The varieties of tangent lines to hypersurfaces in projective spaces

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Abstract

For a hypersurface in a projective space, we consider the set of pairs of a point and a line in the projective space such that the line intersects the hypersurface at the point with a fixed multiplicity. We prove that this set of pairs forms a smooth variety for a general hypersurface.

1 Introduction

Let $\mathbf{P}^n$ be the projective space of dimension n over a field K. We denote by X_F the hypersurface in $\mathbf{P}^n$ defined by a homogeneous polynomial $F \in K[x_0, \ldots, x_n]$ of degree d. Let $\mathbf{G}$ be the Grassmannian variety of all lines in $\mathbf{P}^n$. Then the set

$$Z_F = \{L \in \mathbf{G} \mid L \subset X_F\}$$

forms a closed subscheme of $\mathbf{G}$, and it is called *Fano scheme* of lines in X_F. The Fano schemes for cubic threefolds were first studied by Fano, and they were used by Tjurin [7] and Clemens-Griffiths [3] in the proof of the Torelli theorem and the irrationality for cubic threefolds over the complex numbers. Then the foundations of the Fano schemes of cubic hypersurfaces for any characteristic were given by Altman-Kleiman [1], and the results on the smoothness and connectedness of Z_F for any degree d were proved by Barth-Van de Ven [2] and bettered in the book [6, Chapter V. 4] by Kollár. In this paper, we introduce the following scheme $Y_{F,m}$ as an analogy of the Fano scheme Z_F. For $1 \leq m \leq \infty$, we set

$$Y_{F,m} = \{(p, L) \in \mathbf{P}^n \times \mathbf{G} \mid L \text{ intersects } X_F \text{ at } p \text{ with the multiplicity} \geq m\},$$

which forms a closed subscheme of $\mathbf{P}^n \times \mathbf{G}$. Since $Y_{F,1}$ is a $\mathbf{P}^{n-1}$-bundle over X_F by the first projection and $Y_{F,\infty}$ is a $\mathbf{P}^1$-bundle over Z_F by the second projection, the scheme $Y_{F,m}$ is considered to be an intermediate object between X_F and Z_F. We expect to characterize some geometric properties of X_F by using the Hodge

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structure of $Y_{F,m}$. A computation for the Hodge structure of $Y_{F,m}$ is announced in the summary [5].

In Section 2, following the formulation for the Fano schemes in [1], we define the scheme $Y_{F,m}$ as the zeros of a section of a vector bundle on a flag variety. It enables us to compute the Chern numbers of $Y_{F,m}$ by Schubert calculus. In Section 3, we investigate the smoothness and connectedness of $Y_{F,m}$ for $m \leq d$. If $m \leq 2n-1$ and m is prime to the characteristic of K, then $Y_{F,m}$ is smooth of dimension $2n-m-1$ for a general hypersurface X_F (Theorem 3.2. (iii)). If $m \leq 2n-2$, then $Y_{F,m}$ is connected for any hypersurface X_F (Theorem 3.2. (iv)). Particularly for a cubic hypersurface X_F, the variety $Y_{F,3}$ is smooth of dimension $2n-4$ if and only if X_F is a smooth hypersurface (Theorem 3.5). These results for $Y_{F,m}$ proved in Section 3 corresponds to the results for the Fano scheme Z_F proved in [2] and [6, Chapter V. 4].

