

On the quasi-regularity of non-sectorial Dirichlet forms by processes having the same polar sets

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Abstract. We obtain a criterion for the quasi-regularity of generalized (non-sectorial) Dirichlet forms, which extends the result of P.J. Fitzsimmons on the quasi-regularity of (sectorial) semi-Dirichlet forms. Given the right (Markov) process associated to a semi-Dirichlet form, we present sufficient conditions for a second right process to be a standard one, having the same state space. The above mentioned quasi-regularity criterion is then an application. The conditions are expressed in terms of the associated capacities, nests of compacts, polar sets, and quasi-continuity. A second application is on the quasi-regularity of the generalized Dirichlet forms obtained by perturbing with kernels a semi-Dirichlet form.

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1 Introduction

The theory of Dirichlet forms is a powerful tool in the study of Markov processes, since it combines different areas of mathematics such as probability, potential, and semigroup theory, as well as the theory of partial differential equations (see monographs [9], [14] and references therein). For instance, the classical energy calculus in combination with the potential theory of additive functionals allows to obtain an extension of Itô's formula for only weakly differentiable functions, i.e. functions in the domain of the form. This celebrated extension of Itô's formula where the martingale and the possibly unbounded variation drift part are controlled through the energy is well-known as *Fukushima's decomposition* of additive functionals (see e.g. [9, Theorem 5.2.2]).

Until recent years the applicability of Dirichlet form theory was limited to symmetric Markov processes (see [9]) or, more generally, to Markov processes satisfying a sector condition (cf. [14]; see also sections 7.5 and 7.7 from [1] for the connections with the right processes). Within the theory of *generalized* (non-sectorial) *Dirichlet forms* (see [18], and [21] for the associated stochastic calculus) this limitation has been overcome since in this generalized framework only the existence of a positive measure μ is required for which the transition semigroup of the Markov process operates as a C_0 -semigroup of contractions on $L^2(\mu)$. In particular, as no sector condition has to be verified, the theory of generalized Dirichlet forms is robust and well-suited for heavy perturbation methods.

A central analytic property in the theory of symmetric, sectorial, and non-sectorial Dirichlet forms is the quasi-regularity of the forms, because only such forms can be associated

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with a nice Markov process, i.e. a *standard process* (see [18, IV. Theorem 3.1]). Moreover, after review of [21] and the present paper one can recognize that a stochastic calculus for generalized Dirichlet forms can be developed by only assuming the standardness of an associated process and the existence of an excessive measure for the underlying L^2 -space of the form (such a measure is called an *excessive reference measure*). On the other hand it was shown in [16] that any right continuous Markov process can be related to a generalized Dirichlet form with excessive reference measure. We were hence led to the following question: Given a right Markov process X on a fairly general state space. Under which additional analytic condition is X a standard process? This question has first been investigated in [8] under the additional condition that X is associated with a semi-Dirichlet form (see [13]). A different approach and several extensions have been developed in sections 3.7 and 7.7 from [1]. Moreover as an application in [8] the theory of Revuz measures in the semi-Dirichlet context is developed and it is shown that quasi-regularity is invariant under time change. In particular, if there is an *excessive* reference measure then the developed theory of Revuz measures can be related to the context of classical energy (see [8, (4.19) Remarks]). Hence from the point of view of applications it is interesting to investigate the question which additional analytic condition leads to the standardness in the case of generalized Dirichlet forms or likewise in the case of any right Markov process. This is what we do in subsections 3.1 and 2 respectively.

The definition of standard process is quite abstract and technical but one can say in general that the standardness property and there mainly the quasi-left continuity connects the analytic with the probabilistic potential theory. For a right Markov process that is not quasi-left continuous only capacity zero sets are polar and not vice versa (see Remark 2.3(a) for more on this).

Our main application is an extension of a result in [8] to the case of non-sectorial generalized Dirichlet forms and can roughly and abridged be stated as follows. Given two right Markov processes on the same state space from which one is associated to a semi-Dirichlet form \mathcal{A} and from which the other one is associated to a generalized Dirichlet form whose sectorial part is given by \mathcal{A} . Then the second process is standard if the capacities of both forms are equivalent (cf. Remark 3.1(b), Theorem 3.3(i)). Of course, if the two bilinear forms coincide we may assume that both right Markov processes are the same. Then the condition on the capacity is trivially satisfied and we obtain the original result from [8], i.e. that a right Markov process that is associated with a semi-Dirichlet form is automatically a standard (Markov) process. For the mathematically precise formulation of our main result and why we recover the case of semi-Dirichlet forms see the paragraph right in front of Theorem 3.3 and the theorem itself. It is important to mention that since we compare the second process with an “elliptic” sectorial process associated with a semi-Dirichlet form, we are not able to handle the time-dependent case. We guess that the time-dependent case can be handled by comparing the second process to a standard process associated with a time-dependent Dirichlet form (see [15], [17]). This “guess” might be subject of future investigations.

Once the standardness is shown, as in the original paper [8], we can derive the quasi-regularity of the associated Dirichlet form (see Theorem 3.3(ii)). The tightness, i.e. the existence of a nest of compacts, is automatically satisfied if we only slightly concretize the state space (see Remark 2.3 (b) and (c)), and the special property is in contrast to [14], and [18], not used in order to show that the process resolvent applied to bounded

L^2 -functions is quasi-continuous, i.e. that the process is properly associated in the resolvent sense with the form (see Remark 3.4). For the quasi-regular generalized Dirichlet form and the associated standard process all results from [18], and [21] will be available (cf. first paragraph of section 3).

In subsection 3.2 we present an application where the resolvent of a right Markov process is explicitly given as the perturbation of the resolvent of a quasi-regular semi-Dirichlet form. Typical examples are given through perturbations with the β -potential kernel of a continuous additive functional (see [3]) and by a potential theoretical approach to measure-valued discrete branching (Markov) processes (see [5]). First, in subsection 3.2 we develop general conditions for the perturbed resolvent to be associated with a standard process X (see Proposition 3.7(ii), and (iii)). Then, under the *absolute continuity condition*, in subsection 3.3 we construct explicitly a quasi-regular generalized Dirichlet form that is properly associated in the resolvent sense with X (see Corollary 3.10). As in [16] the generalized Dirichlet form is constructed directly from its generator which is also the generator of the underlying Markov process, i.e. the L^2 -infinitesimal generator of the transition semigroup $(p_t^\beta)_{t \geq 0}$ of the Markov process (cf. (3.7) and paragraph below). In this application we would like to emphasize two points. First, we do not obtain as usual the standard process by showing regularity properties of the form. We rather show that the form is associated to a standard process and from this we derive its quasi-regularity. Second, this application shows that the theory of generalized Dirichlet forms is well-suited for heavy perturbation methods as we do not need any sector condition to be verified after the perturbation.

Our mentioned applications from section 3 are essentially based on the results that we develop in section 2 (see Theorem 2.7) about the sufficient conditions which ensure that the standardness property is transferred from a right (Markov) process to a second one. We use several analytic and probabilistic potential theoretical tools (as implemented in [1], [2], and [4]; see also [6] for further applications) like the capacities associated to a right process and their tightness property, the quasi-continuity, Ray topologies and compactifications, and the fine topology. We complete the second section with the treatment of the weak duality hypothesis frame (Theorem 2.8).

2 Standardness properties

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a Borel right (Markov) process whose state space is a Lusin topological space (E, \mathcal{T}) , and let $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ be its associated sub-Markovian resolvent.

Let $\beta > 0$ be arbitrary. Recall that the Borel σ -algebra \mathcal{B} on E is generated by the set $\mathcal{E}(\mathcal{U}_\beta)$ of all \mathcal{B} -measurable \mathcal{U}_β -excessive functions, where $\mathcal{U}_\beta := (U_{\beta+\alpha})_{\alpha > 0}$. We further recall that a universally measurable function f is said to be \mathcal{U}_β -excessive if $\alpha U_{\beta+\alpha} f \nearrow f$ pointwise as $\alpha \nearrow \infty$.

Let μ be a σ -finite measure on (E, \mathcal{B}) . We say that the right process X is μ -standard if for one (and hence all) finite measures λ which are equivalent to μ it possesses left limits in E P^λ -a.e. on $[0, \zeta)$ and for every increasing sequence $(T_n)_n$ of stopping times with $T_n \nearrow T$ we have $X_{T_n} \rightarrow X_T$ P^λ -a.e. on $\{T < \zeta\}$, ζ being the lifetime of X . If in addition $\mathcal{F}_T^\lambda = \bigvee_n \mathcal{F}_{T_n}^\lambda$ then X is called μ -special standard (cf. Section 16 in [11]).

A *Ray cone* associated with \mathcal{U} is a convex cone \mathcal{R} of bounded \mathcal{B} -measurable, \mathcal{U}_β -excessive functions such that:

- The cone \mathcal{R} contains the positive constant functions and is min-stable.
- $U_\beta((\mathcal{R} - \mathcal{R})_+) \subset \mathcal{R}$ and $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$.
- The cone \mathcal{R} is separable with respect to the uniform norm.
- The σ -algebra on E generated by \mathcal{R} coincides with \mathcal{B} .

One can show that for every countable set \mathcal{A} of bounded \mathcal{B} -measurable, \mathcal{U}_β -excessive functions there exists a Ray cone including \mathcal{A} .

The topology $\mathcal{T}_\mathcal{R}$ on E generated by a Ray cone \mathcal{R} (i.e. the coarsest topology on E for which every function from \mathcal{R} is continuous) is called the *Ray topology* induced by \mathcal{R} .

A Lusin topology on E is called *natural* (with respect to \mathcal{U}) if its Borel σ -algebra is precisely \mathcal{B} and it is smaller than the *fine topology* on E (with respect to \mathcal{U}). We recall that the *fine topology* with respect to \mathcal{U} is the topology on E generated by all \mathcal{U}_β -excessive functions. The initial topology \mathcal{T} as well as any Ray topology are natural. Further note that \mathcal{U} is the resolvent of a right process with respect to any natural topology (cf. [1]).

If $\beta > 0$, then for all $u \in \mathcal{E}(\mathcal{U}_\beta)$ and every subset A of E we consider the function

$$R_\beta^A u := \inf\{v \in \mathcal{E}(\mathcal{U}_\beta) \mid v \geq u \text{ on } A\},$$

called the β -order *reduced function* of u on A . It is known that if $A \in \mathcal{B}$ then $R_\beta^A u$ is universally \mathcal{B} -measurable and if moreover A is finely open and $u \in p\mathcal{B}$ then $R_\beta^A u \in p\mathcal{B}$. If $A = E$ we simply write $R_\beta u$ instead of $R_\beta^E u$.

Let λ be a finite measure on (E, \mathcal{B}) . We also fix a strictly positive, bounded \mathcal{U}_β -excessive function p_o of the form $p_o = U_\beta f_o$, with $f_o \in p\mathcal{B}$, $0 < f_o \leq 1$.

