $\widehat{\widehat{G}} \cong G$, is isomorphic to $\mathcal{A} \otimes \mathcal{K}(L^2(G))$ with the action α on the first factor and the adjoint action of the right regular representation of G on the compact operators. This is true for associative and nonassociative algebras.

Proof. We shall just give the algebraic form of the isomorphism. The technical details justifying the formal calculations involve showing that subspaces of compactly supported continuous functions in the domain and range of the isomorphism are dense, and those details are the same as in the usual case. The associativity of $\mathcal{A}$ is not relevant to the argument, which can even be generalised to the context of monoidal categories.

The first special case can be proved by noting that [5, Theorem 9.2] gives an isomorphism
$$\mathcal{A} \times^\phi_{\widehat{\beta}} \widehat{G} \cong \mathcal{B} \otimes \mathcal{K}_{\overline{\phi}}(L^2(G)),$$
and by [6, Example A.1] the strictification sends $\mathcal{K}_{\overline{\phi}}(L^2(G))$ to $\mathcal{K}_{(}L^2(G))$, so that the result follows.

In the general case we give the direct construction. We first note that for $\mathcal{A} = \mathcal{B} \rtimes G$ the elements of $\mathcal{A}$ are functions on G, with $C_0(G)$ acting by multiplication, and elements of the crossed product can be regarded as functions on $G \times G$, with multiplication
$$\begin{aligned}
(a \star b)(s, x) &= (\phi'.a)_x[(\phi''.b)(x)].\phi'''(x) \\
&= \int \phi'(t) a_x(t, \cdot) \beta_t[\phi'' b(s - t, x)].\phi'''(x) u(t, s - t)\, dt \\
&= \int \phi(t, s - t, x) a_x(t, s - t) \beta_t[b(s - t, x)] u(t, s - t)\, dt.
\end{aligned}$$
We now introduce the transform
$$a(t, x) = \beta_{t+x}[\widetilde{a}(t + x, x)] u(t, x)^{-1}$$

with inverse $\widetilde{a}(w,z) = \beta_w^{-1}[a(w-z,z)u(w-z,z)]$. Then, recalling the relation between u and ϕ and using the antisymmetry properties of ψ,

$$
\begin{aligned}
\beta_w[\widetilde{(a \star b)}(w,z)] &= (a \star b)(w-z,z)u(w-z,z) \\
&= \int (\psi\phi)(t, w-z-t, z)a_z(t, w-z-t) \\
&\qquad \beta_t[b(w-z-t,z)]u(t, w-z-t)u(w-z,z)\,dt \\
&= \int \psi(t, w-z-t, z)a(t, w-t)\beta_t[b(w-z-t,z)u(w-z-t,z)]u(t,w)\,dt \\
&= \int \psi(t, w-t, z)\beta_w[\widetilde{a}(w, w-t)]u(t,w)^{-1}\beta_t[\beta_{w-t}[\widehat{b}(w-z-t,z)]]u(t,w)\,dt \\
&= \int \psi(t, w, z)\beta_w[\widetilde{a}(w, w-t)]\beta_w[b(w-t,z)]\,dt \\
&= \int \psi(t, w, z)\beta_w[\widetilde{a}(w, v)]\beta_w[b(v,z)]\,dv
\end{aligned}
$$

showing that we have ψ-twisted multiplication of compact kernels. When $\psi \equiv 1$ we get the normal untwisted multiplication on the right, and when $\psi = \phi^{-1}$ we get an ordinary crossed product on the left. $\qquad\square$

This result does not depend on associativity of the original algebra, and enables us to confirm that the nonassociative bundles of [5] are $\mathrm{NAP}(T, \phi)$-bundles.

Corollary 4.4. *For $\mathcal{A} = C_0(X, \mathcal{K}_\phi(L^2(\widehat{V}))) \rtimes_{\gamma,u} \widehat{T}$ there is an isomorphism*

$$
\mathcal{A} \rtimes_{\widehat{\gamma}} T \cong C_0(X, \mathcal{K}_\phi(L^2(\widehat{V}))) \otimes \mathcal{K}(L^2(\widehat{T})).
$$

Proof. With $G = \widehat{T}$ and $\psi = \phi^{-1}$, we have

$$
\mathcal{A} \times_{\widehat{\gamma}} T \cong C_0(X, \mathcal{K}_\phi(L^2(\widehat{V}))) \otimes \mathcal{K}_{\overline{\phi}_\Lambda}(L^2(\widehat{T})),
$$

where ϕ_Λ is the restriction of ϕ to $\Lambda^\perp \times \Lambda^\perp \times \Lambda^\perp$ which we showed in the last section was trivial. Thus the result follows. $\qquad\square$