2 Varieties of pairs of a point and a line

Let $\mathbf{P}^n = \mathbf{P}^n_K$ be the projective space of dimension n over a field K, and let V be the K-vector space $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$. We denote by $\mathbf{P} = \operatorname{Grass}(n, V)$ the Grassmannian variety of all n-dimensional subspaces in V, and denote by $\mathcal{Q}_{\mathbf{P}}$ the universal quotient bundle on $\mathbf{P}$. Then $\mathbf{P}$ is naturally identified with $\mathbf{P}^n$, and $\mathcal{Q}_{\mathbf{P}}$ is identified with the tautological line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$. We denote by $\mathbf{G} = \operatorname{Grass}(n-1, V)$ the Grassmannian variety of all $(n-1)$-dimensional subspaces in V, and denote by $\mathcal{Q}_{\mathbf{G}}$ the universal quotient bundle on $\mathbf{G}$. We remark that a point of $\mathbf{G}$ corresponds to a line in $\mathbf{P}^n$. Let $\Gamma \subset \mathbf{P} \times \mathbf{G}$ be the flag variety of all pairs (p, L) of a point $p \in \mathbf{P}^n$ and a line $L \subset \mathbf{P}^n$ containing the point p. The variety Γ is the $\mathbf{P}^{n-1}$-bundle over $\mathbf{P}$ by the first projection $\phi : \Gamma \to \mathbf{P}$, and Γ is the $\mathbf{P}^1$-bundle over $\mathbf{G}$ by the second projection $\pi : \Gamma \to \mathbf{G}$. We define the line bundle $\mathcal{Q}_\phi$ on Γ as the kernel of the natural surjective homomorphism $\pi^*\mathcal{Q}_{\mathbf{G}} \to \phi^*\mathcal{Q}_{\mathbf{P}}$, and define a decreasing filtration

$$\operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}} = \operatorname{Fil}^0 \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}} \supset \cdots \supset \operatorname{Fil}^d \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}} \supset \operatorname{Fil}^\infty \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}} = 0$$

on the d-th symmetric product of $\pi^*\mathcal{Q}_{\mathbf{G}}$, as $\operatorname{Fil}^m \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}}$ being the image of the natural homomorphism

$$\operatorname{Sym}^m \mathcal{Q}_\phi \otimes \operatorname{Sym}^{d-m} \pi^*\mathcal{Q}_{\mathbf{G}} \longrightarrow \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}}$$

for $0 \leq m \leq d$, and $\operatorname{Fil}^\infty \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}} = 0$. Let $F \in \operatorname{Sym}^d V$. We denote by X_F the hypersurface in $\mathbf{P}$ defined as the zeros of the section $[F]_{\mathbf{P}} \in H^0(\mathbf{P}, \operatorname{Sym}^d \mathcal{Q}_{\mathbf{P}})$ which is the image of F by the natural isomorphism

$$\operatorname{Sym}^d V \simeq H^0(\mathbf{P}, \operatorname{Sym}^d \mathcal{Q}_{\mathbf{P}}).$$

We denote by Z_F the subscheme in $\mathbf{G}$ defined as the zeros of the section $[F]_{\mathbf{G}} \in H^0(\mathbf{G}, \operatorname{Sym}^d \mathcal{Q}_{\mathbf{G}})$ which is the image of F by the natural isomorphism

$$\operatorname{Sym}^d V \simeq H^0(\mathbf{G}, \operatorname{Sym}^d \mathcal{Q}_{\mathbf{G}}).$$

Then a point in Z_F corresponds to a line contained in X_F, and Z_F is called the *Fano scheme* of lines in X_F. We denote by $Y_{F,m}$ the subscheme in Γ defined as the zeros of the section $[F]_{\Gamma,m} \in H^0(\Gamma, \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \operatorname{Fil}^m \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}})$ which is the image of F by the natural homomorphism

$$\operatorname{Sym}^d V \simeq H^0(\Gamma, \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}) \longrightarrow H^0(\Gamma, \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \operatorname{Fil}^m \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}).$$

Let L be a line in $\mathbf{P}^n$, and let p be a point on L. The fiber of the line bundle $\mathcal{Q}_\phi$ at $(p, L) \in \Gamma$ is identified with the kernel of the restriction

$$H^0(L, \mathcal{O}_{\mathbf{P}^n}(1)|_L) \longrightarrow H^0(p, \mathcal{O}_{\mathbf{P}^n}(1)|_p).$$

Hence, L intersects X_F at p with the multiplicity $\geq m$ if and only if the pair (p, L) represents a point in $Y_{F,m}$. We have a diagram

$$\begin{array}{ccccccccccccc}
p & \in & \mathbf{P} & \supset & X_F & & & & & & & & \\
\big\uparrow & & \big\uparrow{\scriptstyle \phi} & & \big\uparrow{\scriptstyle \phi|_{Y_{F,1}}} & & & & & & & & \\
(p, L) & \in & \Gamma & \supset & Y_{F,1} & \supset & Y_{F,2} & \supset & \cdots & \supset & Y_{F,d} & \supset & Y_{F,\infty} \\
\big\downarrow & & \big\downarrow{\scriptstyle \pi} & & & & & & & & & & \big\downarrow{\scriptstyle \pi|_{Y_{F,\infty}}} \\
L & \in & \mathbf{G} & & & & & \supset & & & & & Z_F.
\end{array}$$