Since \mathcal{U} is the resolvent of a right process X , the following fundamental result of G. A. Hunt holds for all $A \in \mathcal{B}$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$:

$$R_\beta^A u(x) = E^x(e^{-\beta D_A} u(X_{D_A}))$$

where D_A is the entry time of A , $D_A = \inf\{t \geq 0 \mid X_t \in A\}$; see e.g. [7].

It turns out that the functional $M \mapsto c_\lambda^\beta(M)$, $M \subset E$, defined by

$$c_\lambda^\beta(M) = \inf\{\lambda(R_\beta^G p_o) \mid G \in \mathcal{T}, M \subset G\}$$

is a Choquet capacity on (E, \mathcal{T}) .

The capacity c_λ^β on (E, \mathcal{T}) is named *tight* provided that there exists an increasing sequence $(K_n)_n$ of \mathcal{T} -compact sets such that

$$\inf_n c_\lambda^\beta(E \setminus K_n) = 0$$

(or equivalently $\inf_n R_\beta^{E \setminus K_n} p_o = 0$ λ -a.e.) which is also equivalent to

$$P^\lambda(\lim_n D_{E \setminus K_n} < \zeta) = 0.$$

In particular, if the capacity c_λ^β on (E, \mathcal{T}) is tight for one $\beta > 0$ then this happens for all $\beta > 0$. Similarly, the following assertion holds:

Remark 2.1. If $(G_n)_n$ is a decreasing sequence of \mathcal{T} -open sets such that there exists $\beta > 0$ with $\inf_n c_\lambda^\beta(G_n) = 0$ then the equality holds for all $\beta > 0$.

A set M is called λ -polar with respect to \mathcal{U} provided that there exists $A \in \mathcal{B}$, $M \subset A$, such that $T_A = \infty$ P^λ -a.e., where T_A is the hitting time of A , $T_A = \inf\{t > 0 \mid X_t \in A\}$.

A real valued function $u \in \mathcal{E}(\mathcal{U}_\beta)$ is called *regular* provided that for every sequence $(u_n)_n$ in $\mathcal{E}(\mathcal{U}_\beta)$, $u_n \nearrow u$, we have $\inf_n R_\beta(u - u_n) = 0$; see [1] for more details on regular excessive functions. It is known that (see e.g. [7]) if the process X is transient, then a bounded function $u \in \mathcal{E}(\mathcal{U})$ is regular if and only if there exists a continuous additive functional having u as potential function. A real valued \mathcal{U}_β -excessive function u is called λ -regular with respect to \mathcal{U}_β , provided there exists a regular \mathcal{U}_β -excessive function u' such that $u = u' \lambda \circ U_\beta$ -a.e.

Proposition 2.2. *The following assertions hold.*

(i) $A \in \mathcal{B}$ is λ -polar with respect to \mathcal{U} and λ -negligible if and only if $\lambda(R_\beta^A p_o) = 0$.

Consequently if a Borel set is of c_λ^β -capacity zero then it is λ -polar and λ -negligible

(ii) Assume that one of the following two conditions is satisfied:

(ii.a) The topology \mathcal{T} is a Ray one.

(ii.b) Every \mathcal{U}_β -excessive function dominated by $U_\beta f_o$ is λ -regular.

Then $c_\lambda^\beta(A) = \lambda(R_\beta^A p_o)$ for all $A \in \mathcal{B}$ and in particular the sets which are λ -negligible and λ -polar are precisely those having c_λ^β -capacity zero. If condition (ii.b) holds then the capacity c_λ^β is tight in any natural topology.

Proof. (i) The first statement in (i) is immediate from the definitions. For the second we present two proofs, a direct one for the convenience of the reader, and a shorter alternative proof based on known results:

Direct proof: If $c_\lambda^\beta(A) = 0$ then there are open sets $U_k \supset A$ with $c_\lambda^\beta(U_k) \rightarrow 0$ as $k \rightarrow \infty$. Define $F_k := E \setminus (\bigcap_{l \leq k} U_l)$. Then $A \subset E \setminus F_k$ for all k , hence

$$\begin{aligned} P^\lambda(T_A < \infty) &\leq P^\lambda(D_A < \infty) \\ &\leq \lim_k P^\lambda(D_{E \setminus F_k} < \infty) = P^\lambda(\lim_k D_{E \setminus F_k} < \infty). \end{aligned}$$

We have $c_\lambda^\beta(E \setminus F_k) \leq c_\lambda^\beta(U_k) \rightarrow 0$. Thus by Lebesgue

$$0 = \lim_k c_\lambda^\beta(E \setminus F_k) = \int_E E^x \left(\int_{\lim_k D_{E \setminus F_k}}^\infty e^{-\beta t} f_o(X_t) dt \right) \lambda(dx),$$

and so $P^\lambda(\lim_k D_{E \setminus F_k} < \infty) = 0$. Consequently, A is λ -polar and λ -negligible.

Alternative proof: (a) If the topology \mathcal{T} is a Ray one then the assertion is a direct consequence of Proposition 1.6.3 in [1].

(b) If the topology \mathcal{T} is only natural, then there exists a Ray topology \mathcal{T}_R which is finer than the given topology \mathcal{T} (we use this procedure in the proof of (ii)). Therefore, if a set A has zero capacity w.r.t. the capacity constructed using \mathcal{T} , then A is of zero capacity if we replace \mathcal{T} by \mathcal{T}_R , hence it is λ -polar by (a).

(ii) If (ii.a) holds then the assertion follows from Proposition 1.6.3 in [1].

Assume that (ii.b) is verified. Using Proposition 2.1 in [2] we may consider a Ray cone \mathcal{R} (formed by \mathcal{U}_β -excessive functions) such that the topology $\mathcal{T}_\mathcal{R}$ generated by \mathcal{R} is finer than the given natural topology \mathcal{T} . By (ii.b) and Theorem 3.5.2 in [1] it follows that the capacity c_λ^β is tight in the Ray topology $\mathcal{T}_\mathcal{R}$ and therefore also in the topology \mathcal{T} .

Let $A \in \mathcal{B}$ and $\varepsilon > 0$. We consider a $\mathcal{T}_\mathcal{R}$ -compact set K such that $c_\lambda^\beta(E \setminus K) < \frac{\varepsilon}{2}$. By the above considerations and the definition of c_λ^β there exists a $\mathcal{T}_\mathcal{R}$ -open set G such that $A \subset G$ and $c_\lambda^\beta(G) < \lambda(R_\beta^A p_o) + \frac{\varepsilon}{2}$. Let $G_o = (G \cap K) \cup (E \setminus K)$. Then $A \subset G_o$ and since $\mathcal{T}_\mathcal{R}|_K = \mathcal{T}|_K$, it follows that the set G_o is \mathcal{T} -open. We have $c_\lambda^\beta(G_o) \leq c_\lambda^\beta(G \cap K) + c_\lambda^\beta(E \setminus K) \leq \lambda(R_\beta^A p_o) + \varepsilon$ and we conclude that, considering c_λ^β as a capacity on (E, \mathcal{T}) , we have $c_\lambda^\beta(A) = \lambda(R_\beta^A p_o)$. \square

Remark 2.3. (a) For the converse of the second statement in Proposition 2.2(i) in general one needs at least the quasi-left continuity of X . For instance, if X is λ -standard, then the converse holds. In this sense a λ -standard process connects the analytic capacity related to excessive functions and the process capacity related to polar sets.

(b) Let μ be a σ -finite measure on E . Recall that the right process X is said to be μ -tight provided there exists an increasing sequence $(K_n)_{n \in \mathbb{N}}$ of \mathcal{T} -compact (metrizable) sets in E such that $P^\mu(\lim_{n \rightarrow \infty} D_{E \setminus K_n} < \zeta) = 0$. In particular, if λ is a finite measure on E , then the λ -tightness of X is equivalent to the tightness of c_λ^β for some $\beta > 0$ (see explanations right before Remark 2.1).

(c) Suppose that the right process X has P^μ -a.e. left limits in E up to ζ and that E is a metrizable Lusin space. Then X is automatically μ -tight (see [14, IV. Theorem 1.15]). In particular, X is automatically μ -tight if X is μ -standard (since the existence of the left limits up to ζ P^μ -a.e. is part of the definition of the μ -standardness).

The main argument in the proof of the next result is a modification of the proof of $ii) \implies i)$ from Theorem 1.3 in [2].

Proposition 2.4. Assume that the topology \mathcal{T} is generated by a Ray cone \mathcal{R} and let μ be a σ -finite measure on (E, \mathcal{B}) . Then the following assertions hold.

(i) If the process X is μ -tight, then it is μ -standard.

(ii) If the β -subprocess of X is μ -standard for some $\beta > 0$ then X is also μ -standard.

Proof. (i) Suppose that X is μ -tight and let λ be a finite measure on E which is equivalent with μ . By assertion (b) of Remark 2.3 the capacity c_λ^β is tight and let $(K_n)_n$ be an increasing sequence of \mathcal{T} -compact subsets of E such that $\inf_n R_\beta^{E \setminus K_n} p_o = 0$ λ -a.e. We denote by K the Ray compactification of E with respect to \mathcal{R} (see, e.g., section 1.5 in [1]). Since for every $u \in \mathcal{R}$ the process $(e^{-\beta t} u(X_t))_{t \geq 0}$ is a bounded right continuous supermartingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ it follows that (cf. [7]) this process has left limits P^λ -a.e. Since the Ray cone \mathcal{R} is separable with respect to the uniform norm, it follows that the process $(X_t)_{t \geq 0}$ has left limits in K P^λ -a.e. From $\lim_n R_\beta^{E \setminus K_n} p_o = 0$

λ -a.e. and $\lambda(R_\beta^{E \setminus K_n} p_o) = E^\lambda(\int_{T_{E \setminus K_n}}^\zeta e^{-\beta t} f_o(X_t) dt)$ we deduce that $\sup_n T_{E \setminus K_n} \geq \zeta$ P^λ -a.e. Hence for every $\omega \in \Omega$ with $T_{E \setminus K_n}(\omega) < \zeta(\omega)$ we have $X_t(\omega) \in K_n$ provided that $t < T_{E \setminus K_n}(\omega)$ and so $X_{t-}(\omega) \in K_n$. Consequently the process $(X_t)_{t \geq 0}$ has left limits in E P^λ -a.e. on $[0, \zeta)$. By Theorem (48.15) in [20] we get that the (0)-process is λ -standard.

(ii) If a β -subprocess of X is μ -standard, then from Remark 2.3 (c) (see also [12] and [2]) it follows that X is μ -tight and by (i) we get that X is μ -standard. \square

A set $M \in \mathcal{B}$ is called λ -inessential (with respect to \mathcal{U}) provided that it is λ -negligible and $R_\beta^M 1 = 0$ on $E \setminus M$.

