Exponential instability in the Gel'fand inverse problem on the energy intervals

M.I. Isaev
Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Russia
Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau, France
e-mail: isaev.m.i@gmail.com

Abstract

We consider the Gel'fand inverse problem and continue studies of [Mandache,2001]. We show that the Mandache-type instability remains valid even in the case of Dirichlet-to-Neumann map given on the energy intervals. These instability results show, in particular, that the logarithmic stability estimates of [Alessandrini,1988], [Novikov, Santacesaria,2010] and especially of [Novikov,2010] are optimal (up to the value of the exponent).

1. Introdution

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \tag{1.1}$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2,\ \partial D \in C^2,\ v \in L^\infty(D). \tag{1.2}$$

Consider the map $\Phi(E)$ such that

$$\Phi(E)(\psi|_{\partial D}) = \frac{\partial \psi}{\partial \nu}|_{\partial D}. \tag{1.3}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar{D} = D \cup \partial D$, where ν is the outward normal to ∂D. Here we assume also that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D. \tag{1.4}$$

The map $\Phi(E)$ is called the Dirichlet-to-Neumann map and is considered as boundary measurements.

We consider the following inverse boundary value problem for equation (1.1).

Problem 1.1. Given $\Phi(E)$ on the union of the energy intervals $S = \bigcup_{j=1}^{K} I_j$, find v.

Here we suppose that condition (1.4) is fulfilled for any $E \in S$.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation on the energy intervals (see [2], [6]).

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness for Problem 1.1 was obtained for the first time by Novikov (see Theorem 5.3 in [4]). Some global reconstruction method for Problem 1.1 was proposed for the first time in [4] also. Global uniqueness theorems and global reconstruction methods in the case of fixed energy were given for the first time in [6] in dimension $d \geq 3$ and in [9] in dimension $d = 2$.

Global stability estimates for Problem 1.1 were given for the first time in [1] in dimension $d \geq 3$ and in [8] in dimension $d = 2$. The Alessandrini result of [1] was recently improved by Novikov in [7]. In the case of fixed energy, Mandache showed in [3] that these logarithmic stability results are optimal (up to the value of the exponent). Mandache-type instability estimates for inverse inclusion and scattering problems are given in [12].

In the present work we extend studies of Mandache to the case of Dirichlet-to-Neumann map given on the energy intervals. The stability estimates and our instability results for Problem 1.1 are presented and discussed in Section 2. In Section 5 we prove the main results, using a ball packing and covering by ball arguments. In Section 3 we prove some basic properties of the Dirichlet-to-Neumann map, using some Lemmas about the Bessel functions wich we proved in Section 6.

2. Stability estimates and main results

As in [7] we assume for simplicity that

$$\begin{array}{l} D \text{ is an open bounded domain in } \mathbb{R}^d,\ \partial D \in C^2, \\ v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d,\ \operatorname{supp} v \subset D,\ d \geq 2, \end{array} \tag{2.1}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v:\ \partial^J v \in L^1(\mathbb{R}^d),\ |J| \leq m\},\ m \in \mathbb{N} \cup 0, \tag{2.2}$$

where

$$J \in (\mathbb{N} \cup 0)^d,\ |J| = \sum_{i=1}^{d} J_i,\ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \ldots \partial x_d^{J_d}}. \tag{2.3}$$

Let

$$||v||_{m,1} = \max_{|J| \leq m} ||\partial^J v||_{L^1(\mathbb{R}^d)}. \tag{2.4}$$

We recall that if v_1, v_2 are potentials satisfying (1.4),(1.3), where E and D are fixed, then

$$\Phi_1 - \Phi_2 \text{ is a compact operator in } L^\infty(\partial D), \tag{2.5}$$

where Φ_1, Φ_2 are the DtN maps for v_1, v_2 respectively, see [6]. Note also that (2.1) $\Rightarrow$ (1.2).

Theorem 2.1 (variation of the result of [1], see [7])**.** *Let conditions (1.4), (2.1) hold for potentials v_1 and v_2, where E and D are fixed, $d \geq 3$. Let $||v_j||_{m,1} \leq N$, $j = 1, 2$, for some $N > 0$. Let Φ_1, Φ_2 denote DtN maps for v_1, v_2 respectively. Then*

$$||v_1 - v_2||_{L^\infty(D)} \leq c_1 (\ln(3 + ||\Phi_1 - \Phi_2||^{-1}))^{-\alpha_1}, \tag{2.6}$$

where $c_1 = c_1(N, D, m)$, $\alpha_1 = (m - d)/m$, $||\Phi_1 - \Phi_2|| = ||\Phi_1 - \Phi_2||_{L^\infty(\partial D) \to L^\infty(\partial D)}$.

An analog of stability estimate of [1] for $d = 2$ is given in [8].

A disadvantage of estimate (2.6) is that

$$\alpha_1 < 1 \text{ for any } m > d \text{ even if } m \text{ is very great.} \tag{2.7}$$

Theorem 2.2 (the result of [7]). *Let the assumptions of Theorem 2.1 hold. Then*

$$||v_1 - v_2||_{L^\infty(D)} \le c_2(\ln(3 + ||\Phi_1 - \Phi_2||^{-1}))^{-\alpha_2}, \tag{2.8}$$

where $c_2 = c_2(N, D, m)$, $\alpha_2 = m - d$, $||\Phi_1 - \Phi_2|| = ||\Phi_1 - \Phi_2||_{L^\infty(\partial D)\to L^\infty(\partial D)}$.

A principal advantage of estimate (2.8) in comparison with (2.6) is that

$$\alpha_2 \to +\infty \text{ as } m \to +\infty, \tag{2.9}$$

in contrast with (2.7). Note that strictly speaking Theorem 2.2 was proved in [7] for $E = 0$ with the condition that $\operatorname{supp} v \subset D$, so we cant make use of substitution $v_E = v - E$, since condition $\operatorname{supp} v_E \subset D$ does not hold.

We would like to mention that, under the assumptions of Theorems 2.1 and 2.2, according to the Mandache results of [3], estimate (2.8) can not hold with $\alpha_2 > m(2d-1)/d$ for real-valued potentials and with $\alpha_2 > m$ for complex potentials.