The morphism $\phi|_{Y_{F,1}} : Y_{F,1} \to X_F$ is the $\mathbf{P}^{n-1}$-bundle, whose fiber at $p \in X_F$ is the set of all lines through the point p. If X_F is a smooth hypersurface, then $\phi|_{Y_{F,2}} : Y_{F,2} \to X_F$ is the $\mathbf{P}^{n-2}$-bundle, whose fiber at $p \in X_F$ is the set of all lines through the point p and contained in the projective tangent space of X_F at p. The morphism $\pi|_{Y_{F,\infty}} : Y_{F,\infty} \to Z_F$ is the $\mathbf{P}^1$-bundle, whose fiber at $L \in Z_F$ is the set of all points on the line L.

For $(p, L) \in \Gamma$, there is a basis $(x_0, \ldots, x_n)$ of V such that the point p is defined by $x_1 = \cdots = x_n = 0$ and the line L is defined by $x_2 = \cdots = x_n = 0$ in $\mathbf{P}^n$. Then the map

$$\begin{array}{rcccc}
\mathbf{A}^{2n-1} = & \operatorname{Spec} K[\xi_1, \ldots, \xi_n, \zeta_2, \ldots, \zeta_n] & \xrightarrow{\sim} & U \subset \Gamma; \\
& (\xi_1, \ldots, \xi_n, \zeta_2, \ldots, \zeta_n) & \longmapsto & (p_\xi, L_{(\xi,\zeta)})
\end{array}$$

gives a local coordinate of Γ at (p, L), where p_ξ denotes the point defined by

$$x_1 - \xi_1 x_0 = \cdots = x_n - \xi_n x_0 = 0$$

and $L_{(\xi,\zeta)}$ denotes the line defined by

$$(x_2 - \xi_2 x_0) - \zeta_2(x_1 - \xi_1 x_0) = \cdots = (x_n - \xi_n x_0) - \zeta_n(x_1 - \xi_1 x_0) = 0.$$

On this local coordinate U, $([x_0]_U, [x_1]_U)$ is a local basis of $\pi^* \mathcal{Q}_{\mathbf{G}}$, and $[x_1 - \xi_1 x_0]_U$ is a local basis of $\mathcal{Q}_\phi$, where $[A]_U$ denotes the image of $A \in V$ by the restriction

$V \to H^0(U, \pi^*\mathcal{Q}_{\mathbf{G}})$. Note that $([x_0]_U, [x_1 - \xi_1 x_0]_U)$ is another local basis of $\pi^*\mathcal{Q}_{\mathbf{G}}$. We define the polynomial $f_k(\xi, \zeta) = f_k(\xi_1, \dots, \xi_n, \zeta_2, \dots, \zeta_n)$ by

$$[F]_U = \sum_{k=0}^{d} f_k(\xi, \zeta)[x_1 - \xi_1 x_0]_U^k [x_0]_U^{d-k} \in H^0(U, \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}}).$$

Then $[F]_U$ is contained in $H^0(U, \operatorname{Fil}^m \operatorname{Sym}^d \pi^*\mathcal{Q}_{\mathbf{G}})$ if and only if

$$f_0(\xi, \zeta) = \cdots = f_{m-1}(\xi, \zeta) = 0,$$

and we have

$$Y_{F,m} \cap U \simeq \operatorname{Spec} K[\xi_1, \dots, \xi_n, \zeta_2, \dots, \zeta_n] / \big(f_0(\xi, \zeta), \dots, f_{m-1}(\xi, \zeta)\big).$$