Remark 2.5. If $M \in \mathcal{B}$ is a λ -inessential set then we may consider the restrictions $X|_F$ of the process X to the "absorbing set" $F := E \setminus M$. Note that $U_\beta(1_M) = 0$ on F and the resolvent associated with $X|_F$ is precisely the restriction $\mathcal{U}|_F$ of \mathcal{U} to F . The following assertions hold:

(a) $\mathcal{E}(\mathcal{U}_\beta|_F) = \mathcal{E}(\mathcal{U}_\beta)|_F$. In particular the fine topology on F with respect to $\mathcal{U}|_F$ is the trace on F of the fine topology on E with respect to \mathcal{U} . The process X is λ -standard if and only if $X|_F$ is $\lambda|_F$ -standard.

(b) **Trivial modification.** We consider the trivial modification of \mathcal{U} on M (see e.g. [1] and [4]), namely the sub-Markovian resolvent $\mathcal{U}' = (U'_\alpha)_{\alpha > 0}$ on (E, \mathcal{B}) defined by:

$$U'_\alpha f = 1_E U_\alpha(f 1_F) + \frac{1}{\alpha} f 1_M, \quad \alpha > 0, f \in \mathcal{B}.$$

Then \mathcal{U}' is the resolvent of a right (Markov) process with state space E . A function $u \in p\mathcal{B}$ belongs to $\mathcal{E}(\mathcal{U}'_\beta)$ if and only if $u|_F \in \mathcal{E}(\mathcal{U}_\beta|_F)$. Consequently by (a) we have: a subset Γ of E is finely open with respect to \mathcal{U}' if and only if there exists a finely open set Γ_o with respect to \mathcal{U} , such that $\Gamma \cap F = \Gamma_o \cap F$. In particular every topology on E which is natural with respect to \mathcal{U} is also natural with respect to \mathcal{U}' .

Let $\mathcal{V} = (V_\alpha)_{\alpha > 0}$ be a second sub-Markovian resolvent of kernels on E and assume that it is also the resolvent of a right (Markov) process Y with state space E . Let μ be a σ -finite Borel measure on the Lusin space (E, \mathcal{T}) that has full support. We suppose that there is a semi-Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$ on $L^2(E, \mu)$ with associated $L^2(E, \mu)$ -resolvent $(H_\alpha)_{\alpha > 0}$ (see [13]). We further assume that $V_\alpha f$ is a μ -version of $H_\alpha f$ for any $f \in L^2(E, \mu)$, and $\alpha > 0$, i.e. that the right process Y is associated with $(\mathcal{A}, D(\mathcal{A}))$. In particular

$$H_\alpha(L^2(E, \mu)) \subset D(\mathcal{A}) \text{ densely,}$$

and

$$\mathcal{A}_\alpha(H_\alpha f, u) = (f, u)_{L^2(E, \mu)}$$

for all $\alpha > 0$, $f \in L^2(E, \mu)$, and $u \in D(\mathcal{A})$, where $\mathcal{A}_\alpha(\cdot, \cdot) := \mathcal{A}(\cdot, \cdot) + \alpha(\cdot, \cdot)_{L^2(E, \mu)}$.

Let $K > 0$ be the sector constant of $(\mathcal{A}, D(\mathcal{A}))$, i.e.

$$|\mathcal{A}_1(u, v)| \leq K \mathcal{A}_1(u, u)^{1/2} \mathcal{A}_1(v, v)^{1/2} \quad \text{for all } u, v \in D(\mathcal{A}).$$

We suppose further that the measures λ and μ are equivalent. We may and will assume that $\lambda = f_o \cdot \mu$.

Let cap_λ^β be the capacity corresponding to the resolvent $(V_\alpha)_{\alpha>0}$, i.e. for $M \subset E$, it is defined by

$$\text{cap}_\lambda^\beta(M) = \inf\{\lambda(\overline{R}_\beta^G V_\beta f_o) \mid G \in \mathcal{T}, M \subset G\},$$

where \overline{R}_β^G is defined as R_β^G but w.r.t. \mathcal{V}_β , i.e., $\overline{R}_\beta^G u$ denotes the reduced function of u on G with respect to $\mathcal{E}(\mathcal{V}_\beta)$. Analogously to Remark 2.1 these capacities are all equivalent for any $\beta > 0$. Note further that for open sets G , $\text{cap}_\lambda^\beta(G)$ coincides with the so-called f_o -capacity of G associated with the semi-Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$.

A real valued function on E is called λ -fine with respect to \mathcal{U} provided it is finely continuous with respect to \mathcal{U} outside a λ -inessential set. Note that a real valued function f on E is λ -fine with respect to \mathcal{V} if and only if there exists an increasing sequence $(F_n)_n$ of finely closed sets with respect to \mathcal{V} , such that $\inf_n \text{cap}_\lambda^\beta(E \setminus F_n) = 0$ and $f|_{F_n}$ is finely continuous for all n .

An increasing sequence of \mathcal{T} -closed sets $(F_n)_{n \in \mathbb{N}}$ is called c_λ^β -nest if

$$\lim_{n \rightarrow \infty} c_\lambda^\beta(E \setminus E_n) = 0.$$

We consider the following conditions on \mathcal{U} and \mathcal{V} :

- (A1) The sets which are λ -polar and λ -negligible are the same for \mathcal{U} and \mathcal{V} .
- (A2) Every cap_λ^β -nest is a c_λ^β -nest.
- (B1) The function $U_\beta f$ is λ -fine with respect to \mathcal{V} for every $f \in bp\mathcal{B}$.
- (B2) There exists a bounded strictly positive \mathcal{U}_β -excessive function u_o such that every \mathcal{U}_β -excessive function dominated by u_o is λ -fine with respect to \mathcal{V} .

Remark 2.6. (a) Clearly condition (B1) does not depend on $\beta > 0$ and (B2) for $u_o = U_\beta 1$ implies (B1).

(b) Since the resolvent \mathcal{V} satisfies condition (ii.b) from Proposition 2.2 (cf. [1]), we deduce that the following assertions are equivalent for a set M :

- the set M is μ -polar with respect to \mathcal{V} (in this case M is also μ -negligible);
- the set M is λ -negligible and λ -polar with respect to \mathcal{V} ;
- $\text{cap}_\lambda^\beta(M) = 0$.

(c) By Remark 2.1 it follows that condition (A2) does not depend on $\beta > 0$.

(d) If condition (A2) holds then:

- A set which is λ -polar with respect to \mathcal{V} is also λ -polar with respect to \mathcal{U} ;
- If the capacity cap_λ^β on (E, \mathcal{T}) is tight then c_λ^β is also tight.

A function $g \in \mathcal{B}$ is called c_λ^β -quasi-continuous (in short c_λ^β -q.c.) provided there exists a c_λ^β -nest $(F_n)_{n \in \mathbb{N}}$ such that for each n $g|_{F_n}$ is continuous on F_n .

Theorem 2.7. *Assume that condition (A2) is satisfied. Then the following assertions hold.*

- (i) *If (B1) is verified then there exists a Lusin topology on E which is natural for \mathcal{U} and for a trivial modification \mathcal{V}' of \mathcal{V} , such that the right processes having \mathcal{U} and \mathcal{V}' as associated resolvents respectively are μ -standard.*
- (ii) *If (B2) is verified then the right processes X having \mathcal{U} as associated resolvent is μ -standard in the original topology \mathcal{T} . In addition, $U_\alpha f$ is c_λ^β -q.c. for any $f \in b\mathcal{B}$ and $\alpha > 0$.*

Proof. We have already observed that \mathcal{V} (being the resolvent of a semi-Dirichlet form) satisfies condition (ii.b) from Proposition 2.2:

every \mathcal{V}_β -excessive function dominated by $V_\beta f_o$ is λ -regular.

Consequently (see also Theorem 3.5.2 in [1]), the capacity cap_λ^β is tight and the right process having \mathcal{V} as associated resolvent is μ -standard in any topology which is natural with respect to \mathcal{V} .

(i) According with the usual method of constructing Ray cones (see e.g. [1]) and using hypothesis (B1), there exists a Ray cone \mathcal{R} associated with \mathcal{U} such that every $u \in \mathcal{R}$ is λ -fine with respect to \mathcal{V} . Let \mathcal{R}_o be a countable subset of \mathcal{R} which is dense in \mathcal{R} in the uniform norm. Let M_o be a set which is μ -polar with respect to \mathcal{V} , and such that every u from \mathcal{R}_o is finely continuous with respect to \mathcal{V} outside M_o . From [1], page 168, it follows that there exists a set M which is λ -inessential with respect to \mathcal{V} such that $M_o \subset M$. We consider now the trivial modification \mathcal{V}' of \mathcal{V} on M . Note that \mathcal{V}' is also the resolvent of the (quasi-regular) semi-Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$ on $L^2(E, \mu)$, in particular \mathcal{V}' satisfies condition (ii.b) and if $A \in \mathcal{B}$ then $\text{cap}_\lambda^\beta(A) = \text{cap}_\lambda^\beta(A \setminus M) = \lambda(R_\beta^{A \setminus M} V_\beta' f_o) = \lambda(R_\beta^A V_\beta' f_o)$; here R_β^A denotes the reduction operator on A with respect to \mathcal{V}' . By assertion (b) of Remark 2.5 we deduce that the topology \mathcal{R} is natural with respect to \mathcal{V}' and we conclude that the capacity cap_λ^β is tight in $\mathcal{T}_\mathcal{R}$. Let $(K_n)_n \subset E$ be an increasing sequence of $\mathcal{T}_\mathcal{R}$ -compact sets such that $\text{cap}_\lambda^\beta(E \setminus K_n) = \lambda(R_\beta^{E \setminus K_n} V_\beta f_o) \leq \frac{1}{2^n}$ for all n . From Proposition 2.2 we get for every n a \mathcal{T} -open set G_n such that $G_{n-1} \supset G_n \supset E \setminus K_n$ and $\text{cap}_\lambda^\beta(G_n) \leq \text{cap}_\lambda^\beta(E \setminus K_n) + \frac{1}{2^n}$. By (A2) we deduce that $\inf_n c_\lambda^\beta(G_n) = 0$ and consequently the capacity c_λ^β is also tight in $\mathcal{T}_\mathcal{R}$. The claimed μ -standardness property follows now by the above considerations and Proposition 2.4.

(ii) Passing to the β -level of the resolvent \mathcal{U} and according with Proposition 2.4, we may assume hat $u_o \in \mathcal{E}(\mathcal{U})$. We consider now the "Doob u_o -transform of \mathcal{U} ", namely the sub-Markovian resolvent of kernels $\mathcal{U}^o = (U_\alpha^o)_{\alpha > 0}$ defined by

$$U_\alpha^o f = \frac{1}{u_o} U_\alpha(u_o f), \quad f \in p\mathcal{B}, \quad \alpha > 0.$$

The following assertions hold.

