

# Metric Properties of Euclidean Buildings

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## Abstract

This is a survey on nondiscrete euclidean buildings, with a focus on metric properties of these spaces.

Euclidean buildings are higher dimensional generalizations of trees. Indeed, the euclidean product  $X$  of two (leafless) metric trees  $T_1, T_2$  is already a good “toy example” of a 2-dimensional euclidean building. The space  $X$  contains lots of copies of the euclidean plane  $\mathbb{E}^2$  and has at the same time a complicated local branching.

Euclidean buildings were invented by Jacques Tits in the seventies. Similarly as in the case of spherical buildings, their definition was motivated by group theoretic questions. While spherical buildings are by now a standard tool in the structure theory of reductive algebraic groups over arbitrary fields, euclidean buildings are important for the advanced structure theory of reductive groups over fields with valuations. In particular, they are very much linked to number theory and arithmetic geometry.

In the last 25 years, however, euclidean buildings have also become important in geometry. This is due to the fact that euclidean buildings are spaces of nonpositive curvature. But more is true. Together with the Riemannian symmetric spaces of nonpositive curvature, euclidean buildings could be called the islands of high symmetry in the world of CAT(0) spaces. This claim will be made more precise below. Almost inevitably, questions about symmetry, rigidity, or higher rank for CAT(0) spaces lead to these geometries.

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## 1 The definition of euclidean buildings

We first recall Tits’ definition of a euclidean building. For more details, proofs and further results see Tits [35], Kleiner and Leeb [17], Kramer and Weiss [22] and in particular Parreau [25]. (The axioms used by Kleiner and Leeb [17] look somewhat different from Tits’ definition. They were shown to be equivalent to Tits’ by Parreau.)

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## 1.1 Euclidean buildings

Let  $W$  be a spherical Coxeter group acting in its natural orthogonal representation on euclidean space  $\mathbb{E}^m$ . We call the semidirect product  $W\mathbb{R}^m$  of  $W$  and  $(\mathbb{R}^m, +)$  the *affine Weyl group*. From the reflection hyperplanes of  $W$  we obtain a decomposition of  $\mathbb{E}^m$  into *walls*, *half spaces*, *Weyl chambers* (a Weyl chamber is a fundamental domain for  $W$ —these are Tits’ *chambres vectorielles*) and *Weyl simplices* (Tits’ *facettes vectorielles*).

Let now  $X$  be a metric space. A *chart* is an isometric embedding  $\varphi : \mathbb{E}^m \longrightarrow X$ , and its image is called an (affine) *apartment*. We call two charts  $\varphi, \psi$  *W-compatible* if  $Y = \varphi^{-1}(\psi(\mathbb{E}^m))$  is convex (in the Euclidean sense) and if there is an element  $w \in W\mathbb{R}^m$  such that  $\psi \circ w|_Y = \varphi|_Y$  (this condition is void if  $Y = \emptyset$ ). We call a metric space  $X$  together with a collection  $\mathcal{A}$  of charts a *Euclidean building* if it has the following five properties.

(A1) For all  $\varphi \in \mathcal{A}$  and  $w \in W\mathbb{R}^m$ , the composition  $\varphi \circ w$  is in  $\mathcal{A}$ .

(A2) The charts are  $W$ -compatible.

(A3) Any two points  $p, q \in X$  are contained in some affine apartment.

The charts allow us to map Weyl chambers, walls and half spaces into  $X$ ; their images are also called Weyl chambers, walls and half spaces. The first three axioms guarantee that these notions are coordinate independent.

(A4) If  $C, D \subseteq X$  are Weyl chambers, then there is an affine apartment  $A$  such that the intersections  $A \cap C$  and  $A \cap D$  contain Weyl chambers.

(A5’) For every apartment  $A \subseteq X$  and every  $p \in A$  there is a 1-Lipschitz retraction  $h : X \longrightarrow A$  with  $h^{-1}(p) = \{p\}$ .

Condition (A5’) may be replaced by the following condition:

(A5) If  $A_1, A_2, A_3$  are affine apartments which intersect pairwise in half spaces, then  $A_1 \cap A_2 \cap A_3 \neq \emptyset$ .

See [25] for a thorough discussion of different sets of axioms.