When we consider $F \in \operatorname{Sym}^d V$ as the homogeneous polynomial $F(x_0, \dots, x_n) \in K[x_0, \dots, x_n]$ of degree d, we have

$$F(x_0, x_1, \zeta_2(x_1 - \xi_1 x_0) + \xi_2 x_0, \dots, \zeta_n(x_1 - \xi_1 x_0) + \xi_n x_0) = \sum_{k=0}^{d} f_k(\xi, \zeta)(x_1 - \xi_1 x_0)^k x_0^{d-k},$$

hence the local equation of $X_F \cap L_{(\xi,\zeta)}$ in $L_{(\xi,\zeta)}$ is

$$F(1, t + \xi_1, \zeta_2 t + \xi_2, \dots, \zeta_n t + \xi_n) = \sum_{k=0}^{d} f_k(\xi, \zeta) t^k, \tag{2.1}$$

where $t = \frac{x_1}{x_0} - \xi_1$ is a local parameter of the line $L_{(\xi,\zeta)}$ at the point p_ξ.

3 Smoothness and connectedness

Since $Y_{F,\infty}$ is a $\mathbf{P}^1$-bundle over Z_F, the following theorem is directly induced from the results in [2, Theorem 8] and [6, Chapter V. Theorem 4.3].

Theorem 3.1. *Assume $d \geq 1$.*

(i) *If $d \geq 2n - 2$, then $Y_{F,\infty}$ is empty for general $F \in \operatorname{Sym}^d V$.*

(ii) *If $d \leq 2n - 3$, then $Y_{F,\infty}$ is non-empty for any $F \in \operatorname{Sym}^d V$.*

(iii) *If $d \leq 2n - 3$, then $Y_{F,\infty}$ is smooth of dimension $2n - d - 2$ for general $F \in \operatorname{Sym}^d V$.*

(iv) *If $d \leq 2n - 4$ and $(d, n) \neq (2, 3)$, then $Y_{F,\infty}$ is connected for any $F \in \operatorname{Sym}^d V$.*

In this section, we prove the following theorem;

Theorem 3.2. *Assume $1 \leq m \leq d$.*

(i) *If $m \geq 2n$, then $Y_{F,m}$ is empty for general $F \in \operatorname{Sym}^d V$.*

(ii) *If $m \leq 2n-1$, then $Y_{F,m}$ is non-empty for any $F \in \operatorname{Sym}^d V$.*

(iii) *If $m \leq 2n-1$ and m is prime to the characteristic of K, then $Y_{F,m}$ is smooth of dimension $2n-m-1$ for general $F \in \operatorname{Sym}^d V$.*

(iv) *If $m \leq 2n-2$, then $Y_{F,m}$ is connected for any $F \in \operatorname{Sym}^d V$.*

We denote by $M_d = \operatorname{Grass}(1, \operatorname{Sym}^d V)$ the space of hypersurfaces of degree d in $\mathbf{P}^n$. We set the vector bundle $\mathcal{E}_{d,m}$ on Γ by

$$\mathcal{E}_{d,m} = \operatorname{Ker}(\mathcal{O}_\Gamma \otimes \operatorname{Sym}^d V \longrightarrow \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}} / \operatorname{Fil}^m \operatorname{Sym}^d \pi^* \mathcal{Q}_{\mathbf{G}}),$$

and we consider the projective space bundle $Y_{d,m} = \operatorname{Grass}(1, \mathcal{E}_{d,m}) \to \Gamma$. Then $Y_{d,m}$ is a smooth subvariety of codimension m in $M_d \times \Gamma = \operatorname{Grass}(1, \mathcal{O}_\Gamma \otimes \operatorname{Sym}^d V)$, and the fiber of the projection

$$\psi_{d,m} : Y_{d,m} \longrightarrow M_d; \ \ ([F], p, L) \longmapsto [F]$$

at $[F] \in M_d$ is equal to $Y_{F,m}$. We denote by $Y_{d,m}(p, L) \subset M_d$ the fiber of the projective space bundle $Y_{d,m} \to \Gamma$ at $(p, L) \in \Gamma$. For $(p, L) \in \Gamma$, we fix a basis $(x_0, \ldots, x_n)$ of V such that the point p is defined by $x_1 = \cdots = x_n = 0$ and the line L is defined by $x_2 = \cdots = x_n = 0$ in $\mathbf{P}^n$. Then $Y_{d,m}(p, L)$ is the linear subspace