- If $v \in p\mathcal{B}$ then: $v \in \mathcal{E}(\mathcal{U}_\beta)$ if and only if $\frac{v}{u_o} \in \mathcal{E}(\mathcal{U}_\beta^o)$.
- A σ -finite measure ξ on (E, \mathcal{B}) is \mathcal{U}_β -excessive (i.e., $\xi \circ \alpha U_{\beta+\alpha} \leq \xi$ for all $\alpha > 0$) if and only if the measure $\frac{1}{u_o} \cdot \xi$ is \mathcal{U}_β^o -excessive. Moreover ξ is a potential with respect to \mathcal{U}_β (i.e., $\xi = \nu \circ U_\beta$, where ν is a σ -finite measure on E) if and only if $\frac{1}{u_o} \cdot \xi$ is a potential with respect to \mathcal{U}_β^o . Consequently, \mathcal{U}^o is the resolvent of a Borel right process with state space E (cf. Section 1.8 in [1] and [4]).

– The fine topologies on E with respect to \mathcal{U} and \mathcal{U}^o coincide. In particular, a topology on E is simultaneously natural with respect to \mathcal{U} and \mathcal{U}^o . The λ -inessential sets with respect to \mathcal{U} and \mathcal{U}^o are the same.

– \mathcal{U}^o and \mathcal{V} satisfy condition (A2). Condition (B2) implies that:

(B2') Every bounded \mathcal{U}_β^o -excessive function is λ -fine with respect to \mathcal{V} .

As in the proof of Proposition 2.2 we consider a Ray cone \mathcal{R} (formed by \mathcal{U}_β -excessive functions) such that the topology $\mathcal{T}_\mathcal{R}$ generated by \mathcal{R} is finer than the original topology \mathcal{T} . Similarly, there exists a Ray cone \mathcal{R}^o (formed by \mathcal{U}_β^o -excessive functions) such that the topology $\mathcal{T}_{\mathcal{R}^o}$ generated by \mathcal{R}^o is finer than $\mathcal{T}_\mathcal{R}$. As before, we consider \mathcal{R}_o^o , M_o , M and \mathcal{V}' . From (B2') it follows that $\mathcal{T}_{\mathcal{R}^o}$ is natural with respect to \mathcal{V}' . Like in the proof of assertion (i) we get now that the capacity cap_λ^β is tight in $\mathcal{T}_{\mathcal{R}^o}$ and since $\mathcal{T}_\mathcal{R} \subset \mathcal{T}_{\mathcal{R}^o}$ we deduce by (A2) that the capacity c_λ^β is tight in $\mathcal{T}_\mathcal{R}$. Consequently, Proposition 2.4 implies the λ -standardness property for X in $\mathcal{T}_\mathcal{R}$ and clearly also in the original topology \mathcal{T} .

Let now $f \in bp\mathcal{B}$. It is sufficient to prove the second assertion of (ii) for $\alpha = \beta$. Since $U_\beta f$ is a λ -regular \mathcal{U}_β -excessive function, we deduce by Proposition 3.2.6 from [1] that it is c_λ^β -q.c. in $\mathcal{T}_\mathcal{R}$. Let $(K_n)_{n \in \mathbb{N}}$ be a c_λ^β -nest of $\mathcal{T}_\mathcal{R}$ -compact sets, such that $U_\beta f|_{K_n}$ is continuous for each n . Because $\mathcal{T}|_{K_n} = \mathcal{T}_\mathcal{R}|_{K_n}$ for all n , we conclude that $U_\beta f$ is also c_λ^β -q.c. in \mathcal{T} . \square

Recall that the *weak duality hypothesis* (with respect to the topology \mathcal{T} and a measure m) is satisfied by \mathcal{U} and a sub-Markovian resolvent $\mathcal{U}^* = (U_\alpha^*)_{\alpha > 0}$ provided that \mathcal{U}^* is also the resolvent of a right Markov process with state space E and for all $f, g \in p\mathcal{B}$, $\alpha > 0$ we have

$$\int f U_\alpha g \, dm = \int g U_\alpha^* f \, dm.$$

(see e.g. [11] and [1]).

Note that \mathcal{V} (being the resolvent of a semi-Dirichlet form) fits in the frame of the weak duality, by choosing an appropriate measure m which is equivalent with μ ; see section 7.6 in [1].

We assume further that the measure λ is absolutely continuous with respect to m .

Remark. *If the weak duality hypothesis is satisfied by \mathcal{U} and \mathcal{U}^* (with respect to the topology \mathcal{T} and the measure m) then the capacity c_λ^β is tight in the topology \mathcal{T} (see [2]). Moreover, by the result of J.B. Walsh [24] the process X has càdlàg trajectories.*

The next theorem shows that, under the weak duality hypothesis for \mathcal{U} , it is (A1) the adequate condition leading to the standardness property of the process X .

Theorem 2.8. *Suppose that the weak duality hypothesis (with respect to the topology \mathcal{T} and the measure m) is satisfied by \mathcal{U} and \mathcal{U}^* . Assume that condition (A1) is verified by \mathcal{U}^* and \mathcal{V} , and that every point of E is μ -polar with respect to \mathcal{V} . Then the following assertions hold.*

(i) *The right process X having \mathcal{U} as associated resolvent is μ -standard in the topology \mathcal{T} .*

(ii) *Every \mathcal{U}_β -excessive function dominated by p_o has a c_λ^β -quasi continuous version.*

Proof. We show that the axiom of λ -polarity holds for \mathcal{U}^* (i.e., every semipolar set is λ -polar with respect to \mathcal{U}^*). Let $M \in \mathcal{B}$ be a semipolar set. By Proposition 1.7.27 and Corollary 3.2.16 in [1] there exists a measure ν carried by M such that a subset of M is λ -polar and λ -negligible if and only if it is ν -negligible; such a measure ν is called *Dellacherie measure*. By condition (A1) it follows that ν is a Dellacherie measure with respect to \mathcal{V} and consequently M is a μ -semipolar set. Since the axiom of μ -polarity holds for the semi-Dirichlet forms (see e.g. [8] and Corollary 7.5.20 from [1]) we conclude that the set M is μ -polar and therefore (by Remark 2.6 (b)) it is λ -negligible and λ -polar with respect to \mathcal{V} . Again from (A1) we conclude that M is λ -polar with respect to \mathcal{U}^* .

Theorem 7.2.9 in [1] implies now that assertion (ii) holds. By Theorem 3.5.2 and Proposition 3.5.3 in [1], and Proposition 2.4 we conclude that assertion (i) also holds. \square

Remark 2.9. *The results from Theorem 2.7 and Theorem 2.8 remain valid under the following hypothesis (which is weaker than assuming that \mathcal{V} is the resolvent of a semi-Dirichlet form): condition (ii.b) from Proposition 2.2 holds.*

3 Applications

In this section we derive the quasi-regularity property of generalized Dirichlet forms related to the resolvent \mathcal{U} . Note that we do not aim to derive the μ -speciality of the associated processes. In particular for the quasi-regular generalized Dirichlet forms all results from [18], and [21] will be available as far as the μ -speciality is not concerned. However, this property is not really relevant for stochastic calculus and potential theory. It is mainly used to show the equivalence of a certain class of processes with a certain class of bilinear forms (cf. [14, IV. Theorem 3.5 and Theorem 5.2] and [18, IV. Theorem 3.1 and Corollary 3.2]).

Throughout this section we assume that E is a *metrizable Lusin space*. However, this assumption is only used in order to apply the tightness result of Remark 2.3 (c).

3.1 On the quasi-regularity of generalized Dirichlet forms

If not otherwise stated we maintain the notations of section 2. In particular, μ is a σ -finite Borel measure on the Lusin space (E, \mathcal{T}) that has full support, $(\mathcal{A}, D(\mathcal{A}))$ is a semi-Dirichlet form on $L^2(E, \mu)$ that is associated with a right process Y on E , and $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is the process resolvent of $(\mathcal{A}, D(\mathcal{A}))$. From [8] (see also Theorem 7.6.3 from [1]) we know that then $(\mathcal{A}, D(\mathcal{A}))$ is automatically quasi-regular. We assume that there is a generalized Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \mu)$ with associated $L^2(E, \mu)$ -resolvent $(G_\alpha)_{\alpha>0}$ (see [18]) and let $\lambda = f_o d\mu$ as in the previous section.

In general $(\mathcal{E}, \mathcal{F})$ is written as

$$\mathcal{E}(u, v) := \begin{cases} \mathcal{Q}(u, v) - \langle \Lambda u, v \rangle & \text{for } u \in \mathcal{F}, v \in D(\mathcal{Q}) \\ \mathcal{Q}(u, v) - \langle \widehat{\Lambda} v, u \rangle & \text{for } u \in D(\mathcal{Q}), v \in \widehat{\mathcal{F}}, \end{cases}$$

where $(\mathcal{Q}, D(\mathcal{Q}))$ is a coercive closed form on $L^2(E, \mu)$ with sector constant $K' > 0$,

and $\langle \cdot, \cdot \rangle$ denotes the dualization between $D(\mathcal{Q})'$ and $D(\mathcal{Q})$ (for the definition of further notions cf. again [18]). In particular, by [18, I. Proposition 3.4] we have

$$G_\alpha(L^2(E, \mu)) \subset D(\mathcal{Q}) \text{ densely,}$$

and

$$\mathcal{E}_\alpha(G_\alpha f, u) = (f, u)_{L^2(E, \mu)},$$

for all $\alpha > 0$, $f \in L^2(E, \mu)$, and $u \in D(\mathcal{Q})$, where $\mathcal{E}_\alpha(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + \alpha(\cdot, \cdot)_{L^2(E, \mu)}$. In case $\mathcal{Q} = 0$ it is considered that $D(\mathcal{Q}) = L^2(E, \mu)$. It further holds by [18, I. Lemma 2.5(ii)] that

$$\mathcal{Q}(u, u) \leq \mathcal{E}(u, u) \quad \text{for all } u \in \mathcal{F},$$

and that

$$\|u\|_{\mathcal{F}}^2 = \|u\|_{D(\mathcal{Q})}^2 + \|\Lambda u\|_{D(\mathcal{Q})'}^2, \quad u \in \mathcal{F}.$$

It follows that for all $u \in \mathcal{F}$ and $v \in D(\mathcal{Q})$

$$\begin{aligned} |\mathcal{E}_1(u, v)| &\leq K' \|u\|_{D(\mathcal{Q})} \|v\|_{D(\mathcal{Q})} + \|\Lambda u\|_{D(\mathcal{Q})'} \|v\|_{D(\mathcal{Q})} \\ &\leq K' \|u\|_{\mathcal{F}} \|v\|_{D(\mathcal{Q})} + \|u\|_{\mathcal{F}} \|v\|_{D(\mathcal{Q})} = (K' + 1) \|u\|_{\mathcal{F}} \|v\|_{D(\mathcal{Q})}. \end{aligned}$$

Let us consider the following assumptions on $(\mathcal{E}, \mathcal{F})$:

(C1) There exists a constant $C > 0$ such that

$$\mathcal{A}_1(u, u) \leq C \mathcal{E}_1(u, u) \quad \text{for all } u \in \mathcal{F}.$$

(C2) $G_\gamma(L^2(E, \mu)_b) \subset D(\mathcal{A})$ densely for some $\gamma > 0$.