Axiom (A5’) yields a 1-Lipschitz map  $v : X \longrightarrow \mathbb{E}^m/W$  as follows. We identify  $(A, p)$  by means of a coordinate chart  $\varphi$  with  $(\mathbb{E}^m, 0)$ , and then we quotient out the  $W$ -action. The result  $v(q)$  is called the *vector distance* between  $p$  and  $q$ .

The definition of a euclidean building that we use here is the metric, “nondiscrete” version. It appeared implicitly in [6], and in detail in [35]. There is also the (older) notion of a simplicial affine building; see [1] and in particular [37]. The geometric realization of such a combinatorial affine building is always a euclidean building in our sense (but not vice versa); see §11.2 in [1].

An important invariant of a euclidean building is its *spherical building at infinity*,  $\partial_{\mathcal{A}}X$ . This is a combinatorial simplicial complex which is defined as follows. The simplices are equivalence classes of Weyl simplices. Two Weyl simplices are considered to be equivalent if they have finite Hausdorff distance. It turns out that  $\partial_{\mathcal{A}}X$  is a (weak) *spherical building* of dimension

$n - 1$  (resp., of rank  $n$ ) [25, §1.5]. We refer to [33] and [1] for the definition of a simplicial spherical building.

Another important fact is that a euclidean building is always a CAT(0) space; see [25, §2.3]. This was first observed by Bruhat and Tits in [6], where they proved the CN-inequality.

*Automorphisms* of euclidean buildings are defined in the obvious way; they are bijections which preserve the charts in the given atlas. Clearly, every automorphism is an isometry of  $X$ .

## 2 Basic and less basic properties

We assume that  $X$  is a euclidean building with  $m$ -dimensional apartments. First of all, we remark that the atlas  $\mathcal{A}$  is by no means unique. However, Parreau proved that there is a unique maximal atlas  $\mathcal{A}_{max}$  containing  $\mathcal{A}$ ; see [25, §2.6]. The apartments in the maximal atlas have a simple characterization.

**2.1 Theorem ([25, 2.6] and [17, §4.6])** *Let  $X$  be a euclidean building with  $m$ -dimensional apartments. Suppose that  $F \subseteq X$  is a subspace isometric to some  $\mathbb{E}^\ell$ . Then there exists an apartment in the maximal atlas containing  $F$ . In particular,  $\ell \leq m$  and the apartments of the maximal atlas are precisely the maximal flats in  $X$ .*

The metric realization of the spherical building  $\partial_{\mathcal{A}_{max}} X$  can be identified in a canonical way with the Tits boundary of the CAT(0) space  $X$ . We remark that the dimension  $m$  of the apartments coincides with the covering dimension of  $X$  as a topological space; see [23, Prop. 3.3] or [20, Thm. B]. Moreover,  $X$  is an AR (an absolute retract for the class of metric spaces).

Next, we note that the Weyl group may be “too big”: there might be types of walls which never appear as branchings between apartments. A wall  $M$  in a euclidean building  $X$  is called *thick* if it can be written as the intersection of three apartments. We call a point  $p \in X$  *thick* if every wall passing through  $p$  is thick. Now we can make the statement about the Weyl group being too big more precise: if  $X$  contains no thick points, then there is a (unique) euclidean building  $X_{th}$  (with a smaller Weyl group) containing a thick point, and  $X$  is a euclidean product  $X \cong X_{th} \times \mathbb{E}^k$ , for some  $k \geq 0$ ; see [17, §4.9] and [22, §10]. For the thick part  $X_{th}$ , there is the following trichotomy.

**2.2 Proposition ([22, §10])** *Let  $X$  be an irreducible euclidean building of dimension  $m \geq 2$  containing a thick point. Then there are the following three possibilities.*

- (I) *There is a unique thick point which is contained in every affine apartment of  $X$ . In this case  $X$  is a euclidean cone over a spherical building.*
- (II) *The set of thick points is a closed, discrete and cobounded subset in  $X$  and in every apartment of  $X$ . Then  $X$  is the geometric realization of a simplicial affine building.*
- (III) *The set of thick points is dense in  $X$  and in every apartment of  $X$ .*

A simplicial affine building (type II) is called *thick* if every vertex of the simplicial structure is thick.