$$Y_{d,m}(p, L) = \{[F] \in M_d \mid a_0 = \cdots = a_{m-1} = 0\},$$

where a_i denotes the coefficient of the monomial $x_0^{d-i} x_1^i$ in $F(x_0, \ldots, x_n)$. For $(p, L) \in Y_{m,F}$, we define the matrix $J_m([F], p, L)$ by

$$J_m([F], p, L) = \begin{pmatrix} \frac{\partial f_0}{\partial \xi_1}(0) & \cdots & \frac{\partial f_0}{\partial \xi_n}(0) & \frac{\partial f_0}{\partial \zeta_2}(0) & \cdots & \frac{\partial f_0}{\partial \zeta_n}(0) \\ & \cdots & & & \cdots & \\ \frac{\partial f_{m-1}}{\partial \xi_1}(0) & \cdots & \frac{\partial f_{m-1}}{\partial \xi_n}(0) & \frac{\partial f_{m-1}}{\partial \zeta_2}(0) & \cdots & \frac{\partial f_{m-1}}{\partial \zeta_n}(0) \end{pmatrix},$$

where $(\xi_1, \ldots, \xi_n, \zeta_2, \ldots, \zeta_n)$ is the local coordinate of Γ and $f_0(\xi, \zeta), \ldots, f_{m-1}(\xi, \zeta)$ are the local equations of $Y_{F,m}$ in Section 2. By the equation (2.1), we have

$$J_m([F], p, L) = \begin{pmatrix} 0 & a_{0,2} & \cdots & a_{0,n} & 0 & \cdots & 0 \\ & & & & a_{0,2} & \cdots & a_{0,n} \\ \vdots & & \cdots & & & & \\ & & & & & \cdots & \\ 0 & a_{m-2,2} & \cdots & a_{m-2,n} & & & \\ m a_m & a_{m-1,2} & \cdots & a_{m-1,n} & a_{m-2,2} & \cdots & a_{m-2,n} \end{pmatrix},$$

where $a_{k,j}$ denotes the coefficient of the monomial $x_0^{d-k-1}x_1^k x_j$ in $F(x_0,\dots,x_n)$. We define the degeneracy locus $W_{d,m}$ in $Y_{d,m}$ by

$$\begin{aligned} W_{d,m} &= \{([F],p,L) \in Y_{d,m} \mid \operatorname{rank} J_m([F],p,L) < m\} \\ &= \{([F],p,L) \in Y_{d,m} \mid \operatorname{rank} \mathrm{d}\psi_{d,m}([F],p,L) < \dim M_d\}, \end{aligned}$$

where $\mathrm{d}\psi_{d,m}$ denotes the homomorphism on tangent spaces induced by $\psi_{d,m} : Y_{d,m} \to M_d$. We remark that $W_{d,2} \subset W_{d,m}$ for $m \geq 2$, and we set $W^0_{d,m} = W_{d,m} \setminus W_{d,2}$.

Proposition 3.3. *Assume* $1 \leq m \leq d$.

(i) $\operatorname{codim}_{Y_{d,1}} W_{d,1} = n$ *for* $d \geq 1$.

(ii) $\operatorname{codim}_{Y_{d,2}} W_{d,2} = n-1$ *for* $d \geq 2$.

(iii) *If* $m = 2n-1$ *is prime to the characteristic of* K*, then* $\operatorname{codim}_{Y_{d,2n-1}} W_{d,2n-1} = 1$.