Corollary 4.5. *For a torus $T = V/\Lambda$ a nonassociative principal (T, ϕ) bundle $\mathcal{A}(X)$ satisfies*

$$
\mathcal{A}(X) \otimes \mathcal{K}(L^2(\widehat{V})) \cong C_0(X, \mathcal{K}_\phi(L^2(\widehat{V}))) \rtimes_{\widehat{\alpha}} \Lambda^\perp.
$$

Proof. With the appropriate change of notation (and using $\widehat{\alpha}$ for both the dual action and the equivalent action on kernels) our generalised Takai duality gives

$$
\mathcal{A}(X) \otimes \mathcal{K}(L^2(T)) \cong (\mathcal{A}(X) \rtimes_\alpha T) \rtimes_{\widehat{\alpha}} \widehat{T} \cong C_0(X, \mathcal{K}_\phi(L^2(\widehat{V}))) \rtimes_{\widehat{\alpha}} \Lambda^\perp,
$$

as asserted.

In this last case the double dual action of G is given by $\widehat{\widehat{\alpha}}_v[F](x, \xi) = \xi(v)F(x, \xi)$, which Fourier transforms to

$$
\begin{aligned}
\widetilde{\widehat{\widehat{\alpha}}_v[F]}(x, z) &= \int \xi(v)\alpha_x^{-1}[F(xz^{-1}, \xi)]\xi(z)\,d\xi \\
&= \alpha_v[\int \alpha_{xv}^{-1}[F((xv)(zv)^{-1}, \xi)]\xi(zv)\,d\xi] \\
&= \alpha_v[\widetilde{F}(xv, zv)],
\end{aligned}
$$

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which combines the action of α_v on $\mathcal{A}$ with the adjoint action of the right regular representation on kernels. □

This Corollary shows that the result of [5, Theorem 8.3] is valid for general $\text{NAP}(T, \phi)$-bundles, and not just for the dual bundles considered there.

The strictifiction duality result would enable us to give a more general notion of a nonassociative principle T-bundle by asking that there is a T-action such that the strictified crossed product with T is $C(X, \mathcal{K}(L^2(T)))$. However, this is less useful than at first appears because the nonassociativity factor is obvious from the algebra and, in any case, is not lifted from the torus T.

5. Classification of nonassociative principal torus bundles via nonassociative strict deformation quantization

The next step will be to classify $\text{NAP}(T, \phi)$-bundles more generally, at which point we shall see that they automatically include the noncommutative case of [9, 5], as well as the geometric case of principal bundles.

Theorem 5.1. *Each $\text{NAP}(T, \phi)$-bundle $\mathcal{A}$ can be parametrised by an element $\sigma \in C(X, Z^2(\widehat{T}))$ and a principal T-bundle E over X.*

Proof. The definition of an $\text{NAP}(T, \phi)$-bundle includes ϕ and the fact that there is an action, α, of T. The Corollary tells us that $\mathcal{A}(X)$ is stably equivalent to $C_0(X, \mathcal{K}_\phi(L^2(\widehat{V})) \rtimes_{\widehat{\alpha}} \Lambda^\perp$, where $\widehat{\alpha}$ is the dual action of $\Lambda^\perp \cong \widehat{T}$ action. The action of $\lambda \in \Lambda^\perp$ on the twisted compact operators is given by

$$\widehat{\alpha}_\lambda[K](\xi, \eta) = \phi(\lambda, \xi, \eta)^{-1} K(\xi\lambda, \eta\lambda)$$

and has a multiplier $u(\lambda, \mu)(\xi) = \phi(\lambda, \mu, \xi)^{-1}$, [5]. The cocycle u satisfies

$$\begin{aligned}
\widehat{\alpha}_\lambda \circ \widehat{\alpha}_\mu &= \text{ad}(u(\lambda, \mu)) \circ \widehat{\alpha}_{\lambda\mu}, \\
\phi(\lambda, \mu, \nu)u(\lambda, \mu)u(\lambda\mu, \nu) &= \widehat{\alpha}_\lambda[u(\mu, \nu)]u(\lambda, \mu\nu).
\end{aligned}$$