There are many 2-dimensional euclidean buildings. In fact, there are “free constructions” which show that it is impossible to classify these spaces. In higher dimensions, the picture is

completely different. We call a euclidean building  $X$  a *Bruhat-Tits building* if the spherical building at infinity is a *Moufang building*; see [36]. Roughly speaking, the Moufang condition says that there are certain automorphisms, called *root automorphisms*, that fix a large subset pointwise, and yet act transitively on another subset. The following deep result is again due to Tits [35].

**2.3 Theorem** *Let  $X$  be an irreducible euclidean building of dimension  $m \geq 3$  containing a thick point. Then  $\partial_{\mathcal{A}}X$  is Moufang, and all root automorphisms of  $\partial_{\mathcal{A}}X$  extend to isometries of  $X$ . In particular, the isometry group of  $X$  is transitive on the apartments of  $X$ .*

Tits' article [35] contains in fact a complete classification of these buildings in terms of algebraic data. We remark that if a Bruhat-Tits building is not of type (I), the group generated by the root automorphisms acts with cobounded orbits on  $X$ .

It is by no means clear that every combinatorial automorphism of  $\partial_{\mathcal{A}}X$  extends to an isometry of  $X$ . Surprisingly, the following is true.

**2.4 Theorem ([37, 27.6])** *Let  $X$  be a thick irreducible simplicial Bruhat-Tits building of dimension  $m \geq 2$ . Then every automorphism of  $\partial_{\mathcal{A}_{max}}X$  extends to an isometry of  $X$ . Moreover,  $\partial_{\mathcal{A}_{max}}X$  determines  $X$  up to isomorphism.*

The proof depends on the purely algebraic fact that a field admits at most one discrete complete valuation. It would be interesting to have a geometric proof for this. More generally, there is the following open problem.

**2.5 Question** *Is a thick irreducible simplicial affine building of dimension  $m \geq 2$  uniquely determined by the spherical building  $\partial_{\mathcal{A}_{max}}X$ ? Does every combinatorial automorphism of  $\partial_{\mathcal{A}_{max}}X$  extend to  $X$ ?*

The answer is negative if  $X$  is not assumed to be simplicial.

### 3 Characterizations

The following very general characterization of locally finite (simplicial) euclidean buildings is due to Kleiner.

**3.1 Theorem** *Let  $X$  be a locally compact  $CAT(0)$  space of dimension  $m$ . Suppose that any two points  $x, y \in X$  are contained in some flat  $A \cong \mathbb{E}^m$ . Then  $X$  is a euclidean building.*

This result was not published by Kleiner; a proof was given by Balser and Lytchak in [3, Cor. 1.7]. The dimension may be taken to be the covering dimension; since  $X$  is locally compact, the covering dimension coincides with Kleiner's geometric dimension [16]. The following example shows that local compactness is crucial.

**3.2 Example** Let  $\Gamma_n$ , for  $n \geq 3$ , be a family of thick generalized  $n$ -gons (1-dimensional spherical buildings whose Weyl group is dihedral of order  $2n$ ). Such generalized  $n$ -gons exist by

Tits's free construction [34], see also [32]. Let  $X_n$  be the euclidean cone over  $\Gamma_n$ , with cone point  $o_n$ . Then  $X_n$  is a 2-dimensional euclidean building with precisely one thick point. Now consider the asymptotic cone  $X$  over the family  $\{(X_n, o_n) \mid n \geq 3\}$  (with respect to a constant scaling sequence and a nonprincipal ultrafilter  $\mu$  on the index set  $\mathbb{N}_{\geq 3}$ ). Then  $X$  is a complete CAT(0) space. Any two points in  $X$  are contained in some copy of  $\mathbb{E}^2$ . The “spherical Weyl group”  $W$  that describes the transition functions between these “apartments” is, however, the orthogonal group  $W = O(2)$ . Using a similar argument as in [20, §7], one can show that  $X$  is 2-dimensional. But  $X$  is certainly not a euclidean building. Instead of the cones  $X_n$ , one could also use the euclidean buildings constructed recently by Berenstein and Kapovich [4] in order to get a more interesting asymptotic cone  $X$ .

In a somewhat more combinatorial setting, there is the following result of Charney and Lytchak. A CAT(0) space  $X$  has the *discrete extension property* if for every geodesic  $\gamma = [a, b] \subseteq X$ , the set of the directions of geodesics extending  $\gamma$  beyond  $b$  is nonempty and discrete.

