AHLFORS'S QUASICONFORMAL EXTENSION CONDITION AND Φ-LIKENESS

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Abstract. The notion of Φ-like functions is known to be a necessary and sufficient condition for univalence. By applying the idea, we derive several necessary conditions and sufficient conditions for that an analytic function defined on the unit disk is not only univalent but also has a quasiconformal extension to the Riemann sphere, as generalizations of well-known univalence and quasiconformal extension criteria, in particular, Ahlfors's quasiconformal extension condition.

1. Introduction

Let $\mathbb{C}$ be the complex plane, $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ for $r > 0$ and $\mathbb{D} := \mathbb{D}_1$. We denote by $\mathcal{A}$ the family of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic on $\mathbb{D}$ and $\mathcal{S}$ the subclass of $\mathcal{A}$ whose members are univalent, that is, one-to-one on $\mathbb{D}$. For standard terminology in the theory of univalent functions, see for instance [12]. Let k be a constant in $[0, 1)$. Then a homeomorphism f of $G \subset \mathbb{C}$ is said to be *k-quasiconformal* if $\partial_z f$ and $\partial_{\bar{z}} f$ in the distributional sense are locally integrable on G and fulfill $|\partial_{\bar{z}} f| \leq k|\partial_z f|$ almost everywhere in G. If we do not need to specify k, we will simply call that f is *quasiconformal*.

We shall start our investigation by giving the following considerations about compositions of analytic functions. Let us suppose that f, $g \in \mathcal{A}$ and Q is an analytic function defined on $f(\mathbb{D})$ which satisfies $g = Q \circ f$. We know that a necessary and sufficient condition for univalence of g on $\mathbb{D}$ is that f and Q are univalent on each domain. Therefore if we would like to know whether f is univalent or not, it is enough to see whether so is g or not, namely, existence of Q such that g can be univalent in $\mathbb{D}$.

Let us set one example with the condition for λ-spirallike functions ($|\lambda| < \pi/2$), i.e., $\mathrm{Re}\{e^{-i\lambda} z g'(z)/g(z)\} > 0$. We note that it ensures univalence of g, and therefore f. In

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view of the relationship $g = Q \circ f$, it is equivalent to

$$\operatorname{Re} \frac{zf'(z)}{\Phi(f(z))} > 0, \tag{1}$$

where $\Phi(w) = e^{i\lambda}Q(w)/Q'(w)$. This is the concept of so-called "Φ-like functions";

Definition A. A function $f \in \mathcal{A}$ is said to be Φ-*like* if there exists an analytic function Φ defined on $f(\mathbb{D})$ such that the inequality (1) holds for all $z \in \mathbb{D}$.

Remark 1.1. The inequality (1) implies $\Phi(0) = 0$ and $\operatorname{Re}\Phi'(0) > 0$.

Through the argument above, it can be seen that Φ-likeness is a sufficient condition for univalence. Surprisingly, it turns out also a necessary condition for univalence. In fact, if f is univalent in $\mathbb{D}$ then we can define Φ by means of $Q := g \circ f^{-1}$, where g is a spirallike function. In consequence, we obtain the following;

Theorem B. *A function $f \in \mathcal{A}$ is univalent in $\mathbb{D}$ if and only if f is Φ-like.*

Remark 1.2. If we choose $\Phi(w) = e^{i\lambda}w$ then it immediately follows the condition for λ-spirallikeness.

The notion of Φ-like functions was introduced by Kas'yanyuk [9] and Brickman [6] independently. The reader is referred to [2, §7] which contains some more informations about Φ-like functions. The above simple and instructive observation about Φ-like functions is due to Ruscheweyh [13]. Furthermore, he gave the following two generalizations of well-known univalence conditions by using similar techniques as it;

Theorem C (Generalized Becker condition [13])**.** *Let $f \in \mathcal{A}$. Then f is univalent if and only if there exists an analytic function Ω on $f(\mathbb{D})$ such that*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)} + zf'(z)\Omega(f(z))\right| \leq 1 \tag{2}$$

for all $z \in \mathbb{D}$.