Remark 3.1. (a) If (C2) holds for some $\gamma > 0$, then it holds for all $\gamma > 0$.

(b) The assumption that $(\mathcal{E}, \mathcal{F})$ is related to a semi-Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$ by (C1), and (C2) is quite natural. Indeed, if $(\mathcal{Q}, D(\mathcal{Q})) = (\mathcal{A}, D(\mathcal{A}))$ then (C1), and (C2) hold by definition (see [18]). Typical examples where $(\mathcal{Q}, D(\mathcal{Q})) = (\mathcal{A}, D(\mathcal{A}))$ are the time-dependent Dirichlet forms in [15], and [17]. If $\mathcal{Q} = 0$ on $D(\mathcal{Q}) = L^2(E, \mu)$, then (Λ, \mathcal{F}) is the $L^2(\mu)$ -generator of some C_0 -semigroup of contractions on $L^2(\mu)$. In the applications (cf. e.g. [19], [22], [23]) $\langle -\Lambda u, v \rangle$ can uniquely be extended at least for $u, v \in \mathcal{F}$ to $\mathcal{A}^0(u, v) - \mathcal{N}(u, v)$ where \mathcal{A}^0 is a (quasi-regular) semi-Dirichlet form and \mathcal{N} is some (non-sectorial) positive bilinear form which is represented by a μ -divergence free vector field B . The latter means that $\mathcal{N}(u, v) = \int_E \langle B, \nabla u \rangle v \, d\mu$ and $-\mathcal{N}(u, u) = -\frac{1}{2} \int_E \langle B, \nabla u^2 \rangle \, d\mu = 0$ for enough functions u . Consequently, (C1) holds with $\mathcal{A} = \mathcal{A}^0$, and it turns out in [19], [22], [23], that (C2) also holds.

An element u of $L^2(E, \mu)$ is called 1-excessive w.r.t. \mathcal{E} if $\alpha G_{\alpha+1} u \leq u$ (μ -a.e.) for all $\alpha \geq 0$. Let \mathcal{P} denote the 1-excessive elements w.r.t. \mathcal{E} in $D(\mathcal{Q})$. Define $\mathcal{P}_{\mathcal{F}} := \{u \in \mathcal{P} \mid \exists f \in \mathcal{F}, u \leq f\}$. It can be shown that $\mathcal{P}_{\mathcal{F}} = \{u \text{ 1-excessive} \mid \exists f \in \mathcal{F}, u \leq f\}$ (see [18, III. Lemma 2.1(i)]). For a \mathcal{T} -open set U and an element $u \in L^2(E, \mu)$ such that $u \cdot 1_U \leq f$ for some $f \in \mathcal{F}$, let $u_U := e_{u \cdot 1_U}$ be the 1-reduced function of $u \cdot 1_U$ as defined in [18,

Definition III.1.8.]. By [18, III. Proposition 1.7] and [18, III. Lemma 2.1(ii)] $\mathcal{P}_{\mathcal{F}} \ni u_U \leq u$ and $u_U = u$ on U for any $u \in \mathcal{P}_{\mathcal{F}}$.

For $u \in \mathcal{P}_{\mathcal{F}}$ there exists $u_U^\alpha \in \mathcal{F} \cap \mathcal{P}$ such that $(0 \leq) u_U^\alpha \leq u_U^\beta$, $0 < \alpha \leq \beta$, $u_U^\alpha \rightarrow u_U$, $\alpha \rightarrow \infty$, strongly in $L^2(E, \mu)$ and weakly in $D(\mathcal{Q})$ and

$$\mathcal{E}_1(u_U^\alpha, v) = \alpha((u_U^\alpha - u \cdot 1_U)^-, v)_{L^2(E, \mu)} \text{ for any } v \in D(\mathcal{Q}). \quad (3.1)$$

where $(u_U^\alpha - u \cdot 1_U)^-$ denotes the negative part of $u_U^\alpha - u \cdot 1_U$ (see [18, III. Proposition 1.6 and proof of Proposition 1.7]). Moreover, the solutions u_U^α , $\alpha > 0$, to (3.1) are uniquely determined.

An increasing sequence of closed subsets $(F_k)_{k \geq 1}$ is called an \mathcal{E} -nest, if for every function $u \in \mathcal{P} \cap \mathcal{F}$ it follows that $u_{E \setminus F_k} \rightarrow 0$ in $L^2(E, \mu)$ and weakly in $D(\mathcal{Q})$. Since with $\varphi := f_o$ in [18, III. Proposition 2.10]) we have $\text{Cap}_{f_o}(E \setminus F_k) = \int_E (G_1 f_o)_{E \setminus F_k} f_o d\mu$, it follows by the same proposition that $(F_k)_{k \geq 1}$ is an \mathcal{E} -nest if and only if

$$\lim_{k \rightarrow \infty} \text{Cap}_{f_o}(E \setminus F_k) = \lim_{k \rightarrow \infty} \int_E (G_1 f_o)_{E \setminus F_k} f_o d\mu = 0. \quad (3.2)$$

A subset $N \subset E$ is called \mathcal{E} -exceptional if there is an \mathcal{E} -nest $(F_k)_{k \geq 1}$ with $N \subset \bigcap_{k \geq 1} (E \setminus F_k)$.

A property of points in E holds \mathcal{E} -quasi-everywhere (abbreviated \mathcal{E} -q.e.) if the property holds outside some \mathcal{E} -exceptional set.

A function f defined up to some \mathcal{E} -exceptional set $N \subset E$ is called \mathcal{E} -quasi-continuous (\mathcal{E} -q.c.) if there exists an \mathcal{E} -nest $(F_k)_{k \geq 1}$, such that $\bigcup_{k \geq 1} F_k \subset E \setminus N$ and $f|_{F_k}$ is continuous for all k .

For later purposes we state the following definitions (see [18, IV. Definitions 1.4 and 1.7]):

Definition. X is said to be properly associated in the resolvent sense with $(\mathcal{E}, \mathcal{F})$, if $U_\alpha g$ is \mathcal{E} -q.c. for any $g \in L^2(E; \mu) \cap b\mathcal{B}$, and any $\alpha > 0$.

Definition. The generalized Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \mu)$ is called quasi-regular if:

- (i) There exists an \mathcal{E} -nest $(E_k)_{k \geq 1}$ consisting of compact sets.
- (ii) There exists a dense subset of \mathcal{F} whose elements have \mathcal{E} -q.c. μ -versions.
- (iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having \mathcal{E} -q.c. μ -versions \tilde{u}_n , $n \in \mathbb{N}$, and an \mathcal{E} -exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

Lemma 3.2. Let (C1), (C2) hold. Then we have:

- (i) $\mathcal{P}_{\mathcal{F}} \subset D(\mathcal{A})$, i.e. every 1-excessive function w.r.t. \mathcal{E} that is dominated by an element of \mathcal{F} is in $D(\mathcal{A})$.
- (ii) $u_U \in D(\mathcal{A})$, and $\|u_U\|_{D(\mathcal{A})} \leq (\sqrt{C}(K' + 1) + 1)\|f\|_{\mathcal{F}}$ for any $u \in \mathcal{P}_{\mathcal{F}}$ with $u \leq f \in \mathcal{F}$ and any \mathcal{T} -open set U .
- (iii) Every \mathcal{E} -nest is an \mathcal{A} -nest.

Proof. (i) Since $u_E = u$ for any $u \in \mathcal{P}_{\mathcal{F}}$ (i) follows in particular from (ii) if we put $U = E$. (ii) Let $U \in \mathcal{T}$, $u \in \mathcal{P}_{\mathcal{F}}$, $u \leq f \in \mathcal{F}$, and u_U^α for $\alpha > 0$ be the unique solutions of (3.1). It follows by (C1) that

$$\mathcal{A}_1(u_U^\alpha - f, u_U^\alpha - f) \leq C\mathcal{E}_1(u_U^\alpha - f, u_U^\alpha - f).$$

Since $\mathcal{E}_1(u_U^\alpha, u_U^\alpha - f) \leq 0$ by [18, III. Proposition 1.4.(ii)] we get

$$\mathcal{A}_1(u_U^\alpha - f, u_U^\alpha - f) \leq C(K' + 1)\|f\|_{\mathcal{F}}\|u_U^\alpha - f\|_{D(\mathcal{Q})}.$$

for any $\alpha > 0$. This also holds for \mathcal{A} replaced with \mathcal{Q} and $C = 1$, thus $\|u_U^\alpha - f\|_{D(\mathcal{Q})} \leq (K' + 1)\|f\|_{\mathcal{F}}$, and so

$$\|u_U^\alpha - f\|_{D(\mathcal{A})} \leq \sqrt{C}(K' + 1)\|f\|_{\mathcal{F}}.$$

Since $u_U^\alpha \rightarrow u_U$ in $L^2(E, \mu)$ as $\alpha \rightarrow \infty$ it follows from [14, I. Lemma 2.12] that $u_U \in D(\mathcal{A})$ and

$$\mathcal{A}_1(u_U - f, u_U - f) \leq \liminf_{\alpha \rightarrow \infty} \mathcal{A}_1(u_U^\alpha - f, u_U^\alpha - f).$$

Consequently $\|u_U - f\|_{D(\mathcal{A})} \leq \sqrt{C}(K' + 1)\|f\|_{\mathcal{F}}$. It follows

$$\|u_U\|_{D(\mathcal{A})} \leq \|u_U - f\|_{D(\mathcal{A})} + \|f\|_{\mathcal{F}} \leq (\sqrt{C}(K' + 1) + 1)\|f\|_{\mathcal{F}}$$

as desired.