**3.3 Theorem** *Let  $X$  be a CAT(0) space of dimension  $m \geq 2$  which is a piecewise euclidean cell complex. If  $X$  has the discrete extension property, then  $X$  is a euclidean building.*

We remark that a locally compact euclidean building always admits a euclidean cell structure. This is not true for general euclidean buildings. Finally, we should mention here the following result by Leeb [24].

**3.4 Theorem** *Let  $X$  be a locally compact CAT(0) space with extendible geodesics. If the Tits boundary of  $X$  is an irreducible spherical building of rank at least 2, then  $X$  is either a Riemannian symmetric space of noncompact type or a simplicial euclidean building.*

## 4 Isometries and automorphisms

If  $X$  is a euclidean building containing a thick point, then an isometry of  $X$  is almost the same as an automorphism.

**4.1 Theorem** ([25, §4]) *Let  $g$  be an isometry of a euclidean building  $X$ . Assume that  $X$  contains a thick point. Then there exists an element  $\gamma \in \text{Nor}_{O(m)}(W)$  such that  $g \circ \psi \circ \gamma \in \mathcal{A}_{max}$  holds for all  $\psi \in \mathcal{A}_{max}$ .*

Such a map  $\gamma$  induces a diagram automorphism of the Coxeter group  $W$ ; one also calls such a  $g$  a *non-type-preserving automorphism*.

Suppose that  $g$  is an isometry of a metric space  $(X, d)$ . The *displacement function* of  $g$  is the nonnegative real function  $d_g : x \mapsto d(x, g(x))$ . The infimum of  $d_g(X)$  is the *translation length*  $l_g$  of  $g$ . We call an isometry  $g$

- elliptic* if  $g$  has a fixed point.
- hyperbolic* if  $d_g$  attains a positive minimum.
- parabolic* if  $d_g$  does not attain its minimum.

If  $X$  is a Riemannian symmetric space of nonpositive curvature, then all three types of isometries appear in the isometry group. This is not true for euclidean buildings.

**4.2 Theorem** ([25, §4]) *Let  $g$  be an isometry of a complete euclidean building  $X$  containing a thick point. Then  $g$  is either elliptic or hyperbolic.*

The next result was proved by Rapoport and Zink [27] for the Bruhat-Tits building of  $GL_n$  over a field with discrete valuation, and then extended using Landvogt's Embedding Theorem to other Bruhat-Tits buildings. However, there is a much simpler proof using CAT(0) geometry, which applies to all euclidean buildings, cp. [29]—the author found a somewhat simpler proof (unpublished). We put  $X_r = \{q \in X \mid d_g(q) \leq r\}$ . These sublevel sets form a filtration of  $X$  by convex sets.

**4.3 Theorem** *Let  $g$  be an isometry of a complete euclidean building  $X$  containing a thick point. There exists a positive constant  $c > 0$  (depending only on the Weyl group  $W$ ) such that the following holds. If  $p$  is a point with  $d(p, X_r) = t > 0$ , then*

$$c \cdot t + r \leq d_g(p) \leq 2t + r.$$

The second inequality is trivial, the interesting fact is the lower estimate. We finally note the following (completely elementary) fact.

**4.4 Lemma** *Let  $g$  be a nontrivial isometry of a euclidean building  $X$  containing a thick point. Then  $\sup d_g(X) = \infty$ .*

*Proof.* Suppose  $r = \sup d_g(X) < \infty$ . If  $A$  is an apartment in  $X$ , then  $g(A)$  has Hausdorff distance at most  $r$  from  $A$ . Then  $A$  and  $g(A)$  have the same boundary at infinity. By [25, p. 10],  $A = g(A)$ . Thus  $g$  fixes all apartments setwise, and therefore all thick walls and thick points. Since every apartment contains a thick point,  $g$  fixes every apartment pointwise. Thus  $g = id_X$ .  $\square$

We end this section with some remarks on noncomplete euclidean buildings. Struyve recently proved the following generalization of the Bruhat-Tits Fixed Point Theorem. If a finitely generated group acts isometrically and with bounded orbits on a euclidean building, then it has a fixed point [31]. Moreover, he showed that the main rigidity results in [22] also hold if the completeness assumptions on the euclidean buildings are dropped (unpublished). Finally, he and Schillewaert extended results in [6] about noncomplete Bruhat-Tits buildings, by giving algebraic conditions on the underlying fields (unpublished).