As in [3] in what follows we fix $D = B(0, 1)$, where $B(x, r)$ is the open ball of radius r centred at x. We fix an orthonormal basis in $L^2(S^{d-1}) = L^2(\partial D)$

$$\begin{array}{l}\{f_{jp} : j \ge 0;\ 1 \le p \le p_j\},\\ f_{jp} \text{ is a spherical harmonic of degree } j,\end{array} \tag{2.10}$$

where p_j is the dimension of the space of spherical harmonics of order j,

$$p_j = \binom{j+d-1}{d-1} - \binom{j+d-3}{d-1}, \tag{2.11}$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{for } n \ge 0 \tag{2.12}$$

and

$$\binom{n}{k} = 0 \quad \text{for } n < 0. \tag{2.13}$$

The precise choice of f_{jp} is irrelevant for our purposes. Besides orthonormality, we only need f_{jp} to be the restriction of a homogeneous harmonic polynomial of degree j to the sphere and so $|x|^j f_{jp}(x/|x|)$ is harmonic. In the Sobolev spaces $H^s(S^{d-1})$ we will use the norm

$$||\sum_{j,p} c_{jp} f_{jp}||^2_{H^s} = \sum_{j,p} (1+j)^{2s} |c_{jp}|^2. \tag{2.14}$$

The notation (a_{jpiq}) stands for a multiple sequence. We will drop the subscript

$$0 \le j,\ 1 \le p \le p_j,\ 0 \le i,\ 1 \le q \le p_i. \tag{2.15}$$

We use notations: $|A|$ is the cardinality of a set A, $[a]$ is the integer part of real number a and $(r, \omega) \in \mathbb{R}_+ \times S^{d-1}$ are polar coordinates for $r\omega = x \in \mathbb{R}^d$.

The interval $I = [a, b]$ will be referred as σ-regular interval if for any potential $v \in L^\infty(D)$ with $||v||_{L^\infty(D)} \le \sigma$ and any $E \in I$ condition (1.4) is fulfilled. Note that for any $E \in I$ and any Dirichlet eigenvalue λ for operator $-\Delta$ in D we have that

$$|E - \lambda| \ge \sigma. \tag{2.16}$$

It follows from the definition of σ-regular interval, taking $v \equiv E - \lambda$.

Theorem 2.3. *For $\sigma > 0$ and dimension $d \geq 2$ consider the union $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals. Then for any $m > 0$ and any $s \geq 0$ there is a constant $\beta > 0$, such that for any $\epsilon \in (0, \sigma/3)$ and $v_0 \in C^m(D)$ with $||v_0||_{L^\infty(D)} \leq \sigma/3$ and $\mathrm{supp}\, v_0 \subset B(0, 1/3)$ there are real-valued potentials $v_1, v_2 \in C^m(D)$, also supported in $B(0, 1/3)$, such that*

$$\begin{aligned} &\sup_{E\in S}\Big(||\Phi_1(E) - \Phi_2(E)||_{H^{-s}\to H^s}\Big) \leq \exp\Big(-\epsilon^{-\frac{1}{2m}}\Big),\\ &||v_1 - v_2||_{L^\infty(D)} \geq \epsilon,\\ &||v_i - v_0||_{C^m(D)} \leq \beta, \quad i = 1, 2,\\ &||v_i - v_0||_{L^\infty(D)} \leq \epsilon, \quad i = 1, 2, \end{aligned} \tag{2.17}$$

where $\Phi_1(E)$, $\Phi_2(E)$ are the DtN maps for v_1 and v_2 respectively.

Remark 2.1. We can allow β to be arbitrarily small in Theorem 2.3, if we require $\epsilon \leq \epsilon_0$ and replace the right-hand side in the instability estimate by $\exp(-c\epsilon^{-\frac{1}{2m}})$, with $\epsilon_0 > 0$ and $c > 0$, depending on β.

In addition to Theorem 2.3, we consider explicit instability example with a complex potential given by Mandache in [3]. We show that it gives exponential instability even in case of Dirichlet-to-Neumann map given on the energy intervals. Consider the cylindrical variables $(r_1, \theta, x') \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^{d-2}$, with $x' = (x_3, \ldots, x_d)$, $r_1 \cos\theta = x_1$ and $r_1 \sin\theta = x_2$. Take $\phi \in C^\infty(\mathbb{R}^2)$ with support in $B(0, 1/3) \cap \{x_1 > 1/4\}$ and with $||\phi||_{L^\infty} = 1$.

Theorem 2.4. *For $\sigma > 0$, $m > 0$, integer $n > 0$ and dimension $d \geq 2$ consider the union $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals and define the complex potential*

$$v_{nm}(x) = \frac{\sigma}{3} n^{-m} e^{in\theta} \phi(r_1, |x'|). \tag{2.18}$$

Then $||v_{mn}||_{L^\infty(D)} = \frac{\sigma}{3} n^{-m}$ and for every $s \geq 0$ and $m > 0$ there are constants c, c' such that $||v_{mn}||_{C^m(D)} \leq c$ and for every n

$$\sup_{E\in S}\Big(||\Phi_{mn}(E) - \Phi_0(E)||_{H^{-s}\to H^s}\Big) \leq c' 2^{-n/4}, \tag{2.19}$$

where $\Phi_{mn}(E)$, $\Phi_0(E)$ are the DtN maps for v_{mn} and $v_0 \equiv 0$ respectively.

In some important sense, this is stronger than Theorem 2.3. Indeed, if we take $\epsilon = \frac{\sigma}{3} n^{-m}$ we obtain (2.17) with $\exp(-C\epsilon^{-1/m})$ in the right-hand side. An explicit real-valued counterexample should be difficult to find. This is due to nonlinearity of the map $v \to \Phi$.

Remark 2.2. Note that for sufficient large s one can see that

$$||\Phi_1 - \Phi_2||_{L^\infty(\partial D)\to L^\infty(\partial D)} \leq C||\Phi_1 - \Phi_2||_{H^{-s}\to H^s}. \tag{2.20}$$

So Theorem 2.3 and Theorem 2.4 imply, in particular, that the estimate

$$||v_1 - v_2||_{L^\infty(D)} \leq c_3 \sup_{E\in S} \left(\ln(3 + ||\Phi_1(E) - \Phi_2(E)||^{-1})\right)^{-\alpha_3}, \tag{2.21}$$

where $c_3 = c_3(N, D, m, S)$ and $||\Phi_1(E) - \Phi_2(E)|| = ||\Phi_1(E) - \Phi_2(E)||_{L^\infty(\partial D)\to L^\infty(\partial D)}$, can not hold with $\alpha_3 > 2m$ for real-valued potentials and with $\alpha_3 > m$ for complex potentials. Thus Theorem 2.3 and Theorem 2.4 show optimality of logarithmic stability results of Alessandrini and Novikov in considerably stronger sense that results of Mandache.