Theorem D (Generalized Bazilevič functions [13])**.** *Let $f \in \mathcal{A}$, $p(z)$ with $p(0) = p'(0) - 1 = 0$ be starlike univalent in $\mathbb{D}$ and $s = \alpha + i\beta \in \mathbb{C}$, $\operatorname{Re} s > 0$. Then f is univalent in $\mathbb{D}$ if and only if there exists an analytic function $\Psi(w)$ on $f(\mathbb{D})$ with $\Psi(0) \neq 0$ such that*

$$\operatorname{Re} \frac{f'(z)\left(f(z)/z\right)^{s-1}}{\left(p(z)/z\right)^{\alpha}}\Psi(f(z)) > 0 \tag{3}$$

for all $z \in \mathbb{D}$.

Remark 1.3. The choices $\Omega \equiv 0$ and $\Psi \equiv e^{i\alpha}$ correspond to the original univalence conditions due to Becker [4] and Bazilevič [3] respectively.

Indeed, we can show the case (2) from the fact that $g(z) = Q(f(z))$ with $Q''/Q' = \Omega$ satisfies original Becker's univalence condition, and the case (3) from that the function $g(z) = Q(f(z))$ with $Q(w) = \left(s\int_0^w t^{s-1}\Psi(t)dt\right)^{1/s} = \Psi(0)w + \cdots$ is Bazilevič and hence univalent in $\mathbb{D}$. The other directions of Theorem C and Theorem D can be easily proved to define Ω and Ψ by $Q(w) = g(f^{-1}(w))$, $w \in f(\mathbb{D})$, where g is a suitable function which satisfies Becker's condition or a Bazilevič function, respectively. Of course this observation is valid for not only the above two univalence criteria but also many other ones. In other words, the following is true;

Proposition 1. *A necessary and sufficient condition for that $f \in \mathcal{A}$ is univalent in $\mathbb{D}$ is that there exists an analytic function Q on $f(\mathbb{D})$ such that $g := Q \circ f$ satisfies some univalence criteria.*

The main aim of this paper is to derive several necessary conditions and sufficient conditions for that a function $f \in \mathcal{A}$ is univalent in $\mathbb{D}$ and extendible to a quasiconformal mapping to the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ as an application of Ruscheweyh's observation. Those results are based on well-known univalence and quasiconformal extension criteria. For instance, the next theorem which is a generalization of Ahlfors's quasiconformal extension condition [1] (see also [5]) will be obtained. Here $\mathcal{S}(k)$, $0 \leq k < 1$, is the family of functions which are in $\mathcal{S}$ and can be extended to k-quasiconformal mappings to $\widehat{\mathbb{C}}$.

Theorem 2. *Let $f \in \mathcal{A}$ and $k \in [0, 1)$. If there exists a $k_0 \in [0, k)$ and an analytic function Q defined on $f(\mathbb{D})$ which is univalent in $f(\mathbb{D})$, has a $(k-k_0)/(1-kk_0)$-quasiconformal extension to $\widehat{\mathbb{C}}$ and satisfies $Q'(0) \neq 0$ such that for a constant $c \in \mathbb{C}$ and for all $z \in \mathbb{D}$*

$$\left| c|z|^2 + (1 - |z|^2)\left\{ \frac{zf''(z)}{f'(z)} + zf'(z)\Omega(f(z)) \right\} \right| \leq k_0, \tag{4}$$

then $f \in \mathcal{S}(k)$, where $\Omega = Q''/Q'$. Conversely, if $f \in \mathcal{S}(k)$ then there exists a $k_0 \in [0, 1)$ and an analytic function Q defined on $f(\mathbb{D})$ which is univalent in $f(\mathbb{D})$, has a $(k + k_0)/(1 + kk_0)$-quasiconformal extension to $\widehat{\mathbb{C}}$ and satisfies $Q'(0) \neq 0$ such that the inequality (4) holds for a constant $c \in \mathbb{C}$ and for all $z \in \mathbb{D}$.

Remark 1.4. In Theorem 2, if the extended quasiconformal mapping of Q does not take the value ∞ in $\mathbb{C}$ then $\mathcal{S}(k)$ can be replaced by $\mathcal{S}_0(k)$, where $\mathcal{S}_0(k)$ is the family of functions which belong to $\mathcal{S}(k)$ and can be extended to k-quasiconformal automorphisms of $\mathbb{C}$.