## 5 Kostant convexity

We first recall the statement of Kostant's Convexity Theorem [18] for Riemannian symmetric spaces. Let  $G$  be a simple noncompact Lie group with Iwasawa decomposition  $G = KAU$ . The group  $K$  is maximal compact,  $A$  is diagonalizable, and  $U$  is unipotent. The group  $W = \text{Nor}_K(A)/\text{Cen}_K(A)$  is the associated Weyl group.

The solvable group  $AU$  acts regularly on the Riemannian symmetric space  $X = G/K$ . Let  $o \in X$  denote the point stabilized by  $K$ . The  $A$ -orbit  $E = A(o) \subseteq X$  is a maximal flat in  $X$ . The projection  $AU \longrightarrow AU/U \cong A$  induces a natural 1-Lipschitz map  $\rho_U : X \longrightarrow E$  which we call the *Iwasawa projection*. Let  $p \in E$ . The Convexity Theorem says that

$$\rho_U(K(p)) = \text{conv}(W(p)),$$

the image of the  $K$ -orbit of  $p$  in  $X$  under the Iwasawa projection is the convex hull of the  $W$ -orbit of  $p$  in  $E$ .

Geometrically, the Iwasawa decomposition can also be described as follows. The group  $U$  determines a chamber  $C$  of the spherical building at infinity of  $X$ . The maximal flats in  $X$  containing  $C$  in their boundary form a foliation of  $X$ . The Iwasawa projection identifies each leaf by means of the  $U$ -action with the leaf  $A(o)$ .

Suppose now that  $X$  is a euclidean building and that  $C$  is a chamber at infinity. We fix an apartment  $E \subseteq X$  containing  $C$  in its boundary. If  $E' \subseteq X$  is any other apartment containing  $C$  in its boundary, then  $E \cap E'$  contains a Weyl chamber representing  $C$ . Thus, there is a canonical isometry  $E' \longrightarrow E$  fixing  $E \cap E'$  pointwise. These isometries fit together to a 1-Lipschitz retraction  $\rho_C : X \longrightarrow E$ . Suppose now that  $o, p \in E$  are vertices. Let  $S \subseteq X$  denote the set of all special vertices in  $X$  that have the same vector distance from  $o$  as  $p$ . This set  $S$  corresponds to the orbit  $K(p)$  in the Riemannian symmetric case. If the euclidean building  $X$  happens to be a Bruhat-Tits building, then  $S$  is indeed the  $K$ -orbit of  $p$ , where  $K$  is the stabilizer of  $p$ . The following result was proved by Hitzelberger [13] in 2007.

**5.1 Theorem** *Suppose that  $X$  is a thick simplicial euclidean building. With the same notation as above, suppose that  $o, p$  are special vertices (see [6] for the definition of a special vertex). Then*

$$\rho_C(S) = \{q \in \text{conv}(W(p)) \mid q \text{ has the same type as } p\},$$

*the image of  $S$  is the set of all vertices in  $E$  which are in the convex hull of the  $W$ -orbit of  $p$  in  $E$  and have the same type as  $p$ .*

This result had been announced by Silberger [30] for the special case that  $X$  is the Bruhat-Tits building of a simple  $p$ -adic algebraic group (but the proof, which relied on a case-by-case analysis, was never published). The difficult part of the proof is to show that the map is onto. For the special case of Bruhat-Tits buildings, the theorem may be restated as a fact about intersections of certain double cosets in the group. The result was recently extended by Hitzelberger to general euclidean buildings [14].

## 6 Rigidity

We first recall some notions from coarse geometry [28]. A map  $f : X \longrightarrow Y$  between metric spaces is called *controlled* if there is a monotonic real function  $\rho : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$  such that

$$d_Y(f(x), f(y)) \leq \rho(d_X(x, y))$$

holds for all  $x, y \in X$ . If in addition the preimage of every bounded set is bounded, then  $f$  is called a *coarse map*. Neither  $f$  nor  $\rho$  is required to be continuous. Note that the image of a bounded set under a controlled map is bounded. Two maps  $g, f : X \rightrightarrows Y$  between metric spaces have *finite distance* if the set  $\{d_Y(f(x), g(x)) \mid x \in X\}$  is bounded. This is an equivalence relation which leads to the *coarse metric category* whose objects are metric spaces and whose morphisms are equivalence classes of coarse maps. A *coarse equivalence* is an isomorphism in this category. We remark that a coarse equivalence between geodesic metric spaces is the same as a quasi-isometric equivalence.