3. Some basic properties of Dirichlet-to-Neumann map

We continue to consider $D = B(0,1)$ and also to use polar coordinates $(r,\omega) \in \mathbb{R}_+ \times S^{d-1}$, with $x = r\omega$. Solutions of equation $-\Delta\psi = E\psi$ in D can be expressed by the Bessel functions J_α and Y_α with integer or half-integer order α, see definitions of Section 6. Here we state some Lemmas about these functions (Lemma 3.1, Lemma 3.2 and Lemma 3.3).

Lemma 3.1. *Suppose $k \neq 0$ and k^2 is not a Dirichlet eigenvalue for operator $-\Delta$ in D. Then*

$$\psi_0(r,\omega) = r^{-\frac{d-2}{2}} \frac{J_{j+\frac{d-2}{2}}(kr)}{J_{j+\frac{d-2}{2}}(k)} f_{jp}(\omega) \tag{3.1}$$

is the solution of equation (1.1) with $v \equiv 0$, $E = k^2$ and boundary condition $\psi|_{\partial D} = f_{jp}$.

Remark 3.1. Note that the assumptions of Lemma 3.1 imply $J_{j+\frac{d-2}{2}}(k) \neq 0$.

Lemma 3.2. *Let the assumptions of Lemma 3.1 hold. Then system of functions*

$$\{\psi_{jp}(r,\omega) = R_j(k,r) f_{jp}(\omega) : j \geq 0; 1 \leq p \leq p_j\}, \tag{3.2}$$

where

$$R_j(k,r) = r^{-\frac{d-2}{2}} \Big(Y_{j+\frac{d-2}{2}}(kr) J_{j+\frac{d-2}{2}}(k) - J_{j+\frac{d-2}{2}}(kr) Y_{j+\frac{d-2}{2}}(k) \Big), \tag{3.3}$$

is complete orthogonal system (in the sense of L_2) in the space of solutions of equation (1.1) in $D' = B(0,1) \setminus B(0,1/3)$ with $v \equiv 0$, $E = k^2$ and boundary condition $\psi|_{r=1} = 0$.

Lemma 3.3. *For any $C > 0$ and integer $d \geq 2$ there is a constant $N > 3$ depending on C such that for any integer $n \geq N$ and any $|z| \leq C$*

$$\frac{1}{2}\frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)} \leq |J_\alpha(z)| \leq \frac{3}{2}\frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)}, \tag{3.4}$$

$$|J'_\alpha(z)| \leq 3\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.5}$$

$$\frac{1}{2\pi}(|z|/2)^{-\alpha}\Gamma(\alpha) \leq |Y_\alpha(z)| \leq \frac{3}{2\pi}(|z|/2)^{-\alpha}\Gamma(\alpha) \tag{3.6}$$

$$|Y'_\alpha(z)| \leq \frac{3}{\pi}(|z|/2)^{-\alpha-1}\Gamma(\alpha+1) \tag{3.7}$$

where $'$ denotes derivation with respect to z, $\alpha = n + \frac{d-2}{2}$ and $\Gamma(x)$ is the Gamma function.

Proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3 are given in Section 6.

Lemma 3.4. *Consider a compact $W \subset \mathbb{C}$. Suppose, that v is bounded, $\operatorname{supp} v \subset B(0,1/3)$ and condition (1.4) is fulfilled for any $E \in W$ and potentials v and v_0, where $v_0 \equiv 0$. Denote $\Lambda_{v,E} = \Phi(E) - \Phi_0(E)$. Then there is a constant $\rho = \rho(W,d)$, such that for any $0 \leq j, 1 \leq p \leq p_j$, $0 \leq i, 1 \leq q \leq p_i$, we have*

$$|\langle \Lambda_{v,E} f_{jp}, f_{iq} \rangle| \leq \rho\, 2^{-\max(j,i)} ||v||_{L^\infty(D)} ||(-\Delta + v - E)^{-1}||_{L^2(D)}, \tag{3.8}$$

where $\Phi(E)$, $\Phi_0(E)$ are the DtN maps for v and v_0 respectively and $(-\Delta + v - E)^{-1}$ is considered with the Dirichlet boundary condition.

Proof of Lemma 3.4. For simplicity we give first a proof under the additional assumtions that $0 \notin W$ and there is a holomorphic germ $\sqrt{E}$ for $E \in W$. Since W is compact there is $C > 0$ such that for any $z \in W$ we have $|z| \le C$. We take N from Lemma 3.3 for this C. We fix indeces j, p. Consider solutions $\psi(E)$, $\psi_0(E)$ of equation (1.1) with $E \in W$, boundary condition $\psi|_{\partial D} = f_{jp}$ and potentials v and v_0 respectively. Then $\psi(E) - \psi_0(E)$ has zero boundary values, so it is domain of $-\Delta + v - E$, and since

$$(-\Delta + v - E)\left(\psi(E) - \psi_0(E)\right) = -v\psi_0(E) \ \text{ in } D, \tag{3.9}$$

we obtain that

$$\psi(E) - \psi_0(E) = -(-\Delta + v - E)^{-1} v\psi_0(E). \tag{3.10}$$

If $j \ge N$ from Lemma 3.1 and Lemma 3.3 we have that

$$\begin{aligned} ||\psi_0(E)||^2_{L^2(B(0,1/3))} &= ||f_{jp}||^2_{L^2(S^{d-1})} \int_0^{1/3} \left| r^{-\frac{d-2}{2}} \frac{J_{j+\frac{d-2}{2}}(\sqrt{E}\, r)}{J_{j+\frac{d-2}{2}}(\sqrt{E})} \right|^2 r^{d-1} dr \le \\ \le \int_0^{1/3} &\left(\frac{3}{2} \frac{(|E|^{1/2} r/2)^{j+\frac{d-2}{2}}}{\Gamma(j + \frac{d-2}{2} + 1)} \right)^2 \Big/ \left(\frac{1}{2} \frac{(|E|^{1/2}/2)^{j+\frac{d-2}{2}}}{\Gamma(j + \frac{d-2}{2} + 1)} \right)^2 r\, dr = \\ = 3 \int_0^{1/3} &r^{2j+d-1} dr = \frac{3}{2j+d} \left(\frac{1}{3} \right)^{2j+d} < 2^{-2j}. \end{aligned} \tag{3.11}$$