The freedom in the choice of u is very limited since, in the first condition, two choices u_1 and u_2 will give

$$\text{ad}(u_1(\lambda, \mu)) \circ \widehat{\alpha}_{\lambda\mu} = \widehat{\alpha}_\lambda \circ \widehat{\alpha}_\mu = \text{ad}(u_2(\lambda, \mu)) \circ \widehat{\alpha}_{\lambda\mu},$$

forcing $\sigma(\lambda, \mu) = u_1(\lambda, \mu)u_2(\lambda, \mu)^{-1}$ to be central, that is in $C_b(X)$, and so it is also $\widehat{\alpha}$-invariant. Comparison of the other condition $\phi(\lambda, \mu, \nu)u_j(\lambda, \mu)u_j(\lambda\mu, \nu) = \widehat{\alpha}_\lambda[u_j(\mu, \nu)]u_j(\lambda, \mu\nu)$, for $j = 1, 2$, then tells us that σ is an ordinary cocycle: $\sigma(\lambda, \mu)\sigma(\lambda\mu, \nu) = \sigma(\mu, \nu)\sigma(\lambda, \mu\nu)$. We then argue as in [9, Section 2] or [11] to see that the only remaining freedom is given by $H^1(X, \widehat{T})$, which is the choice of an ordinary principal T-bundle over X. □

Appendix A. Octonions as a nonassociative strict deformation quantization and nonassociative principal bundle

As noted in [5, 6, 1, 16, pp669-670], it is known that the octonions $\mathbb{O}$ can be described as a twisted group algebra of $\widehat{T} = \mathbf{Z}_2^3$, which is generated by $\{e(\mathbf{a}) : \mathbf{a} \in \mathbf{Z}_2^3\}$ so that

$$e(\mathbf{a})e(\mathbf{b}) = u(\mathbf{a}, \mathbf{b})e(\mathbf{a} + \mathbf{b}),$$

where

$$u(\mathbf{a}, \mathbf{b}) = (-1)^{\sum_{i<j} a_i b_j + a_1 a_2 b_3 + a_3 a_1 b_2 + a_2 a_3 b_1}.$$

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Thus the octonions have the form of a nonassociative strict deformation quantization, and since

$$
\begin{aligned}
(e(\mathbf{a})e(\mathbf{b}))e(\mathbf{c}) &= u(\mathbf{a},\mathbf{b})e(\mathbf{a}+\mathbf{b})e(\mathbf{c}) \\
&= u(\mathbf{a},\mathbf{b})u(\mathbf{a}+\mathbf{b},\mathbf{c})e(\mathbf{a}+\mathbf{b}+\mathbf{c}) \\
&= \frac{u(\mathbf{a},\mathbf{b})u(\mathbf{a}+\mathbf{b},\mathbf{c})}{u(\mathbf{a},\mathbf{b}+\mathbf{c})u(\mathbf{b},\mathbf{c})}e(\mathbf{a})(e(\mathbf{b})e(\mathbf{c})),
\end{aligned}
$$

the associativity factor can be calculated to be

$$
\phi(\mathbf{a},\mathbf{b},\mathbf{c}) = \frac{u(\mathbf{a},\mathbf{b}+\mathbf{c})u(\mathbf{b},\mathbf{c})}{u(\mathbf{a},\mathbf{b})u(\mathbf{a}+\mathbf{b},\mathbf{c})} = (-1)^{\mathbf{a}.(\mathbf{b}\times\mathbf{c})}.
$$

This is a finite group version of Example 3.2 with $\{U(\xi) : \xi \in \widehat{T}\}$ replaced by $\{e(\mathbf{a}) : \mathbf{a} \in \mathbf{Z}_2^3\}$, and there is an action of $\mathbf{p} \in T = \widehat{\mathbf{Z}_2^3} \cong \mathbf{Z}_2^3$ by

$$
\mathbf{p}[e(\mathbf{a})] = (-1)^{\mathbf{p}\cdot\mathbf{a}}e(\mathbf{a}).
$$

From this we can construct the twisted crossed product $\mathbb{O} \rtimes^{\phi} \mathbf{Z}_2^3$ which, using a real version of Theorem 4.3, is $\mathbf{R} \otimes \mathcal{K}(\ell^2(\mathbf{Z}_2^3))$ which reduces to the matrix algebra $M_8(\mathbf{R})$. Theorem 4.3 also tells us that the crossed product $\mathbb{O} \rtimes^{\phi} \mathbf{Z}_2^3$ is the ϕ-nonassociative matrix algebra. (See also the comment at the end of [6, Appendix 1].) This means that the octonions are an $\mathrm{NAP}(\mathbf{Z}_2^3, \phi)$-bundle over a point.