In contrast to the case of univalent functions, it is not always true that if $g \circ f$ has a quasiconformal extension then so do f and g as well (consider the case, for instance, $f(z) = z - z^2/2$ and $g = f^{-1}$). This is the reason why Q is required some bothersome assumptions in Theorem 2. On the other hand, if we give a specific form of Q, then it can be obtained some new quasiconformal extension criteria. We will discuss this problem in the last section.

2. Preliminaries

Let $f_t(z) = f(z,t) = \sum_{n=1}^{\infty} a_n(t) z^n$, $a_1(t) \neq 0$, be a function defined on $\mathbb{D} \times [0,\infty)$ and analytic in $\mathbb{D}$ for each $t \in [0,\infty)$, where $a_1(t)$ is a locally absolutely continuous function on $[0,\infty)$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. f_t is said to be a *Löwner chain* if f_t is univalent on $\mathbb{D}$ for each $t \in [0,\infty)$ and satisfies $f_s(\mathbb{D}) \subsetneq f_t(\mathbb{D})$ for $0 \leq s < t < \infty$.

The following necessary and sufficient condition for Löwner chains due to Pommerenke is well-known;

Theorem E ([11])**.** *Let $0 < r_0 \leq 1$. Let $f(z,t)$ be a function defined above. Then the function $f(z,t)$ is a Löwner chain if and only if the following conditions are satisfied;*

1. *The function $f(z,t)$ is analytic in $\mathbb{D}_{r_0}$ for each $t \in [0,\infty)$, locally absolutely continuous in $[0,\infty)$ for each $z \in \mathbb{D}_{r_0}$ and*
$$|f(z,t)| \leq K_0 |a_1(t)| \qquad (z \in \mathbb{D}_{r_1}, \text{ a.e. } t \in [0,\infty))$$
for some positive constants K_0.
2. *There exists a function $p(z,t)$ analytic in $\mathbb{D}$ for each $t \in [0,\infty)$ and measurable in $[0,\infty)$ for each $z \in \mathbb{D}$ satisfying*
$$\operatorname{Re} p(z,t) > 0 \qquad (z \in \mathbb{D}, \; t \in [0,\infty))$$
such that
$$\partial_t f(z,t) = z \partial_z f(z,t) p(z,t) \qquad (z \in \mathbb{D}_{r_1}, \text{ a.e. } t \in [0,\infty)).$$

Remark 2.1. It is known that $a_1(t)$ is admitted to be a complex-valued function ([7]). Also it should be noted here about constant terms of Löwner chains. If $f(z,t)$ is a Löwner chain then $f(z,t) + c$ satisfies all the conditions of the definition of Löwner chains and the sufficient conditions of Theorem E with a modification of K_0, where c is a complex constant which does not depend on t. For this reason here and hereafter we shall treat also such functions as Löwner chains.

The next theorem which is due to Becker plays a central role in our argument;

Theorem F ([4, 5])**.** *Suppose that $f_t(z) = f(z,t)$ is a Löwner chain for which $p(z,t)$ in (2) satisfies the condition*

$$\begin{aligned} p(z,t) \in U(k) \; &:= \; \left\{ w \in \mathbb{C} : \left| \frac{w-1}{w+1} \right| \leq k \right\} \\ &= \; \left\{ w \in \mathbb{C} : \left| w - \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2} \right\} \end{aligned}$$

for all $z \in \mathbb{D}$ and almost all $t \in [0, \infty)$. Then $f(z,t)$ admits a continuous extension to $\overline{\mathbb{D}}$ for each $t \geq 0$ and the map $\hat{f}$ defined by

$$\hat{f}(re^{i\theta}) = \begin{cases} f(re^{i\theta}, 0), & \text{if} \quad r < 1, \\ f(e^{i\theta}, \log r), & \text{if} \quad r \geq 1, \end{cases}$$

is a k-quasiconformal extension of f_0 to $\mathbb{C}$.

In other words, if f_t is normalized by $f_t(0) = 0$ then the above theorem gives a sufficient condition for $f_0 \in \mathcal{S}_0(k)$.