For $j < N$ we use fact that $||\psi_0(E)||_{L^2(B(0,1))}$ is continuous function on compact W and, since N depends only on W, we get that there is a constant $\rho_1 = \rho_1(W, d)$ such that

$$||\psi_0(E)||_{L^2(B(0,1/3))} \le \rho_1 2^{-j}. \tag{3.12}$$

Since v has support in $B(0, 1/3)$ from (3.10) we get that

$$||\psi(E) - \psi_0(E)||_{L^2(B(0,1))} \le \rho_1 2^{-j} ||v||_{L^\infty(D)} ||(-\Delta + v - E)^{-1}||_{L^2(D)}. \tag{3.13}$$

Note that $\psi(E) - \psi_0(E)$ is the solution of equation (1.1) in $D' = B(0,1) \setminus B(0, 1/3)$ with potential $v_0 \equiv 0$ and boundary condition $\psi|_{r=1} = 0$. From Lemma 3.2 we have that

$$\psi(E) - \psi_0(E) = \sum_{0 \le i, 1 \le q \le p_i} c_{iq}(E) \psi_{iq}(E) \ \text{ in } D' \tag{3.14}$$

for some c_{iq}, where

$$\psi_{iq}(E)(r, \omega) = R_i(\sqrt{E}, r) f_{iq}(\omega). \tag{3.15}$$

Since $R_i(\sqrt{E}, 1) = 0$

$$\left. \frac{\partial R_i(\sqrt{E}, r)}{\partial r} \right|_{r=1} = \left. \frac{\partial \left(r^{\frac{d-2}{2}} R_i(\sqrt{E}, r) \right)}{\partial r} \right|_{r=1}. \tag{3.16}$$

For $i \ge N$ from Lemma 3.3 we have that

$$\begin{aligned} \left| \frac{\left. \frac{\partial R_i(\sqrt{E}, r)}{\partial r} \right|_{r=1}}{Y_\alpha(\sqrt{E}) J_\alpha(\sqrt{E})} \right| &= |E|^{1/2} \left| \frac{Y'_\alpha(\sqrt{E})}{Y_\alpha(\sqrt{E})} - \frac{J'_\alpha(\sqrt{E})}{J_\alpha(\sqrt{E})} \right| \le \\ \le 6|E|^{1/2} &\left(\frac{(|E|^{1/2}/2)^{-\alpha-1} \Gamma(\alpha+1)}{(|E|^{1/2}/2)^{-\alpha} \Gamma(\alpha)} + \frac{(|E|^{1/2}/2)^{\alpha-1} \Gamma(\alpha+1)}{(|E|^{1/2}/2)^{\alpha} \Gamma(\alpha)} \right) = 6\alpha, \end{aligned} \tag{3.17}$$

$$\left(\frac{||r^{-\frac{d-2}{2}}Y_\alpha(\sqrt{E}r)||_{L^2(\{1/3<|x|<2/5\})}}{|Y_\alpha(\sqrt{E})|}\right)^2 \geq \int_{1/3}^{2/5}\left(\frac{1}{3}\frac{(|E|^{1/2}r/2)^{-\alpha}\Gamma(\alpha)}{(|E|^{1/2}/2)^{-\alpha}\Gamma(\alpha)}\right)^2 r\,dr$$
$$\geq \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(\frac{1}{3}(5/2)^\alpha\right)^2, \tag{3.18}$$

$$\left(\frac{||r^{-\frac{d-2}{2}}J_\alpha(\sqrt{E}r)||_{L^2(\{1/3<|x|<2/5\})}}{|J_\alpha(\sqrt{E})|}\right)^2 \leq \int_{1/3}^{2/5}\left(3\frac{(|E|^{1/2}r/2)^{\alpha}\Gamma(\alpha)}{(|E|^{1/2}/2)^{\alpha}\Gamma(\alpha)}\right)^2 r\,dr$$
$$\leq \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(3(2/5)^\alpha\right)^2, \tag{3.19}$$

where $\alpha = i+\frac{d-2}{2}$. Since $N>3$ we have that $\alpha>3$. Using (3.18) and (3.19) we get that

$$\frac{||\psi_{iq}(E)||_{L^2(\{1/3<|x|<2/5\})}}{\left|Y_\alpha(\sqrt{E})J_\alpha(\sqrt{E})\right|} \geq \left(\left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\right)^{1/2}\left(\frac{1}{3}(5/2)^\alpha - 3(2/5)^\alpha\right) \geq \frac{1}{1000}(5/2)^\alpha. \tag{3.20}$$

For $i \geq N$ we get that

$$\left|\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}\right| \leq 1000\alpha(5/2)^{-\alpha}||\psi_{iq}(E)||_{L^2(\{1/3<|x|<1\})}. \tag{3.21}$$

For $i<N$ we use the fact that $\left|\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}\right| / ||\psi_{iq}(E)||_{L^2(\{1/3<|x|<1\})}$ is continuous function on compact W and get that for any $i \geq 0$ there is a constant $\rho_2 = \rho_2(W,d)$ such that

$$\left|\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}\right| \leq \rho_2\, 2^{-i}||\psi_{iq}(E)||_{L^2(\{1/3<|x|<1\})}. \tag{3.22}$$

Proceeding from (3.14) and using the Cauchy–Schwarz inequality we get that

$$|c_{iq}(E)| = \left|\frac{\left\langle \psi(E)-\psi_0(E), \psi_{iq}(E)\right\rangle_{L^2(\{1/3<|x|<1\})}}{||\psi_{iq}(E)||^2_{L^2(\{1/3<|x|<1\})}}\right| \leq \frac{||\psi(E)-\psi_0(E)||_{L^2(B(0,1))}}{||\psi_{iq}(E)||_{L^2(\{1/3<|x|<1\})}}. \tag{3.23}$$

Taking into account

$$\langle \Lambda_{v,E} f_{jp}, f_{iq}\rangle = \left\langle \left.\frac{\partial(\psi(E)-\psi_0(E))}{\partial \nu}\right|_{\partial D}, f_{iq}\right\rangle = c_{iq}(E)\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1} \tag{3.24}$$

and combining (3.22) and (3.23) we obtain that

$$|\langle \Lambda_{v,E} f_{jp}, f_{iq}\rangle| \leq \rho_2 2^{-i}||\psi(E)-\psi_0(E)||_{L^2(B(0,1))}. \tag{3.25}$$

From (3.13) and (3.25) we get (3.8).