Prasad proved in 1978 the following analog of Mostow's Rigidity Theorem.

**6.1 Theorem ([26])** *Let  $X$  and  $Y$  be thick simplicial, irreducible and locally finite Bruhat-Tits buildings of rank at least 2. Suppose that a group  $\Gamma$  acts cocompactly and properly discontinuously on both spaces. Then there is a  $\Gamma$ -equivariant simplicial isomorphism between  $X$  and  $Y$ .*

The group  $\Gamma$  appearing in Prasad's Theorem is finitely presentable. From the  $\Gamma$ -action, one obtains a  $\Gamma$ -equivariant coarse equivalence  $f : X \longrightarrow Y$  which plays a crucial role in the proof. About twenty years later, Kleiner and Leeb [17] proved the following generalization of Prasad's Theorem.

**6.2 Theorem ([17])** *Let  $X$  and  $Y$  be complete Bruhat-Tits buildings whose de Rham factors all have rank at least 2. Suppose that  $f : X \longrightarrow Y$  is a coarse equivalence. Then there is an isometry  $\bar{f} : X \longrightarrow Y$  (possibly after rescaling the metrics on the de Rham factors of  $Y$ ) which has finite distance from  $f$ .*

The strategy of their proof is roughly as follows. Using *asymptotic cones*, Kleiner and Leeb show that the  $f$ -image of a maximal flat  $E \subseteq X$  has finite Hausdorff distance from a (necessarily unique) maximal flat  $E' \subseteq Y$ . This fact is then used to set up a one-to-one correspondence between the maximal bounded subgroups of the isometry groups of the two Bruhat-Tits buildings. The maximal bounded subgroups, in turn, correspond to (certain) points in the buildings. In this way, they construct an equivariant isometry.

Weiss and the author proved in 2009 a more general result which is valid for all euclidean buildings.

**6.3 Theorem ([22, Thm. III])** *Let  $X$  and  $Y$  be complete euclidean buildings containing thick points, and without rank 1 de Rham factors. Suppose that  $f : X \longrightarrow Y$  is a coarse equivalence. Then there is an isometry  $\bar{f} : X \longrightarrow Y$ . If no de Rham factor of  $X$  is a euclidean cone, then  $f$  has finite distance from  $\bar{f}$ .*

The proof relies, among other things, on the following result about trees.

**6.4 Theorem ([22, Thm. I])** *Let  $T, T'$  be two complete  $\mathbb{R}$ -trees without leaves. Suppose that a group  $G$  acts isometrically on both trees, and that this action is 2-transitive on the ends. Suppose that  $f : T \longrightarrow T'$  is a coarse equivalence whose induced boundary map  $\partial T \longrightarrow \partial T'$  is  $G$ -equivariant. Then  $T$  and  $T'$  are  $G$ -equivariantly isometric.*



The proof of 6.3 proceeds roughly as follows. The first step is a result due to Kleiner and Leeb which was already mentioned: the  $f$ -image of an apartment  $E \subseteq X$  has finite Hausdorff distance from a (unique) apartment  $E' \subseteq Y$ . But then we follow a different line. We show directly that  $f$  induced a combinatorial isomorphism  $f_*$  between the Tits boundaries of  $X$  and  $Y$ . (For the case of simplicial Bruhat-Tits buildings, this implies by 2.4 already that  $X$  and  $Y$  are combinatorially isomorphic.) Next, we show that we obtain a coarse bijection between the so-called *wall trees* of  $X$  and  $Y$ . Since these trees have large holonomy groups, we may apply 6.4. In this way we get equivariant isomorphisms between the wall trees, and thus, by Tits [35], an isometry between the euclidean buildings. We remark that the main results in [35] also enter as important ingredients into the proof of [17].

## 7 Locally compact Bruhat-Tits buildings

In the mid-nineties, Grundhöfer, Knarr and the author completed the classification of all compact connected spherical buildings admitting a chamber transitive automorphism group. Such buildings arise for example as boundaries of Riemannian symmetric spaces. The proof and the method of the classification built on earlier work by Salzmann, Löwen, Burns and Spatzier. Briefly, it may be stated as follows.