(iii) Let $(F_k)_{k \geq 1}$ be an \mathcal{E} -nest. By definition of \mathcal{E} -nest we have that $g_{E \setminus F_k} \rightarrow 0$ in $L^2(E, \mu)$ as $k \rightarrow \infty$ for any 1-excessive function $g \in \mathcal{F}$. By (ii) we obtain that

$$\sup_{k \geq 1} \|g_{E \setminus F_k}\|_{D(\mathcal{A})} \leq (\sqrt{C}(K' + 1) + 1)\|g\|_{\mathcal{F}} < \infty.$$

Hence we conclude that $g_{E \setminus F_k} \rightarrow 0$ weakly in $D(\mathcal{A})$ as $k \rightarrow \infty$ for any 1-excessive function $g \in \mathcal{F}$. Now let $f \in L^2(E, \mu)_b$ be arbitrary. By the preliminary considerations we know that

$$f_k := G_1 f - (G_1 f^+)_{E \setminus F_k} + (G_1 f^-)_{E \setminus F_k} \in D(\mathcal{A}),$$

and that $f_k \rightarrow G_1 f$ weakly in $D(\mathcal{A})$ as $k \rightarrow \infty$ (f^+ , f^- denote respectively the positive and negative parts of f). By Banach-Saks theorem we know that there is a subsequence $(n_k)_{k \geq 1}$ such that the Cesaro means $v_N := \frac{1}{N} \sum_{k=1}^N f_{n_k}$ converge strongly to $G_1 f$ in $D(\mathcal{A})$. Note that for each N there is some k (e.g. $k = n_N$) such that $v_N \in D(\mathcal{A})_{F_k} := \{u \in D(\mathcal{A}) \mid u = 0 \text{ } \mu\text{-a.e. on } E \setminus F_k\}$. Using (C2), and Remark 3.1(a) we then see that

$$\bigcup_{k \geq 1} D(\mathcal{A})_{F_k} \subset D(\mathcal{A}) \text{ densely.}$$

Thus $(F_k)_{k \geq 1}$ is an \mathcal{A} -nest by [13, Definition 2.9(i)]. \square

Let (as in section 2) $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a Borel right Markov process whose state space is (E, \mathcal{T}) and whose resolvent is $\mathcal{U} = (U_\alpha)_{\alpha > 0}$.

Now, we assume that $U_\alpha f$ is a μ -version of $G_\alpha f$ for any $\alpha > 0$ and $f \in L^2(E, \mu)$, i.e., we assume that the right process X is associated with the generalized Dirichlet form

$(\mathcal{E}, \mathcal{F})$.

As already mentioned in Remark 3.1(b) in [19], [22], [23], assumptions (C1), (C2) hold with $C = 1$ and equality in (C1). This suggests that the capacities corresponding to \mathcal{E} and \mathcal{A} should be equivalent and this is indeed the case in [19], [22], [23]. A general statement, however, even with additional assumptions is yet unshown. Therefore we assume (A2) in the following theorem and repeat that the conditions (A2), (C1), and (C2) are satisfied by the "elliptic" generalized Dirichlet forms given in [19], [22], [23], and moreover by any semi-Dirichlet form. Note however, that even though (C1), and (C2) are trivially satisfied for the time-dependent Dirichlet forms in [15], and [17], (A2) is not.

Theorem 3.3. *Suppose that (A2), (C1), and (C2) hold. Then:*

- (i) (B2) holds for any $\beta > 0$. In particular, the right process X associated to the generalized Dirichlet form $(\mathcal{E}, \mathcal{F})$ is μ -tight μ -standard in the original topology \mathcal{T} . In addition, X is properly associated in the resolvent sense with $(\mathcal{E}, \mathcal{F})$.
- (ii) The generalized Dirichlet form $(\mathcal{E}, \mathcal{F})$ is quasi-regular.

Proof. (i) It is enough to show the statement for $\beta = 1$. Let $\varphi \in L^1(E, \mu)$, $0 < \varphi \leq 1$. Define $u_o := U_1\varphi$. Let u be an \mathcal{U}_1 -excessive function such that $u \leq u_o$. Since $U_1\varphi$ is a μ -version of some element in \mathcal{F} , u is a μ -version of some element in $\mathcal{P}_{\mathcal{F}}$. Thus by Lemma 3.2(i) u is a μ -version of some element in $D(\mathcal{A})$. Since the semi-Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$ is quasi-regular, u has an \mathcal{A} -q.c. μ -version \tilde{u} . In particular by (A2) \tilde{u} is \mathcal{E} -q.c, thus λ -fine w.r.t. \mathcal{U} . Since u is also λ -fine w.r.t. \mathcal{U} and $\tilde{u} = u$ μ -a.e, we have that $\tilde{u} = u$ \mathcal{E} -q.e. But then by Lemma 3.2 (iii) $\tilde{u} = u$ \mathcal{A} -q.e. and therefore u is \mathcal{A} -q.c. It follows that u is μ -fine with respect to \mathcal{V} . Consequently (B2) holds as desired.

The second assertion follows by Theorem 2.7 (ii) and Remark 2.3 (c).

(ii) By (i) we have μ -tightness, and so there exists an increasing sequence $(K_n)_n$ of \mathcal{T} -compact sets such that

$$\lim_{n \rightarrow \infty} \int_E (R_1^{E \setminus K_n} U_1 f_o) f_o d\mu = 0.$$

Since $R_1^{E \setminus K_n} U_1 f_o \leq U_1 f_o$ and $R_1^{E \setminus K_n} U_1 f_o \in \mathcal{E}(\mathcal{U}_1)$ we know that $R_1^{E \setminus K_n} U_1 f_o$ it is a μ -version of some element in $\mathcal{P}_{\mathcal{F}}$. Since moreover $R_1^{E \setminus K_n} U_1 f_o = U_1 f_o$ on $E \setminus K_n$ we conclude by [18, III. Proposition 1.7 (ii)] that $(G_1 f_o)_{E \setminus E_k} \leq R_1^{E \setminus K_n} U_1 f_o$ μ -a.e. Hence by (3.2)

$$\lim_{n \rightarrow \infty} \text{Cap}_{f_o}(E \setminus K_n) = 0,$$

and so there exists an \mathcal{E} -nest of \mathcal{T} -compact sets. As in [18, I. Remark 3.5] we conclude that $G_1(L^2(E; \mu)_b) \subset \mathcal{F}$ densely. By part (i), X is properly associated in the resolvent sense with $(\mathcal{E}, \mathcal{F})$, i.e. $U_1 g$ is \mathcal{E} -q.c. for any $g \in L^2(E; \mu) \cap b\mathcal{B}$. Thus every element of the dense subset $G_1(L^2(E; \mu)_b)$ in \mathcal{F} admits an \mathcal{E} -q.c. μ -version.

Since again X is properly associated in the resolvent sense with $(\mathcal{E}, \mathcal{F})$ we obtain as in [18, IV. Lemma 3.9] that $R_1^U U_1 f_o$ is \mathcal{E} -q.l.s.c. Let $(K_n)_n$ be a sequence of \mathcal{T} -compact sets as at the beginning of this proof. As in [18, IV. Lemma 3.10] we show that $P^x(\lim_{n \rightarrow \infty} D_{E \setminus K_n} <$

$\zeta) = 0$ for \mathcal{E} -q.e. $x \in E$. The countable family of \mathcal{E} -q.c. elements of \mathcal{F} that separates the points of E up to an \mathcal{E} -exceptional set can then e.g. be constructed as in [18, IV. Lemma 3.11 and paragraph below]. □

Remark 3.4. In [14], and [18] the μ -special property of the associated process is used in order to show that the process resolvent is quasi-continuous. Note that we did not use any μ -special property in order to show that X is properly associated in the resolvent sense with $(\mathcal{E}, \mathcal{F})$. This is because no μ -special property is used in the proof of Theorem 3.3(ii).

3.2 Perturbation with kernels

In this subsection again, if not otherwise stated we maintain the notations of section 2. Thus $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is the process resolvent of a quasi-regular semi-Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$ on $L^2(E, \mu)$, and \overline{R}_β^M is the reduction operator on M w.r.t. $\mathcal{V}_\beta = (V_{\beta+\alpha})_{\alpha>0}$.

Let P be a kernel on (E, \mathcal{B}) such that:

(p.1) $Pf \in \mathcal{E}(\mathcal{V})$ for all $f \in p\mathcal{B}$,

(p.2) $1 - P1 \in \mathcal{E}(\mathcal{V})$.

For $\alpha > 0$ define the kernel P_α by

$$P_\alpha f := Pf - \alpha V_\alpha Pf, \quad f \in bp\mathcal{B},$$

and

$$U_\alpha := \left(\sum_{n=0}^{\infty} P_\alpha^n \right) \circ V_\alpha.$$

Let $Q_\alpha := \sum_{n=1}^{\infty} P_\alpha^n$ be the associated α -level "potential kernel". Assume that for some $\beta > 0$ the kernel Q_β is bounded. Then the following assertions hold (see Proposition 5.2.4 and 5.2.5. in [1]):

- (i) The family $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent of a right process with state space E .
- (ii) If $M \in \mathcal{B}$ and R_β^M (resp. \overline{R}_β^M) denotes the kernel on E induced by the reduction operator on M w.r.t. \mathcal{U}_β (resp. w.r.t. \mathcal{V}_β), then

$$R_\beta^M = \overline{R}_\beta^M + Q_\beta \overline{R}_\beta^M - R_\beta^M Q_\beta \overline{R}_\beta^M. \tag{3.3}$$

Examples. A first typical example of perturbing kernel (see [3]) is produced by the β -potential kernel V_A^β of a continuous additive functional $A = (A_t)_{t \geq 0}$ of the process Y associated with \mathcal{V} ,

$$Q_\beta f(x) = V_A^\beta f(x) := E^x \int_0^\infty e^{-\beta t} f(Y_t) dA_t, \quad f \in p\mathcal{B}, x \in E.$$

A second example is emphasized by a potential theoretical approach for the measure-valued discrete branching (Markov) processes; see [5] for details.

Remark 3.5. (a) We have $V_\alpha \leq U_\alpha$ for all $\alpha > 0$ (i.e. \mathcal{V} is subordinate to \mathcal{U} ; see, e.g., [1]) and $\mathcal{E}(\mathcal{U}_\beta) \subset \mathcal{E}(\mathcal{V}_\beta)$ for any $\beta > 0$.

(b) The fine topologies of \mathcal{V} and \mathcal{U} coincide. This clearly implies that conditions (B1) and (B2) are satisfied.

(c) Assume that instead of \mathcal{V} we start with the β -level resolvent \mathcal{V}_β (which is also the process resolvent of a quasi-regular semi-Dirichlet form) for some $\beta > 0$. Then the kernel P_β satisfies conditions (p.1) and (p.2) w.r.t. \mathcal{V}_β and by the resolvent equation we get $(P_\beta)_\alpha := P_\beta - \alpha V_{\beta+\alpha} P_\beta = P_{\beta+\alpha}$ for all $\alpha > 0$. Consequently, the corresponding α -level potential kernel is $Q_{\beta+\alpha}$ which is bounded because $Q_{\beta+\alpha} \leq Q_\beta$ and the induced perturbed resolvent is \mathcal{U}_β .