For the general case we consider two compacts

$$W_\pm = W \cap \{z \mid \pm \operatorname{Im} z \geq 0\}. \tag{3.26}$$

Note that $\frac{J_{j+\frac{d-2}{2}}(\sqrt{E}r)}{J_{j+\frac{d-2}{2}}(\sqrt{E})}$ and $\frac{Y_{j+\frac{d-2}{2}}(\sqrt{E}r)}{Y_{j+\frac{d-2}{2}}(\sqrt{E})}$ have removable singularity in $E = 0$ or, more precisely,

$$\begin{gathered}\frac{J_{j+\frac{d-2}{2}}(\sqrt{E}r)}{J_{j+\frac{d-2}{2}}(\sqrt{E})} \longrightarrow r^{j+\frac{d-2}{2}},\\ \frac{Y_{j+\frac{d-2}{2}}(\sqrt{E}r)}{Y_{j+\frac{d-2}{2}}(\sqrt{E})} \longrightarrow r^{-j-\frac{d-2}{2}}\\ \text{as } E \longrightarrow 0.\end{gathered} \tag{3.27}$$

Considering the limit as $E \to 0$ we get that (3.13), (3.25) and consequently (3.8) are valid for $W_{\pm}$. To complete proof we can take $\rho = \max\{\rho_+, \rho_-\}$. ∎

Remark 3.2. From (3.1) and (3.10) we get that

$$\langle \Lambda_{v,E} f_{jp}, f_{iq} \rangle \quad \text{is holomorphic function in } W. \tag{3.28}$$

4. A fat metric space and a thin metric space

Definition 4.1. Let $(X, dist)$ be a metric space and $\epsilon > 0$. We say that a set $Y \subset X$ is an ϵ-net for $X_1 \subset X$ if for any $x \in X_1$ there is $y \in Y$ such that $dist(x, y) \le \epsilon$. We call ϵ-entropy of the set X_1 the number $\mathcal{H}_\epsilon(X_1) := \log_2 \min\{|Y| : Y \text{ is an } \epsilon\text{-net fot } X_1\}$.

A set $Z \subset X$ is called ϵ-discrete if for any distinct $z_1, z_2 \in Z$, we have $dist(z_1, z_2) \ge \epsilon$. We call ϵ-capacity of the set X_1 the number $\mathcal{C}_\epsilon := \log_2 \max\{|Z| : Z \subset X_1 \text{ and } Z \text{ is } \epsilon\text{-discrete}\}$.

The use of ϵ-entropy and ϵ-capacity to derive properties of mappings between metric spaces goes back to Vitushkin and Kolmogorov (see [10] and references therein). One notable application was Hilbert's 13th problem (about representing a function of several variables as a composition of functions of a smaller number of variables). In essence, Lemma 4.1 and Lemma 4.2 are parts of the Theorem XIV and the Theorem XVII in [10].

Lemma 4.1. *Let $d \ge 2$ и $m > 0$. For $\epsilon, \beta > 0$, consider the real metric space*

$$X_{m\epsilon\beta} = \{f \in C^m(D) \mid supp\, f \subset B(0, 1/3),\ ||f||_{L^\infty(D)} \le \epsilon,\ ||f||_{C^m(D)} \le \beta\},$$

with the metric induced by L^∞. Then there is a $\mu > 0$ such that for any $\beta > 0$ and $\epsilon \in (0, \mu\beta)$, there is an ϵ-discrete set $Z \subset X_{m\epsilon\beta}$ with at least $\exp\left(2^{-d-1}(\mu\beta/\epsilon)^{d/m}\right)$ elements.

Lemma 4.1 was also formulated and proved in [3].

Lemma 4.2. *For the interval $I = [a, b]$ with $a < b$ and $\gamma > 0$ consider ellipse $W_{I,\gamma} \in \mathbb{C}$*

$$W_{I,\gamma} = \{\frac{a+b}{2} + \frac{a-b}{2}\cos z \mid |Im\, z| \le \gamma\}. \tag{4.1}$$

Then there is a constant $\nu = \nu(C, \gamma) > 0$, such that for every $\delta \in (0, e^{-1})$, there is a δ-net for the space functions on I with L^∞-norm, having holomorphic continuation to $W_{I,\gamma}$ with module bounded above on $W_{I,\gamma}$ by the constant C, with at most $\exp(\nu(\ln \delta^{-1})^2)$ elements.

Proof of Lemma 4.2. Theorem XVII in [10] provides asymptotic behaviour of the entropy of this space with respect to $\delta \to 0$. Here we get upper estimate of it. Suppose $g(z)$ is holomorphic function in $W_{I,\gamma}$ with module bounded above by the constant C. Consider the function $f(z) = g(\frac{a+b}{2} + \frac{a-b}{2}\cos z)$. By the choise of $W_{I,\gamma}$ we get that $f(z)$ is 2π-periodic holomorphic function in the stripe $|\mathrm{Im}\, z| \le \gamma$. Then for any integer n

$$|c_n| = \left|\int_0^{2\pi} e^{inx} f(x)dx\right| \le \int_0^{2\pi} e^{-|n|\gamma} C dx \le 2\pi C e^{-|n|\gamma}. \tag{4.2}$$

Let n_δ be the smallest natural number such that $2\pi C e^{-n\gamma} \le 6\pi^{-2}(n+1)^{-2}\delta$ for any $n \ge n_\delta$. Taking natural logarithm and using $\ln\delta^{-1} \ge 1$, we get that

$$n_\delta \le C' \ln \delta^{-1}, \tag{4.3}$$

where C' depends only on C and γ. We denote $\delta' = 3\pi^{-2}(n_\delta+1)^{-2}\delta$. Consider the set

$$Y_\delta = \delta'\mathbb{Z}\bigcap[-2\pi C, 2\pi C] + i\cdot\delta'\mathbb{Z}\bigcap[-2\pi C, 2\pi C]. \tag{4.4}$$

Using (4.3), we have that

$$|Y_\delta| = (1 + 2[2\pi C/\delta'])^2 \le C''\delta^{-2}\ln^4\delta^{-1}, \tag{4.5}$$

with C'' depending only on C and γ. We set

$$Y = \left\{ \sum_{n=0}^{\infty} d_n \cos\left(n \arccos\frac{x - \frac{a+b}{2}}{\frac{a-b}{2}}\right) \mid d_n \in Y_\delta \text{ for } n \le n_\delta,\ d_n = 0 \text{ otherwise} \right\}. \tag{4.6}$$

For given $f(z)$ in case of $n \le n_\delta$ we take d_n to be one of the closest elements of Y_δ to c_n. Since $|c_n| \le 2\pi C$, this ensures $|c_n - d_n| \le 2\delta'$. For $n > n_\delta$ we take $d_n = 0$. We have then

$$|c_n - d_n| \le 6\pi^{-2}(n+1)^{-2}\delta. \tag{4.7}$$

For $n > n_\delta$ this is true by the construction of n_δ, otherwise by the choise of δ'. Since $f(x)$ is 2π-periodic even function, we get $g_Y(x) \in Y$ such that

$$||g(x) - g_Y(x)||_{L^\infty(a,b)} \le \sum_{n=0}^{\infty}|c_n - d_n| \le 6\pi^{-2}\delta\sum_{n=1}^{\infty}\frac{1}{n^2} = \delta. \tag{4.8}$$