(iv) *If* $3 \leq m \leq 2n-2$ *and* m *is prime to the characteristic of* K*, then*

$$\begin{cases} \operatorname{codim}_{Y_{d,m}} W_{d,m} = \min\{n-1, 2n-m\}, \\ \operatorname{codim}_{Y_{d,m}} W^0_{d,m} = 2n-m. \end{cases}$$

(v) *If* $3 \leq m \leq 2n-2$ *and* m *is divisible by the characteristic of* K*, then*

$$\begin{cases} \operatorname{codim}_{Y_{d,m}} W_{d,m} = \min\{n-1, 2n-m-1\}, \\ \operatorname{codim}_{Y_{d,m}} W^0_{d,m} = 2n-m-1. \end{cases}$$

We denote by $\mathrm{Mat}\,(l,r)$ the K-vector space of $l \times r$ matrices. We define a subscheme $\Delta(l,r)$ in $\mathrm{Grass}\,(1, \mathrm{Mat}\,(l,r))$ by

$$\Delta(l,r) = \{[B] \in \mathrm{Grass}\,(1,\mathrm{Mat}\,(l,r)) \mid \operatorname{rank} \tilde{B} < l\},$$

where we set

$$\tilde{B} = \begin{pmatrix} b_{1,1} & \cdots & b_{1,r} & 0 & \cdots & 0 \\ & & & b_{1,1} & \cdots & b_{1,r} \\ & \cdots & & & & \\ & & & & \cdots & \\ b_{l-1,1} & \cdots & b_{l-1,r} & & & \\ b_{l,1} & \cdots & b_{l,r} & b_{l-1,1} & \cdots & b_{l-1,r} \end{pmatrix} \in \mathrm{Mat}\,(l,2r)$$

for a matrix

$$B = \begin{pmatrix} b_{1,1} & \cdots & b_{1,r} \\ & \cdots & \\ b_{l,1} & \cdots & b_{l,r} \end{pmatrix} \in \mathrm{Mat}\,(l,r).$$

We set an open subset $\Delta^0(l,r)$ of $\Delta(l,r)$ by

$$\Delta^0(l,r) = \{[B] \in \Delta(l,r) \mid (b_{1,1},\dots,b_{1,r}) \neq (0,\dots,0)\}.$$

Lemma 3.4. *For* $3 \le l \le 2r$,

$$\begin{cases} \operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(l,r))} \Delta(l,r) = \min\{r, 2r-l+1\}, \\ \operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(l,r))} \Delta^0(l,r) = 2r-l+1. \end{cases}$$

Proof. For $2 \le i \le l$, we set

$$\Delta_i(l,r) = \left\{ [B] \in \Delta(l,r) \mid \operatorname{rank} \begin{pmatrix} b_{1,1} & \cdots & b_{1,r} & 0 & \cdots & 0 \\ & & & b_{1,1} & \cdots & b_{1,r} \\ & \cdots & & & & \\ & & & & \cdots & \\ b_{i-1,1} & \cdots & b_{i-1,r} & & & \\ b_{i,1} & \cdots & b_{i,r} & b_{i-1,1} & \cdots & b_{i-1,r} \end{pmatrix} < i \right\}.$$

Then

$$\Delta_2(l,r) \subset \cdots \subset \Delta_l(l,r) = \Delta(l,r),$$

hence we have $\Delta(l,r) = \Delta_2(l,r) \amalg \Delta^0(l,r)$ and

$$\Delta^0(l,r) = \coprod_{i=3}^{l} \big(\Delta_i(l,r) \setminus \Delta_{i-1}(l,r)\big).$$

Since

$$\Delta_2(l,r) = \{[B] \in \Delta(l,r) \mid (b_{1,1}, \ldots, b_{1,r}) = (0, \ldots, 0)\},$$

we have $\operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(l,r))} \Delta_2(l,r) = r$. For $3 \le i \le l$, there is an open immersion

$$\Delta_i(l,r) \setminus \Delta_{i-1}(l,r) \longrightarrow \mathrm{Grass}\,(1, \mathrm{Mat}\,(l-2,r)) \times \mathbf{A}^{i-1};$$

$$\begin{bmatrix} b_{1,1} & \cdots & b_{1,r} \\ & \cdots & \\ b_{l,1} & \cdots & b_{l,r} \end{bmatrix} \longmapsto \left(\begin{bmatrix} b_{1,1} & \cdots & b_{1,r} \\ & \cdots & \\ b_{i-2,1} & \cdots & b_{i-2,r} \\ b_{i+1,1} & \cdots & b_{i+1,r} \\ & \cdots & \\ b_{l,1} & \cdots & b_{l,r} \end{bmatrix}, (\alpha_1, \ldots, \alpha_{i-1}) \right),$$