(d) Recall that a function $v \in \mathcal{E}(\mathcal{U}_\beta)$ is called universally quasi bounded in $\mathcal{E}(\mathcal{U}_\beta)$ provided that for every strictly positive function $u \in \mathcal{E}(\mathcal{U}_\beta)$, there exists a sequence $(v_i)_{i \in \mathbb{N}} \subset \mathcal{E}(\mathcal{V}_\beta)$ such that $v = \sum_{i \in \mathbb{N}} v_i$ and $v_i \leq u$ for all $i \in \mathbb{N}$. We denote by $Qbd(\mathcal{U}_\beta)$ the set of all quasi bounded elements from $\mathcal{E}(\mathcal{U}_\beta)$. The following assertions holds (see [1] for details):

- If $v \in Qbd(\mathcal{U}_\beta)$ and $w \in \mathcal{E}(\mathcal{U}_\beta)$ with $w \leq v$, then $w \in Qbd(\mathcal{U}_\beta)$.
- Every regular \mathcal{U}_β -excessive functions is universally quasi bounded in $\mathcal{E}(\mathcal{U}_\beta)$.
- A function $v \in \mathcal{E}(\mathcal{U}_\beta)$ belongs to $Qbd(\mathcal{U}_\beta)$ if and only if exists a sequence $(v_i)_{i \in \mathbb{N}} \subset \mathcal{E}(\mathcal{U}_\beta)$ such that $v = \sum_{i \in \mathbb{N}} v_i$ and $v_i \leq U_\beta f_o$ for all $i \in \mathbb{N}$.
- If $\beta < \beta'$ then $Qbd(\mathcal{U}_\beta) \subset Qbd(\mathcal{U}_{\beta'})$.

Let ν be any finite measure on (E, \mathcal{B}) which is equivalent with μ . Note that in this subsection neither ν nor λ is assumed to have the density f_o with respect to μ . Let

$$c_\nu^\beta(M) := \inf\{\nu(R_\beta^G p_o) \mid G \in \mathcal{T}, M \subset G\}.$$

as before, but

$$\text{cap}_{\nu, p_o}^\beta(M) := \inf\{\nu(\overline{R}_\beta^G p_o) \mid G \in \mathcal{T}, M \subset G\},$$

where $p_o = U_\beta f_o$ is as in section 2. Note that the second set function $\text{cap}_{\nu, p_o}^\beta$ is defined w.r.t. p_o and not w.r.t. $V_\beta f_o$ as in section 2. We therefore make the following remark:

Remark 3.6. Let $u \in b\mathcal{E}(\mathcal{U}_\beta)$. The functional $M \mapsto c_{\nu, u}^\beta(M)$, $M \subset E$, defined by

$$c_{\nu, u}^\beta(M) = \inf\{\nu(R_\beta^G u) \mid G \in \mathcal{T}, M \subset G\}$$

is a Choquet capacity on (E, \mathcal{T}) . Clearly, if $u = p_o$ then $c_{\nu, u}^\beta = c_\nu^\beta$.

Then the following assertions hold.

(a) Assume that u is a strictly positive bounded function and $u \in Qbd(\mathcal{U}_\beta)$. (Such a function u always exists, e.g., take $u = U_\beta 1$ and use assertion (d) of Remark 3.5.) Then the conclusion of Proposition 2.2 holds if we replace c_ν^β with $c_{\nu, u}^\beta$ and condition (ii.a) with the following one:

(ii.a') The topology \mathcal{T} is a Ray one, $\mathcal{T} = \mathcal{T}_\mathcal{R}$, and $u \in \mathcal{R}$.

The assertion follows since by Proposition 1.6.3 in [1], if (ii.a') holds and $A \in \mathcal{B}$, then $c_{\nu,u}^\beta(A) = \nu(R_\beta^A u)$. Note that Theorem 3.5.2 from [1] may be applied for the capacity $c_{\nu,u}^\beta$ because condition (ii.b) is equivalent with:

(ii.b') every \mathcal{U}_β -excessive function dominated by u is ν -regular.

The equivalence between (ii.b) and (ii.b') is a consequence of the following facts (for details see sections 2.4, 3.1 and 3.2 from [1]): Since $u \in Qbd(\mathcal{U}_\beta)$ and $U_\beta f_o > 0$, there exists a sequence $(u_i)_{i \in \mathbb{N}}$ in $b\mathcal{E}(\mathcal{U}_\beta)$ such that $u = \sum_{i \in \mathbb{N}} u_i$ and $u_i \leq U_\beta f_o$ for all $i \in \mathbb{N}$. The Riesz decomposition property from the cone of potentials $b\mathcal{E}(\mathcal{U}_\beta)$ is also used: if $v \in b\mathcal{E}(\mathcal{U}_\beta)$ and $v \leq \sum_{i \in \mathbb{N}} u_i$, then there exists a sequence $(v_i)_{i \in \mathbb{N}} \subset b\mathcal{E}(\mathcal{U}_\beta)$ such that $v = \sum_{i \in \mathbb{N}} v_i$ and $v_i \leq u_i$ for all $i \in \mathbb{N}$.

(b) Suppose that $p_o = U_\beta f_o$ belongs to $Qbd(\mathcal{V}_\beta)$. Then the assertions from Theorem 2.7 hold if we assume that condition (A2) is satisfied by cap_{ν,p_o}^β and c_ν^β . Indeed, using the above assertion (a) for the resolvent \mathcal{V} instead of \mathcal{U} , we can apply Proposition 2.2 for cap_{ν,p_o}^β , considering the Ray cone \mathcal{R} such that $p_o \in \mathcal{R}$.

By (3.3) it follows for open G that

$$cap_{\nu,p_o}^\beta(G) \leq c_\nu^\beta(G) \leq cap_{\nu+\nu \circ Q_\beta, p_o}^\beta(G). \quad (3.4)$$

The next result shows that condition (A2) holds in the case of perturbation with kernels, allowing a second application of Theorem 2.7.

Proposition 3.7. (i) Let λ be the finite measure defined on E by $\lambda := \nu + \nu \circ Q_\beta$. Then condition (A2) is satisfied by cap_{λ,p_o}^β and c_λ^β . More precisely, if $(G_n)_{n \in \mathbb{N}} \subset \mathcal{T}$ is decreasing then:

$$\inf_{n \in \mathbb{N}} cap_{\lambda,p_o}^\beta(G_n) = 0 \iff \inf_{n \in \mathbb{N}} c_\lambda^\beta(G_n) = 0.$$

(ii) Assume that $Q_\beta 1$ belongs to $Qbd(\mathcal{V}_\beta)$ and that $\mu \circ P \ll \mu$. Then the right process having \mathcal{U} as associated resolvent is μ -standard in the original topology \mathcal{T} .

(iii) Assume that the measure $\mu \circ Q_\beta$ charges no \mathcal{A} -exceptional set. Then condition (A2) is satisfied by cap_{λ,p_o}^β and c_λ^β for any finite measure λ equivalent with μ . If in addition $Q_\beta 1 \in Qbd(\mathcal{V}_\beta)$, then the right process having \mathcal{U} as associated resolvent is μ -standard in the original topology \mathcal{T} .

Proof. (i) The implication " \Leftarrow " is clear since $cap_{\lambda,p_o}^\beta \leq c_\lambda^\beta$. We show now that

$$\lambda \circ Q_\beta \ll \lambda. \quad (3.5)$$

Indeed, if $f \geq 0$ and $\lambda(f) = 0$ then $\nu(f) = 0$ and $\nu(Q_\beta f) = 0$. Since $P_\beta Q_\beta \leq Q_\beta$ we get $0 \leq P_\beta^n Q_\beta f \leq Q_\beta f = 0$ ν -a.e. for any n , hence $Q_\beta(Q_\beta f) = \sum_{n=1}^\infty P_\beta^n Q_\beta f = 0$ ν -a.e, i.e. $\nu \circ Q_\beta(Q_\beta f) = 0$. Hence $\lambda(Q_\beta f) = \nu(Q_\beta f) + \nu \circ Q_\beta(Q_\beta f) = 0$.

Note that

$$\inf_{n \in \mathbb{N}} cap_{\lambda,p_o}^\beta(G_n) = 0 \iff \inf_{n \in \mathbb{N}} \overline{R}_\beta^{G_n} p_o = 0 \quad \lambda\text{-a.e.} \quad (3.6)$$

Let $(G_n)_{n \in \mathbb{N}} \subset \mathcal{T}$ be such that $\inf_{n \in \mathbb{N}} \text{cap}_{\lambda, p_o}^\beta(G_n) = 0$. Then by (3.6) we get that $\inf_{n \in \mathbb{N}} \overline{R}_\beta^{G_n} p_o = 0$ λ -a.e. By (3.5) the last equality holds $(\lambda + \lambda \circ Q_\beta)$ -a.e, hence again by (3.6) we get

$$\inf_{n \in \mathbb{N}} \text{cap}_{\lambda + \lambda \circ Q_\beta, p_o}^\beta(G_n) = 0.$$

From (3.4) applied to λ we conclude that $\inf_{n \in \mathbb{N}} c_\lambda^\beta(G_n) \leq \inf_{n \in \mathbb{N}} \text{cap}_{\lambda + \lambda \circ Q_\beta, p_o}^\beta(G_n) = 0$.

(ii) Let λ be as in (i). Observe that from the assumption $\mu \circ P \ll \mu$ we deduce that the measures λ and μ are equivalent. Since $U_\beta = V_\beta + Q_\beta V_\beta$, we have $p_o = V_\beta f_o + Q_\beta V_\beta f_o$. As a consequence, using the hypothesis on $Q_\beta 1$ and Remark 3.5 (d), it turns out that $p_o \in \text{Qbd}(\mathcal{V}_\beta)$. Remark 3.5 (b) implies that (B2) holds, while by assertion (i) it follows that (A2) is satisfied by $\text{cap}_{\lambda, p_o}^\beta$ and c_λ^β . From Remark 3.6 (b) and Theorem 2.7 (ii) we conclude now that assertion (ii) holds.

(iii) Recall that by definition $\lambda \circ Q_\beta$ charges no \mathcal{A} -exceptional set if $\lambda \circ Q_\beta(M) = 0$ for any $M \subset \bigcap_{k \geq 1} (E \setminus F_k)$, where $(F_k)_{k \geq 1}$ is an \mathcal{A} -nest. Since \mathcal{A} is quasi-regular, this is equivalent to saying that $\lambda \circ Q_\beta$ charges no ν -polar sets w.r.t. \mathcal{V} .