We have that $|Y| = |Y_\delta|^{n_\delta}$. Taking into account (4.3),(4.5) and $\ln\delta^{-1} \ge 1$, we get

$$|Y| \le (C''\delta^{-2}\ln^4\delta^{-1})^{C'\ln\delta^{-1}} \le \exp\left(C'''\ln\delta^{-1}C'\ln\delta^{-1}\right) \le \exp(\nu(\ln\delta^{-1})^2). \tag{4.9}$$

■

Remark 4.1. The assertion is valid even in the case of $a = b$. As δ-net we can take

$$Y = \frac{\delta}{2}\mathbb{Z}\bigcap[-C, C] + i\cdot\frac{\delta}{2}\mathbb{Z}\bigcap[-C, C]. \tag{4.10}$$

Consider an operator $A : H^{-s}(S^{d-1}) \to H^s(S^{d-1})$. We denote its matrix elements in the basis $\{f_{jp}\}$ by $a_{jpiq} = \langle Af_{jp}, f_{iq}\rangle$. From [3] we have that

$$||A||_{H^{-s}\to H^s} \le 4\sup_{j,p,i,q}(1 + \max(j,i))^{2s+d}|a_{jpiq}|. \tag{4.11}$$

Consider system $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals. We introduce the Banach space

$$X_{S,s} = \left\{ \Big(a_{jpiq}(E)\Big) \mid \left\| \Big(a_{jpiq}(E)\Big) \right\|_{X_{S,s}} := \sup_{j,p,i,q} \left((1+\max(j,i))^{2s+d} \sup_{E\in S} |a_{jpiq}(E)| \right) < \infty \right\}.$$

Denote by B^∞ the ball of centre 0 and radius $2\sigma/3$ in $L^\infty(B(0,1/3))$. We identify in the sequel an operator $A(E) : H^{-s}(S^{d-1}) \to H^s(S^{d-1})$ with its matrix $\Big(a_{jpiq}(E)\Big)$. Note that the estimate (4.11) implies that

$$\sup_{E\in S} \|A(E)\|_{H^{-s}\to H^s} \le 4 \left\| \Big(a_{jpiq}(E)\Big) \right\|_{X_{S,s}}. \tag{4.12}$$

We consider operator $\Lambda_{v,E}$ from Lemma 3.4 as

$$\Lambda : B^\infty \to \left\{ \Big(a_{jpiq}(E)\Big) \right\}, \tag{4.13}$$

where $a_{jpiq}(E)$ are matrix elements in the basis $\{f_{jp}\}$ of operator $\Lambda_{v,E}$.

Lemma 4.3. *Λ maps B^∞ into $X_{S,s}$ for any s. There is a constant $0 < \eta = \eta(S,s,d)$, such that for every $\delta \in (0, e^{-1})$, there is a δ-net Y for $\Lambda(B^\infty)$ in $X_{S,s}$ with at most $\exp(\eta(\ln\delta^{-1})^{2d})$ elements.*

Proof of Lemma 4.3. For simplicity we give first a proof in case of S consists of only one σ-regular interval I. From (4.1) we take $W_I = W_{I,\gamma}$, where constant $\gamma > 0$ is such as for any $E \in W_I$ there is E_I in I such as $|E - E_I| < \sigma/6$. From (2.16) we get that

$$|E - \lambda| \ge |E_I - \lambda| - |E - E_I| \ge 5\sigma/6, \tag{4.14}$$

with λ being Dirichlet eigenvalue for operator $-\Delta$ in D which is closest to E. Then for potential $v \in B^\infty$ and $E \in W_I$ we have that

$$||(-\Delta + v - E)^{-1}||_{L^2(D)} \le (|\lambda - E| - 2\sigma/3)^{-1} \le (5\sigma/6 - 2\sigma/3)^{-1} = 6/\sigma \tag{4.15}$$

and

$$||v||_{L^\infty(D)} ||(-\Delta + v - E)^{-1}||_{L^2(D)} \le (2\sigma/3)(6/\sigma) = 4, \tag{4.16}$$

where $(-\Delta + v - E)^{-1}$ is considered with the Dirichlet boundary condition. We obtain from Lemma 3.4 that

$$|a_{jpiq}(E)| \le 4\rho\, 2^{-\max(j,i)}, \tag{4.17}$$

where $\rho = \rho(W_I, d)$. Hence $||(a_{jpiq}(E))||_{X_{S,s}} \le \sup_l (1+l)^{2s+d} 4\rho\, 2^{-l} < \infty$ for any s and d and so the first assertion of the Lemma 4.3 is proved.

Let $l_{\delta s}$ be the smallest natural number such that $(1+l)^{2s+d} 4\rho\, 2^{-l} \le \delta$ for any $l \ge l_{\delta s}$. Taking natural logarithm and using $\ln\delta^{-1} \ge 1$, we get that

$$l_{\delta s} \le C' \ln\delta^{-1}, \tag{4.18}$$

where the constant C' depends only on s, d and I. Denote Y_{jpiq} is δ_{jpiq}-net from Lemma 4.2 with constant $C = \sup_l (1+l)^{2s+d} 4\rho\, 2^{-l}$, where $\delta_{jpiq} = (1+\max(j,i))^{-2s-d}\delta$. We set

$$Y = \{(a_{jpiq}(E)) \mid a_{jpiq}(E) \in Y_{jpiq} \text{ for } \max(j,i) \le l_{\delta s},\ a_{jpiq}(E) = 0 \text{ otherwise}\}. \tag{4.19}$$

For any $(a_{jpiq}(E)) \in \Lambda(B^\infty)$ there is an element $(b_{jpiq}(E)) \in Y$ such that

$$(1+\max(j,i))^{2s+d}|a_{jpiq}(E) - b_{jpiq}(E)| \le (1+\max(j,i))^{2s+d}\delta_{jpiq} = \delta, \tag{4.20}$$

in case of $\max(j,i) \le l_{\delta s}$ and

$$(1+\max(j,i))^{2s+d}|a_{jpiq}(E) - b_{jpiq}(E)| \le (1+\max(j,i))^{2s+d} 2\rho\, 2^{-\max(j,i)} \le \delta, \tag{4.21}$$

otherwise.