REFERENCES

[1] Baez, J.C., The octonions. Bull. Amer. Math. Soc. (N.S.) **39** (2002), 145–205

[2] Bouwknegt, P., Carey, A., Mathai, V., Murray, M., Stevenson, D.: Twisted K-theory and K-theory of bundle gerbes, Comm. Math. Phys. **228** (2002) 17-45, [`arXiv:hep-th/0106194`].

[3] Bouwknegt, P., Evslin, J., Mathai, V.: T-duality: Topology change from H-flux. Commun. Math. Phys. **249** (2004) 383-415, [`arXiv:hep-th/0306062`].

[4] Bouwknegt, P., Evslin, J., Mathai, V.: On the topology and H-flux of T-dual manifolds. Phys. Rev. Lett. **92** (2004) 181601, [`arXiv:hep-th/0312052`].

[5] Bouwknegt, P., Hannabuss, K.C. and Mathai, V.: Nonassociative tori and applications to T-duality. Comm. Math. Phys. **264** (2006) 41–69, [`arXiv:hep-th/0412092`].

[6] Bouwknegt, P., Hannabuss, K.C. and Mathai, V.: C*-algebras in tensor categories. Clay Mathematics Proceedings. **12** (2011) 127-165. [`arXiv:math.QA/0702802`]

[7] Bouwknegt, P., Mathai, V.: D-branes, B-fields and twisted K-theory. J. High Energy Phys. **03** (2000) 007, [`arXiv:hep-th/0002023`].

[8] Cornalba, L., Schiappa, R.: Nonassociative star product deformations for D-brane world-volumes in curved backgrounds. Comm. Math. Phys. **225** (2002), no. 1, 33-66.

[9] Echterhoff, S., Nest, R. and Oyono-Oyono, H.: Principal non-commutative torus bundles. Proc. London Math. Soc. (3) **99**, (2009) 1–31.

[10] Hannabuss, K.C. and Mathai, V.: Noncommutative principal torus bundles via parametrised strict deformation quantization. AMS Proceedings of Symposia in Pure Mathematics, **81** (2010) 133-148, [`arXiv:0911.1886`]

[11] Hannabuss, K.C. and Mathai, V.: Parametrised strict deformation quantization of C*-bundles and Hilbert C*-modules, 13 pages, [`arXiv:1007.4696`]

[12] Herbst, M., Kling, A., Kreuzer, M.: Cyclicity of non-associative products on D-branes. J. High Energy Phys. 2004, no.**3**, 003, 20 pp.

[13] Kasprzak, P.: Rieffel deformation via crossed products. J. Funct. Anal. **257** (2009) 1288–1332.

[14] Landstad, M.B.: Duality theory for covariant systems. Trans. Amer. Math. Soc. **248** (1979) 223–267.

[15] Landstad, M.B.: Quantization arising from abelian subgroups. Internat. J. Math. **5** (1994) 897–936.

[16] Majid, S., Foundations of quantum group theory, Cambridge University Press, Cambridge, 1995.

[17] Mathai, V., Rosenberg, J.: T-duality for torus bundles via noncommutative topology, Comm. Math. Phys. **253** (2005) 705-721, [`arXiv:hep-th/0401168`].

[18] Mathai, V., Rosenberg, J.: On mysteriously missing T-duals, H-flux and the T-duality group. In "Differential Geometry and Physics", editors Mo-Lin Ge and Weiping Zhang, Nankai Tracts in Mathematics, Volume **10**, World Scientific 2006, pages 350-358, [`arXiv:hep-th/0409073`].

[19] Mathai, V., Rosenberg, J.: T-duality for torus bundles with H-fluxes via noncommutative topology, II: the high-dimensional case and the T-duality group. Adv. Theor. Math. Phys. **10** (2006) 123-158, [arXiv:/hep-th/0508084].

[20] Nesterov, A. I., Sabinin, L. V.: Nonassociative geometry: Towards discrete structure of spacetime. Phys. Rev. 2000, D62, 081501.

[21] Paal, E.: Note on operadic non-associative deformations. J. Nonlinear Math. Phys. **13** (2006), suppl. 1, 87-92.

[22] M.A. Rieffel, M.A.: Deformation quantization for actions of $\mathbf{R}^d$. Memoirs of the Amer. Math. Soc. **106** (1993), no. 506, 93 pp.

[23] Strominger, A., Yau, S-T., Zaslow, E.: Mirror symmetry is T-duality. Nuclear Phys. B, **479** (1996), no. 1-2, 243259.

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