Let $(E_n)_{n \geq 1}$ be a $\text{cap}_{\lambda, p_o}^\beta$ -nest, i.e., $\text{cap}_{\lambda, p_o}^\beta(G_n) \searrow 0$ as $n \rightarrow \infty$, with $G_n := E \setminus E_n$. By the quasi-regularity of \mathcal{A} and monotonicity of $\overline{R}_\beta^{G_n} U_\beta f_o$, there exists a second nest $(\overline{E}_k)_{k \geq 1}$ such that pointwise $\overline{R}_\beta^{G_n} U_\beta f_o \searrow 0$ as $n \rightarrow \infty$ on each \overline{E}_k . Since by hypothesis the finite measure $\lambda \circ Q_\beta$ charges no \mathcal{A} -exceptional set, we get that $(\overline{R}_\beta^{G_n} U_\beta f_o)_n$ is a sequence of bounded functions decreasing to zero $\lambda \circ Q_\beta$ -a.e. Consequently, we have $\lambda \circ Q_\beta(\overline{R}_\beta^{G_n} U_\beta f_o) \searrow 0$ as $n \rightarrow \infty$. By (3.3)

$$R_\beta^{G_n} U_\beta f_o \leq \overline{R}_\beta^{G_n} U_\beta f_o + Q_\beta \overline{R}_\beta^{G_n} U_\beta f_o,$$

hence

$$c_\lambda^\beta(G_n) \leq \text{cap}_{\lambda, p_o}^\beta(G_n) + \int_E Q_\beta \overline{R}_\beta^{G_n} U_\beta f_o(x) \lambda(dx).$$

It follows that $c_\lambda^\beta(G_n) \searrow 0$ as $n \rightarrow \infty$, i.e., $(E_n)_{n \geq 1}$ is a c_λ^β -nest.

The proof of the last assertion of (iii) is similar to that of (ii). □

3.3 Quasi-regularity of generalized Dirichlet forms obtained by perturbation with kernels

In this subsection we want to show that there exists a quasi-regular generalized Dirichlet form that is associated to the perturbation with kernels of the quasi-regular semi-Dirichlet form $(\mathcal{A}, D(\mathcal{A}))$ on $L^2(E, \mu)$. Our results are in particular related to [16, Remark 3.3.(iv)].

Let $\beta > 0$ be as in subsection 3.2.

Definition. We say that \mathcal{V} satisfies the **absolute continuity condition** if

$$V_\beta(x, \cdot) \ll \mu \text{ for all } x \in E,$$

i.e. $V_\beta(x, \cdot)$ is absolutely continuous w.r.t. μ for each x .

Remark. In finite dimensions the absolute continuity condition is typically guaranteed through embedding theorems of (weighted) Sobolev spaces in Hölder spaces. For instance, it is satisfied if $H_\alpha f$ (the $L^p(E, \mu)$ -version of $V_\alpha f$), admits a Hölder continuous μ -version for any $\alpha > 0$ and $f \in L^p(E; \mu)$.

For the rest of the section we assume that E is a separable (and metrizable) space.

Let $\{x_k; k \geq 1\} \subset E$ be any dense subset and define the finite measure ν w.r.t. $\{x_k; k \geq 1\}$ as

$$\nu := \sum_{k \geq 1} 2^{-k} \delta_{x_k}.$$

Consider now the potential \mathcal{V}_β -excessive measure ξ (resp. the \mathcal{U}_β -excessive measure η) defined as

$$\xi := \nu \circ V_\beta \quad (\text{resp. } \eta := \nu \circ U_\beta).$$

Let further

$$p_t f(x) := p_t(x, f) := E^x[f(X_t)], \quad x \in E, \quad t \geq 0, \quad f \in p\mathcal{B},$$

denote the transition semigroup of the right process $(X_t)_{t \geq 0}$ with state space E , corresponding to $\mathcal{U} = (U_\alpha)_{\alpha > 0}$.

The right process $(X_t)_{t \geq 0}$ is said to be *transient*, if

- (**T**) there exists $\varphi > 0$, universally Borel measurable and $U_0 \varphi(x) = E^x[\int_0^\infty \varphi(X_t) dt] < \infty$ for all $x \in E$.

For more details about this hypothesis see, e.g., [10].

If (**T**) holds we define the \mathcal{U} -excessive measure η_0 on E as

$$\eta_0 := \nu_0 \circ U_0, \quad \text{where} \quad \nu_0 := \sum_{k \geq 1} \frac{2^{-k}}{U_0 \varphi(x_k)} \delta_{x_k}.$$

Remark 3.8. (a) Clearly, ξ and η are finite measures while η_0 is σ -finite, provided that (**T**) holds. In addition, all these measures have full support.

(b) Assume that (**T**) holds. If W is a kernel on E , then because ν and ν_0 are equivalent measures, it follows that the measures $\nu \circ W$ and $\nu_0 \circ W$ are also equivalent. It is also clear that if κ is a σ -finite measure then $\kappa \circ U_0$ and $\kappa \circ U_\alpha$ are equivalent for any $\alpha > 0$. In particular, all the measures $\eta = \nu \circ U_\beta$, $\nu_0 \circ U_\beta$, $\eta_0 = \nu_0 \circ U_0$, and $\nu \circ U_0$ are mutually equivalent

(c) By the complete maximum principle (cf. e.g. [10, (2.2) Proposition]), it is possible to choose the function φ such that $U_0 \varphi$ is bounded. Consequently, the measure $\lambda \circ U_0$ is σ -finite for every finite measure λ on E . Therefore, one could consider the measure $\nu \circ U_0$, without normalizing constants $U_0 \varphi(x_k)$, instead of η_0 . This measure is \mathcal{U} -excessive and could have been equally used in what follows. However, we want to apply results from [16] and these are given w.r.t. the measure η_0 .

Define

$$p_t^\alpha := e^{-\alpha t} p_t; \quad t \geq 0, \alpha \geq 0. \quad (3.7)$$

It is known from [16, Proposition 2.4] that there are unique extensions $(p_t^\beta)_{t \geq 0}$ on $L^2(E; \eta)$ (resp. $(p_t)_{t \geq 0}$ on $L^2(E; \eta_0)$ in the transient case) as strongly continuous semigroups of contractions on the respective L^2 -spaces. Moreover the adjoint semigroup of the respective extensions on the L^2 -spaces are sub-Markovian (see [16, Remark 3.1(ii)]). Using these extended semigroups one can define uniquely the L^2 -generator which in turn determines a generalized Dirichlet form (see [16, Section 3]). We shall denote these generalized Dirichlet forms by $(\mathcal{E}^\beta, \mathcal{F}^\beta)$, resp. $(\mathcal{E}^0, \mathcal{F}^0)$. So, in particular \mathcal{U}_β is associated to $(\mathcal{E}^\beta, \mathcal{F}^\beta)$, and \mathcal{U} is associated to $(\mathcal{E}^0, \mathcal{F}^0)$ in the transient case.

In the next lemma we will assume that the set $\{x_k; k \geq 1\}$ has a special form as follows: since the semi-Dirichlet form with process resolvent \mathcal{V} is quasi-regular there exists a nest $(F_k)_{k \geq 1}$ of compacts. Since E is separable each F_k is also separable. Then choose for each F_k a countable dense set $\{x_n^k; n \geq 1\}$, and a countable dense set $\{x_n^0; n \geq 1\}$ in E . Let

$$\{y_n; n \geq 1\} := \bigcup_{k \geq 0} \{x_n^k; n \geq 1\}.$$

$\{y_n; n \geq 1\}$ is a special countable and dense subset of E . We have the following:

Lemma 3.9. *Suppose that the absolute continuity condition holds.*

(i) *It always holds $\xi \ll \eta \ll \mu$.*

(ii) *Let ν be defined w.r.t. the special dense subset $\{y_n; n \geq 1\}$. Then ξ, η , and μ are mutually equivalent measures.*

Proof. (i) Clearly $\xi \ll \eta$. If $\mu(N) = 0$ then $V_\beta 1_N = 0$ by the absolute continuity condition and thus $U_\beta 1_N = (I + Q_\beta)V_\beta 1_N = 0$, and so $\eta(N) = \nu(U_\beta 1_N) = 0$. Hence $\eta \ll \mu$.

(ii) Let $\xi(N) = 0$. Then $V_\beta 1_N(x_n^k) = 0$ for all n, k . Since μ is σ -finite we may assume that $1_N \in L^2(E, \mu)$, otherwise we choose $D_l \nearrow E$ with $\mu(D_l) < \infty$ for any l and show the following for $N \cap D_l$ and any l . By \mathcal{A} -quasi-continuity of $V_\beta 1_N$ there exists an \mathcal{A} -nest $(E_k)_{k \geq 1}$ such that $V_\beta 1_N$ is continuous on each E_k hence on each $\overline{F}_k := E_k \cap F_k$. Therefore by approximation $V_\beta 1_N = 0$ on $\bigcup_{k \geq 1} \overline{F}_k$. It follows that $V_\beta 1_N = 0$ \mathcal{A} -quasi-everywhere and so, $\mu(N) = 0$ since $(V_\alpha)_{\alpha > 0}$ is a C_0 -resolvent on $L^2(E, \mu)$. \square

The following result offers the claimed example of quasi-regular generalized Dirichlet form obtained by perturbation with kernels. It is a corollary of Proposition 3.7.

Corollary 3.10. *Let η , and in the transient case η_0 , be defined w.r.t. any dense subset $\{x_n; n \geq 1\} \subset E$, suppose that the absolute continuity condition holds, and that $Q_\beta 1 \in Qbd(\mathcal{V}_\beta)$. Consider the assumptions:*

(i) $\mu \circ P \ll \mu$.

(ii) *The measure $\mu \circ Q_\beta$ charges no \mathcal{A} -exceptional set.*

Suppose that either (i) or (ii) holds. Then the generalized Dirichlet form $(\mathcal{E}^\beta, \mathcal{F}^\beta)$ associated with \mathcal{U}_β on $L^2(E, \eta)$ is quasi-regular. If (\mathbf{T}) holds then the generalized Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ associated with \mathcal{U} on $L^2(E, \eta_0)$ is quasi-regular. Moreover in either case the corresponding process is properly associated in the resolvent sense with the corresponding generalized Dirichlet form.

Proof. By Lemma 3.9 we have $\eta \ll \mu$ and by Remark 3.8(b) η is equivalent to η_0 if (\mathbf{T}) holds, hence we also have $\eta_0 \ll \mu$ if (\mathbf{T}) holds. We have $\mu \circ P_\beta \ll \mu \circ P$, and \mathcal{A} -exceptional sets are \mathcal{A}_β -exceptional. According to assertions (ii) and (iii) of Proposition 3.7 applied to \mathcal{V}_β instead of \mathcal{V} (this is possible taking into account assertions (c) and (d) of Remark 3.5; in particular, from the hypothesis on $Q_\beta 1$ we get $Q_{2\beta} 1 \leq Q_\beta 1 \in \text{Qbd}(\mathcal{V}_\beta) \subset \text{Qbd}(\mathcal{V}_{2\beta})$ and thus $Q_{2\beta} 1 \in \text{Qbd}(\mathcal{V}_{2\beta})$), the processes corresponding to \mathcal{U}_β are μ -standard, hence η -standard since $\eta \ll \mu$. From Remark 2.3(c) they are also η -tight. The quasi-regularity as well as the proper association with the forms $(\mathcal{E}^\beta, \mathcal{F}^\beta)$ follows with the help of Theorem 2.7(ii) as in the proof of Theorem 3.3(i) and (ii). The transient case is similar. \square

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