It remains to count the elements of Y. Using again the fact that $\ln \delta^{-1} \ge 1$ and (4.18) we get for $\max(j,i) \le l_{\delta s}$

$$|Y_{jpiq}| \le \exp(\nu(\ln \delta_{jpiq}^{-1})^2) \le \exp(\nu'(\ln \delta^{-1})^2). \tag{4.22}$$

From [3] we have that $n_{\delta s} \le 8(1+l_{\delta s})^{2d-2}$, where $n_{\delta s}$ is the number of four-tuples (j,p,i,q) with $\max(j,i) \le l_{\delta s}$. Taking η to be big enough we get that

$$\begin{aligned} |Y| &\le \left(\exp(\nu'(\ln \delta^{-1})^2)\right)^{n_{\delta s}} \\ &\le \exp\left(\nu'(\ln \delta^{-1})^2 8(1 + C' \ln \delta^{-1})^{2d-2}\right) \\ &\le \exp\left(\eta(\ln \delta^{-1})^{2d}\right). \end{aligned} \tag{4.23}$$

For $S = \bigcup\limits_{j=1}^{K} I_j$ assertion follows immediately, taking η to be in K times more and Y as composition $(Y_1, \ldots, Y_K)$ of δ-nets for each interval. ■

5. Proofs of the main results

In this section we give proofs of Theorem 2.3 and Theorem 2.4.

Proof of Theorem 2.3. Take $v_0 \in L^\infty(B(0,1/3))$, $||v_0||_{L^\infty(D)} \le \sigma/3$ and $\epsilon \in (0,\sigma/3)$. By Lemma 4.1, the set $v_0 + X_{m\epsilon\beta}$ has an ϵ-discrete subset $v_0 + Z$. Since for $\epsilon \in (0,\sigma/3)$ we have $v_0 + X_{m\epsilon\beta} \subset B^\infty$, where B^∞ is the ball of centre 0 and radius $2\sigma/3$ in $L^\infty(B(0,1/3))$. The set Y constructed in Lemma 4.3 is also δ-net for $\Lambda(v_0 + X_{m\epsilon\beta})$. We take δ such that $8\delta = \exp\left(-\epsilon^{-\frac{1}{2m}}\right)$. Note that inequalities of (2.17) follow from

$$|v_0 + Z| > |Y|. \tag{5.1}$$

In fact, if $|v_0 + Z| > |Y|$, then there are two potentials $v_1, v_2 \in v_0 + Z$ with images under Λ in the same $X_{S,s}$-ball radius δ centered at a point of Y, so we get from (4.12)

$$\sup_{E\in S} ||\Phi_1(E) - \Phi_2(E)||_{H^{-s}\to H^s} \le 4||\Lambda_{v_1,E} - \Lambda_{v_2,E}||_{X_{S,s}} \le 8\delta = \exp\left(-\epsilon^{-\frac{1}{2m}}\right). \tag{5.2}$$

It remains to find β such as (5.1) is fullfiled. By Lemma 4.3

$$|Y| \le \exp\left(\eta\left(\ln 8 + \epsilon^{-\frac{1}{2m}}\right)^{2d}\right) \le \max\left(\exp\left((2\ln 8)^{2d}\eta\right), \exp\left(2^{2d}\eta\epsilon^{-d/m}\right)\right). \tag{5.3}$$

Now we take

$$\beta > \mu^{-1} \max\left(\sigma/3, \eta^{m/d}2^{3m}, \frac{\sigma}{3}\eta^{m/d}2^m(2\ln 8)^{2m}\right) \tag{5.4}$$

This fulfils requirement $\epsilon < \mu\beta$ in Lemma 4.1, which gives

$$\begin{aligned} |v_0 + Z| = |Z| \geq \exp\left(2^{-d-1}(\mu\beta/\epsilon)^{d/m}\right) \overset{(5.4)}{>} \\ > \max\left(\exp\left(2^{-d-1}(\eta^{m/d}2^{3m}/\epsilon)^{d/m}\right), \exp\left(2^{-d-1}(\eta^{m/d}2^m(2\ln 8)^{2m})^{d/m}\right)\right) \overset{(5.3)}{\geq} |Y|. \end{aligned} \tag{5.5}$$

■

Proof of Theorem 2.4. In a similar way with the proof of Theorem 2 of [3] we obtain that

$$\langle (\Phi_{mn}(E) - \Phi_0(E))\, f_{jp}, f_{iq}\rangle = 0 \tag{5.6}$$

for $j, i \leq \left[\frac{n-1}{2}\right]$. The only difference is that instead of the operator $-\Delta$ we consider the operator $-\Delta - E$. From (4.11), (4.17) and (5.6) we get

$$||\Phi_{mn}(E) - \Phi_0(E)||_{H^{-s}\to H^s} \leq 16\rho \sup_{l\geq n/2} (1+l)^{2s+d}2^{-l} \leq c' 2^{-n/4}. \tag{5.7}$$

The fact that $||v_{mn}||_{C^m(D)}$ is bounded as $n \to \infty$ is also a part of Theorem 2 of [3]. ■

6. Bessel functions

In this section we prove Lemma 3.1, Lemma 3.2 and Lemma 3.3 about the Bessel functions. Consider the problem of finding solutions of the form $\psi(r,\omega) = R(r) f_{jp}(\omega)$ of equation (1.1) with $v \equiv 0$. We have that

$$\Delta = \frac{\partial^2}{(\partial r)^2} + (d-1)r^{-1}\frac{\partial}{\partial r} + r^{-2}\Delta_{S^{d-1}}, \tag{6.1}$$

where $\Delta_{S^{d-1}}$ is Laplace-Beltrami operator on S^{d-1}. We have that

$$\Delta_{S^{d-1}} f_{jp} = -j(j+d-2) f_{jp}. \tag{6.2}$$

Then we have the following equation for $R(r)$:

$$-R'' - \frac{d-1}{r}R' + \frac{j(j+d-2)}{r^2}R = ER. \tag{6.3}$$

Taking $R(r) = r^{-\frac{d-2}{2}}\tilde{R}(r)$, we get

$$r^2\tilde{R}'' + r\tilde{R}' + \left(Er^2 - \left(j + \frac{d-2}{2}\right)^2\right)\tilde{R} = 0. \tag{6.4}$$

This equation is known as Bessel's equation. For $E = k^2 \neq 0$ it has two linearly independent solutions $J_{j+\frac{d-2}{2}}(kr)$ and $Y_{j+\frac{d-2}{2}}(kr)$, where

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\alpha}}{\Gamma(m+1)\Gamma(m+\alpha+1)}, \tag{6.5}$$

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos\pi\alpha - J_{-\alpha}(z)}{\sin\pi\alpha} \text{ for } \alpha \notin \mathbb{Z}, \tag{6.6}$$

and

$$Y_\alpha(z) = \lim_{\alpha'\to\alpha} Y_{\alpha'}(z) \text{ for } \alpha \in \mathbb{Z}. \tag{6.7}$$

The following Lemma is called the Nielsen inequality. A proof can be found in [5]

Lemma 6.1.

$$\begin{aligned} J_\alpha(z) &= \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}(1+\theta), \\ |\theta| &< \exp\left(\frac{|z|^2/4}{|\alpha_0+1|}\right) - 1, \end{aligned} \tag{6.8}$$

where $|\alpha_0+1|$ *is the least of numbers* $|\alpha+1|, |\alpha+2|, |\alpha+3|, \dots$.