3. Proof of Theorem 2

Firstly we shall show the first part of Theorem 2. Set

$$F(z,t) := Q(f(e^{-t}z)) + (1+c)^{-1}(e^t - e^{-t})zQ'(f(e^{-t}z))f'(e^{-t}z). \tag{5}$$

We note that $1 + c \neq 0$ since the inequality (4) implies $|c| \leq (k - k_0)/(1 - kk_0) < 1$ (see [8, Remark 1.1 and 1.2]). Then we have

$$\begin{aligned} &\left| \frac{\partial_t F(z,t) - z\partial_z F(z,t)}{\partial_t F(z,t) + z\partial_z F(z,t)} \right| \\ &\quad = \left| e^{-2t}c + (1 - e^{-2t}) \left\{ e^{-t}z \frac{f''(e^{-t}z)}{f'(e^{-t}z)} + e^{-t}zf'(e^{-t}z)\Omega(f(e^{-t}z)) \right\} \right| \end{aligned} \tag{6}$$

where $\Omega = Q''/Q'$. The right-hand side of (6) is always less than or equal to k_0 from (4) and hence $g := Q \circ f$ can be extended to a k_0-quasiconformal mapping to $\widehat{\mathbb{C}}$ by Theorem E and Theorem F. Since Q has a $(k - k_0)/(1 - kk_0)$-quasiconformal extension to $\widehat{\mathbb{C}}$ we conclude that $f = Q^{-1} \circ g \in \mathcal{S}(k)$.

The second part of Theorem 2 easily follows to define $Q := g \circ f^{-1}$, where g is an analytic function defined on $\mathbb{D}$ which satisfies original Ahlfors's k_0-quasiconformal extension condition. □

4. Further results

It can be derived similar necessary conditions and sufficient conditions for quasiconformal extensions as Theorem 2. Here we select one example out of a large variety of possibilities. This is based on the Noshiro-Warschawski theorem [10, 15];

Theorem 3. *Let $f \in \mathcal{A}$ and $k \in [0,1)$. If there exists a $k_0 \in [0,k)$ and an analytic function Q defined on $f(\mathbb{D})$ which is univalent in $f(\mathbb{D})$, has a $(k-k_0)/(1-kk_0)$-quasiconformal extension to $\widehat{\mathbb{C}}$ and satisfies $Q'(0) \neq 0$ such that for all $z \in \mathbb{D}$*

$$f'(z)Q'(f(z)) \in U(k_0) \tag{7}$$

then $f \in \mathcal{S}(k)$, where $U(k_0)$ is the disk defined in Theorem F. Conversely, if $f \in \mathcal{S}(k)$ then there exists a $k_0 \in [0,1)$ and an analytic function Q defined on $f(\mathbb{D})$ which is univalent in $f(\mathbb{D})$, has a $(k+k_0)/(1+kk_0)$-quasiconformal extension to $\widehat{\mathbb{C}}$ and satisfies $Q'(0) \neq 0$ such that the inequality (7) holds for all $z \in \mathbb{D}$.

Proof. Let us put

$$F(z,t) := Q(f(z)) + (e^t - 1)z. \tag{8}$$

Then calculations show that

$$\frac{z\partial_z F(z,t)}{\partial_t F(z,t)} = \frac{1}{e^t} Q'(f(z))f'(z) + \left(1 - \frac{1}{e^t}\right).$$

Following the lines of the proof of Theorem 2 one can deduce all the assertions of Theorem 3. □

Of course various similar results as Theorem 3 can be proved to choose the other univalence criterion and set a suitable Löwner chain. For example, the condition

$$\frac{zf'(z)}{\Phi(f(z))} \in U(k_0)$$

which is based on the definition of Φ-like functions is given by the Löwner chain

$$F(z,t) = e^t Q(f(z)),$$

or

$$\frac{f'(z)\,(f(z)/z)^{s-1}}{(p(z)/z)^{\alpha}} \Psi(f(z)) \in U(k_0)$$

which is based on the definition of Bazilevič functions is given by

$$F(z,t) = \left\{Q(f(z))^s + s(e^t - 1)p(z)^{\alpha} z^{i\beta}\right\}^{1/s},$$

where Φ and Ψ are functions defined in Section 1.

5. Applications

In this section we consider several applications of theorems we have obtained in previous sections, in particular, Theorem 2 and Theorem 3. Two specific forms of the function Q which is univalent on a certain domain and can be extended to a quasiconformal mapping to $\widehat{\mathbb{C}}$ are given.