**7.1 Theorem** ([10, 11, 19]) *Let  $B$  be a compact spherical building (in the sense of [7]) without rank 1 factors. Suppose that  $B$  is (locally) connected and admits a chamber transitive group of continuous automorphisms. Then  $B$  is the Tits boundary of a Riemannian symmetric space of noncompact type.*

There should be an analog of this result, corresponding to the boundaries of locally compact euclidean buildings. The following conjecture is wide open (even for buildings of type  $A_2$ , i.e. compact projective planes).

**7.2 Conjecture** *Let  $B$  be a compact spherical building (in the sense of [7]) without rank 1 factors. Suppose that  $B$  is totally disconnected and admits a chamber transitive groups of continuous automorphisms. Then  $B$  is the Tits boundary of a locally finite simplicial Bruhat-Tits building.*

The problem is that in comparison to 7.1, no homotopy theory is available. Presently, a proof of this conjecture seems to be out of reach. Assuming the Moufang property, we showed however the following.

**7.3 Theorem** ([12]) *Let  $B$  be a compact spherical building (in the sense of [7]) without rank 1 factors. Suppose that  $B$  is totally disconnected and Moufang. Then  $B$  is the Tits boundary of a locally finite simplicial Bruhat-Tits building.*

We recall that the Moufang property is automatically satisfied if all irreducible factors of  $B$  have rank at least 3; see [33] and [36]. The proof of 7.3 relies very much on the classification of spherical Moufang buildings due to Tits and Weiss.

## 8 Lattices

Let  $X$  be a complete and locally compact CAT(0) space and let  $\Gamma$  be a group of isometries. We call  $\Gamma$  a *uniform lattice* if  $\Gamma$  acts properly discontinuously and cocompactly on  $X$  (such groups are also called CAT(0) groups). Borel’s Density Theorem says that Riemannian symmetric spaces of noncompact type admit (many) uniform lattices. Such a uniform lattice is always finitely presentable. However, very few presentations of lattices are known. Essert observed the following correspondence between uniform lattices acting regularly on the 1-simplices of a given type of a 2-dimensional locally finite simplicial euclidean building and Singer groups. A *Singer group* is a subgroup of the automorphism group a finite generalized polygon (a 1-dimensional spherical building) which acts regularly on the vertices of a given type. Singer groups are studied by finite geometers and group theorists, and quite a few constructions are known. Essert showed that from a collection of Singer groups, one can construct a 2-dimensional *complex of groups* which unfolds to a lattice  $\Gamma$  acting on such a 2-dimensional euclidean building. Specific examples are presentations such as

$$\langle a, b, c \mid a^7 = b^7 = c^7 = abc = a^3b^3c^3 = 1 \rangle$$

or

$$\langle a, b, c \mid a^{13} = b^{13} = c^{13} = ab^3c^9 = a^3b^9c = a^9bc^3 = 1 \rangle.$$

These explicit representations allow, for example, to compute the group homology of the lattices. It is clear that “most” of the buildings  $X$  that he constructed in this way are “exotic”, i.e. they are not Bruhat-Tits buildings. There are presently many open questions about these lattices  $\Gamma$ , eg. about commensurability, quasi-isometric type, or the covolume. We refer to [9] for details and more results.

## 9 Noncrystallographic Weyl groups

The Weyl group of a Bruhat-Tits building arising from a reductive algebraic group over a field with valuation is always crystallographic. Also, the Weyl group of a simplicial euclidean building is necessarily a crystallographic group. But in the definition of a euclidean buildings, there is no reason to assume that  $W$  satisfies the crystallographic condition. It was remarked (without giving details) by Tits [35] that there are Bruhat-Tits buildings with non-crystallographic Weyl groups. An explicit construction of such euclidean buildings, defined over certain, very special fields, can be found in [15]. Their Weyl groups are dihedral groups of order 16, and their Tits boundaries are so-called Moufang generalized octagons.

In a completely different way, Berenstein and Kapovich constructed “wild” 2-dimensional euclidean buildings whose Weyl groups are dihedral groups of arbitrary order [4]. It would be interesting to see if the construction can be done in such a way that it yields highly transitive automorphism groups, as was the case for the 1-dimensional spherical buildings constructed by Tent in [32] for 1-dimensional spherical buildings.

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