Lemma 6.1 implies that $r^{-\frac{d-2}{2}}J_{j+\frac{d-2}{2}}(kr)$ has removable singularity at $r=0$. Using the boundary conditions $R(1)=1$ and $R(1)=0$, we obtain assertions of Lemma 3.1 and Lemma 3.2, respectively.

Proof of Lemma 3.3 Formula (3.4) follows immediately from Lemma 6.1. We have from [5] that

$$J'_\alpha(z) = J_{\alpha-1}(z) - \frac{\alpha}{z}J_\alpha(z). \tag{6.9}$$

Further, taking α big enough we get

$$|J'_\alpha(z)| \le |J_{\alpha-1}(z)| + |\frac{\alpha}{z}J_\alpha(z)| \le \frac{3}{2}\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)} + \frac{3\alpha}{2|z|}\frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)} \le 3\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)}. \tag{6.10}$$

For $\alpha = n+1/2$ we have $Y_\alpha = (-1)^{n+1}J_{-\alpha}$. Consider its series expansion, see (6.5).

$$J_{-\alpha}(z) = \sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{2m-\alpha}}{m!\,\Gamma(m-\alpha+1)} = \sum_{m=0}^{\infty} c_m(z/2)^{2m-\alpha}. \tag{6.11}$$

Note that $|c_m/c_{m+1}| = (m+1)|m-\alpha+1| \ge n/2$. As corollary we obtain that

$$\begin{aligned} |Y_\alpha(z)| &= \frac{(|z|/2)^{-\alpha}}{|\Gamma(-\alpha+1)|}(1+\theta) = \frac{1}{\pi}(|z|/2)^{-\alpha}\Gamma(\alpha)(1+\theta), \\ |\theta| &\le \sum_{m=1}^{\infty}\left(\frac{|z|^2}{2n}\right)^{2m} \le \frac{|z|^2/2n}{1-|z|^2/2n}. \end{aligned} \tag{6.12}$$

For $\alpha = n$ we have from [5] that

$$\begin{aligned} Y_n(z) &= \frac{2}{\pi}J_n(z)\ln\left(\frac{z}{2}\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\left(\frac{z}{2}\right)^{2m-n}\frac{(n-m-1)!}{m!} - \\ &-\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{2m+n}}{m!(m+n)!}\left(\frac{\Gamma'(m+1)}{\Gamma(m+1)} + \frac{\Gamma'(m+n+1)}{\Gamma(m+n+1)}\right) = \\ &= \frac{2}{\pi}J_n(z)\ln\left(\frac{z}{2}\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\tilde{c}_m(z/2)^{2m-n} - \frac{1}{\pi}\sum_{m=0}^{\infty}b_m(z/2)^{2m+n}. \end{aligned} \tag{6.13}$$

Using well-known equality $\Gamma'(x)/\Gamma(x) < \ln x$, $x > 1$, see [11], we get following estimation for the coefficients b_m are defined in (6.13).

$$|b_m| < \frac{\ln(m+1) + \ln(n+m+1)}{m!(n+m)!} < \frac{2(n+m)}{m!(n+m)!} < \frac{1}{m!}. \tag{6.14}$$

Note also that $|\tilde{c}_m/\tilde{c}_{m+1}| = (m+1)(n-m-1) \geq n/2$. Combining it with (6.13) and (6.14), we obtain that

$$\begin{gathered} |Y_n(z)| = \frac{1}{\pi}(|z|/2)^{-n}\Gamma(n)(1+\theta), \\ |\theta| \leq 3\frac{(|z|/2)^{2n}|\ln(z/2)|}{\Gamma(n)} + \sum_{m=1}^{n-1}\left(\frac{|z|^2}{2n}\right)^{2m} + \frac{(|z|/2)^{2n}}{\Gamma(n)}\sum_{m=0}^{\infty}\frac{(|z|/2)^{2m}}{m!} \leq \\ \leq 3\pi\frac{\max\left(1,(|z|/2)^{2n+1}\right)}{\Gamma(n)} + \frac{|z|^2/2n}{1-|z|^2/2n} + \frac{(|z|/2)^{2n}e^{|z|^2/4}}{\Gamma(n)}. \end{gathered} \tag{6.15}$$

Formula (3.6) follows from (6.12) and (6.15). We have from [5] that

$$Y'_\alpha(z) = Y_{\alpha-1}(z) - \frac{\alpha}{z}Y_\alpha(z). \tag{6.16}$$

Taking n big enough, we get that

$$\begin{gathered} |Y'_\alpha(z)| \leq |Y_{\alpha-1}(z)| + |\frac{\alpha}{z}Y_\alpha(z)| \leq \\ \leq \frac{3}{2\pi}\left((|z|/2)^{-\alpha+1}\,\Gamma(\alpha-1) + \frac{\alpha}{|z|}(|z|/2)^{\alpha}\Gamma(\alpha)\right) \leq \frac{3}{\pi}(|z|/2)^{-\alpha-1}\Gamma(\alpha+1). \end{gathered} \tag{6.17}$$

Combining reqirements for n, stated above, we get that for any $n \geq N+1$ all inequalities of Lemma 3.3 are fullfiled, where N such that

$$\begin{cases} N > 3, \\ \exp\left(\dfrac{C^2/4}{N+1}\right) - 1 \leq 1/2, \\ 3\pi\dfrac{\max\left(1,(C/2)^{2N+1}\right)}{\Gamma(N)} + \dfrac{C^2}{2N-C^2} + \dfrac{(C/2)^{2N}e^{C^2/4}}{\Gamma(N)} \leq 1/2. \end{cases} \tag{6.18}$$

■

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