Here we shall remark that in the cases below Q does not need to be normalized by $Q(0) = 0$. For Löwner chains $F(z,t)$ defined in (5) and (8) we have $F(0,t) = Q(0)$ which implies that in both cases a constant term of $F(z,t)$ does not depend on t. Hence, as we noted in Remark 2.1, $F(z,t)$ is a Löwner chain even though $Q(0) \neq 0$. This fact allows us to avoid some technical complications.

5.1. **Möbius transformations.** Let Q_1 be the Möbius transformation given by

$$Q_1(w) := \frac{\alpha w + \beta}{\gamma w + \delta} \quad (\alpha,\, \beta,\, \gamma,\, \delta \in \mathbb{C},\, \gamma \neq 0,\, \alpha\delta - \beta\gamma = 1).$$

For a given $f \in \mathcal{A}$, we suppose that $-\delta/\gamma \notin f(\mathbb{D})$, otherwise Q_1 is no longer analytic on $f(\mathbb{D})$. Thus Q_1 is considered as a function which is analytic and univalent on $f(\mathbb{D})$ and has a 0-quasiconformal extension to $\widehat{\mathbb{C}}$. Q_1 is the unique function which it can be chosen as Q in Theorem 2 and Theorem 3 without any restrictions on the shape of $f(\mathbb{D})$.

Simple calculations show that $Q_1'(w) = 1/(\gamma w + \delta)^2$ and $Q_1''(w)/Q_1'(w) = -2/(w + (\delta/\gamma))$, and hence by defining $Q := Q_1$ we obtain the following new quasiconformal extension criteria as corollaries of Theorem 2 and Theorem 3;

Corollary 4. *Let $f \in \mathcal{A}$ and $k \in [0,1)$. If f satisfies*

$$\left| c_1|z|^2 + (1-|z|^2)\left\{ \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)-c_2} \right\} \right| \leq k$$

for constants c_1, $c_2 \in \mathbb{C}$, $c_2 \notin f(\mathbb{D})$, and for all $z \in \mathbb{D}$, then $f \in \mathcal{S}(k)$.

Corollary 5. *Let $f \in \mathcal{A}$ and $k \in [0,1)$. If f satisfies*

$$\frac{f'(z)}{(\gamma f(z) + \delta)^2} \in U(k)$$

for constants γ, $\delta \in \mathbb{C}$, $\gamma \neq 0$, $-\delta/\gamma \notin f(\mathbb{D})$, and for all $z \in \mathbb{D}$, then $f \in \mathcal{S}(k)$.

Remark 5.1. We assumed that $\gamma \neq 0$ because in the case when $\gamma = 0$ the function Q_1 is an Affine transformation and Corollary 4 and Corollary 5 are nothing but well-known quasiconformal extension criteria [1, 14].

Remark 5.2. In the above Corollaries $\mathcal{S}(k)$ cannot be replaced by $\mathcal{S}_0(k)$ since Q_1 does not fix ∞ because of $\gamma \neq 0$.

5.2. **Sector domain.** We may set the function Q under the assumption that the image of $\mathbb{D}$ under $f \in \mathcal{A}$ is contained in a quasidisk which has a special shape. For instance, we assume that $f(\mathbb{D})$ lies in the sector domain $\Delta(w_0, \lambda_0, a) := \{w \in \mathbb{C} : \pi\lambda_0 < \arg(w - w_0) < \pi\lambda_1, |\lambda_1 - \lambda_0| < a\}$ for $w_0 \in \mathbb{C}\backslash f(\mathbb{D})$, $\lambda_0 \in [0, 2)$ and $a \in (0, 2)$. Then, we define Q in Theorem 2 and Theorem 3 by Q_2,

$$Q_2(w) := \left(e^{-i\pi\lambda_0}(w - w_0)\right)^{1/a},$$

which maps $\Delta(w_0, \lambda_0, a)$ conformally onto the upper half plane. It is verified that Q_2 can be extended to a $|1 - a|$-quasiconformal automorphism of $\mathbb{C}$ as follows: Let us set

$$P_1(z) := z^{1/(2-a)}, \quad P_2(z) := |z|^{(2-a)/a}\frac{z}{|z|},$$

respectively. Then the function P defined by

$$P(z) := \begin{cases} z^{1/a}, & \textit{if} \quad z \in \Delta(0, 0, a), \\ -(P_2 \circ P_1)(e^{-\pi a} z), & \textit{if} \quad z \in \overline{\Delta(0, a, 2 - a)}, \end{cases}$$

is a $|1 - a|$-quasiconformal automorphism of $\mathbb{C}$. After composing proper Affine transformations we obtain the desired extension of Q_2.

Since $Q_2''(w)/Q_2'(w) = ((1/a) - 1)/(w - w_0)$, we deduce the next corollary;

Corollary 6. *Let $f \in \mathcal{A}$ and $k \in [0, 1)$. We assume that $f(\mathbb{D})$ is contained in the sector domain $\Delta(w_0, \lambda_0, a)$. If f satisfies*

$$\left| c|z|^2 + (1 - |z|^2) \left\{ \frac{zf''(z)}{f'(z)} + \left(\frac{1}{a} - 1\right) \frac{zf'(z)}{f(z) - w_0} \right\} \right| \leq k$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{S}_0(\ell)$, where $\ell = (k + |1 - a|)/(1 + k|1 - a|)$.

To state the next corollary we shall put $Q_3(w) := Q_2(w)/Q_2'(0)$, so that $Q_3'(w) = (1 - (w/w_0))^{(1/a)-1}$ and hence $f'(0)Q_3'(0) = 1 \in U(k)$ for any $k \in [0, 1)$.

Corollary 7. *Let $f \in \mathcal{A}$ and $k \in [0, 1)$. We assume that $f(\mathbb{D})$ is contained in the sector domain $\Delta(w_0, \lambda_0, a)$. If f satisfies*

$$f'(z) \left(1 - \frac{f(z)}{w_0}\right)^{(1/a)-1} \in U(k)$$

for all $z \in \mathbb{D}$, then $f \in \mathcal{S}_0(\ell)$, where $\ell = (k + |1 - a|)/(1 + k|1 - a|)$.

As a special case of Corollary 7, if we can choose $w_0 = 1$ and $a = 1/2$, then we will have a $(2k + 1)/(k + 2)$-quasiconformal extension criterion $f'(z)(1 - f(z)) \in U(k)$.

5.3. **Bounded functions.** The hypothesis of Corollary 6 and Corollary 7 that $f(\mathbb{D})$ lies in a sector domain may seem to be less useful in practical situations. It will, however, make a valuable contribution when $f \in \mathcal{A}$ is a bounded function on $\mathbb{D}$ because $f(\mathbb{D})$ is contained in a disk and thus there must exist a sector domain which includes the disk.

It is important to consider the case when f is bounded because if $f \in \mathcal{S}$ is unbounded then $f \notin \mathcal{S}_0(k)$ since an extended quasiconformal mapping of f takes ∞ at a point on the boundary of $\mathbb{D}$. Therefore, it is enough to deal with only bounded components of $\mathcal{S}$ when it is investigated whether $f \in \mathcal{S}$ is contained in the class $\mathcal{S}_0(k)$ or not.

It is not difficult to find a precise sector domain which includes $f(\mathbb{D})$. In fact, if $f \in \mathcal{A}$ is bounded on $\mathbb{D}$, the there exists a constant $M := \sup_{z\in\mathbb{D}} |f(z)|$. Since $f(\mathbb{D}) \subset \mathbb{D}_M$, $f(\mathbb{D})$ is contained in $\Delta(w_0, \lambda_2, 2\arcsin(M/|w_0|))$ for $w_0 \in \mathbb{C}\backslash\mathbb{D}_M$ and a suitable λ_2 which will be given below. Of course, one may choose the other disk $\{z \in \mathbb{C} : |z - z_0| < R\}$ which contains $f(\mathbb{D})$ and put a sector by $\Delta(w_0, \lambda_3, 2\arcsin(R/|w_0 - z_0|))$ for $w_0 \in \mathbb{C}\backslash\{|z - z_0| < R\}$ and $\lambda_3 \in [0, 2)$. Here $\lambda_3 = \arg(z_0 - w_0) + \arg(\sqrt{|w_0 - z_0|^2 - R^2} - iR)$. λ_2 is given by the case when $z_0 = 0$ and $R = M$.

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