where $(\alpha_1, \ldots, \alpha_{i-1})$ is determined by

$$\begin{aligned} &(b_{i,1}, \ldots, b_{i,r}, b_{i-1,1}, \ldots, b_{i-1,r}) \\ &\qquad = (\alpha_1, \ldots, \alpha_{i-1}) \begin{pmatrix} b_{1,1} & \cdots & b_{1,r} & 0 & \cdots & 0 \\ & & & b_{1,1} & \cdots & b_{1,r} \\ & \cdots & & & & \\ & & & & \cdots & \\ b_{i-2,1} & \cdots & b_{i-2,r} & & & \\ b_{i-1,1} & \cdots & b_{i-1,r} & b_{i-2,1} & \cdots & b_{i-2,r} \end{pmatrix}. \end{aligned}$$

Hence we have $\operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(l,r))} \big(\Delta_i(l,r) \setminus \Delta_{i-1}(l,r)\big) = 2r - i + 1$. $\square$

Proof of Proposition 3.3. We set $W_{d,m}(p,L) = Y_{d,m}(p,L) \cap W_{d,m}$ and $W^0_{d,m}(p,L) = Y_{d,m}(p,L) \cap W^0_{d,m}$, and we compute their codimension in $Y_{d,m}(p,L)$. It is clear that $\operatorname{codim}_{Y_{d,1}(p,L)} W_{d,1}(p,L) = n$ and $\operatorname{codim}_{Y_{d,2}(p,L)} W_{d,2}(p,L) = n-1$. Since $W_{d,2n-1}(p,L)$ is defined by $\det J_{2n-1}([F],p,L) = 0$ in $Y_{d,2n-1}(p,L)$, if $m = 2n-1$ is prime to the characteristic of K, then we have $\operatorname{codim}_{Y_{d,2n-1}(p,L)} W_{d,2n-1}(p,L) = 1$. We assume $3 \le m \le 2n-2$. We define the hyperplane $T_{d,m}(p,L)$ in $Y_{d,m}(p,L)$ by

$$T_{d,m}(p,L) = \{[F] \in Y_{d,m}(p,L) \mid a_m = 0\}.$$

If m is prime to the characteristic of K, then by Lemma 3.4, we have

$$\begin{aligned}
&\operatorname{codim}_{Y_{d,m}(p,L)} (W_{d,m}(p,L) \setminus T_{d,m}(p,L)) \\
&= \operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(m-1,n-1))} \Delta(m-1,n-1) = \min\{n-1, 2n-m\}, \\
&\operatorname{codim}_{Y_{d,m}(p,L)} (W_{d,m}(p,L) \cap T_{d,m}(p,L)) \\
&= 1 + \operatorname{codim}_{Y_{d,m}(p,L)\cap T_{d,m}(p,L)} (W_{d,m}(p,L) \cap T_{d,m}(p,L)) \\
&= 1 + \operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(m,n-1))} \Delta(m,n-1) = \min\{n, 2n-m\}
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{codim}_{Y_{d,m}(p,L)} (W^0_{d,m}(p,L) \setminus T_{d,m}(p,L)) \\
&= \operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(m-1,n-1))} \Delta^0(m-1,n-1) = 2n-m, \\
&\operatorname{codim}_{Y_{d,m}(p,L)} (W^0_{d,m}(p,L) \cap T_{d,m}(p,L)) \\
&= 1 + \operatorname{codim}_{Y_{d,m}(p,L)\cap T_{d,m}(p,L)} (W^0_{d,m}(p,L) \cap T_{d,m}(p,L)) \\
&= 1 + \operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(m,n-1))} \Delta^0(m,n-1) = 2n-m.
\end{aligned}$$

If m is divisible by the characteristic of K, then by Lemma 3.4, we have

$$\begin{aligned}
&\operatorname{codim}_{Y_{d,m}(p,L)} W_{d,m}(p,L) \\
&= \operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(m,n-1))} \Delta(m,n-1) = \min\{n-1, 2n-m-1\}, \\
&\operatorname{codim}_{Y_{d,m}(p,L)} W^0_{d,m}(p,L) \\
&= \operatorname{codim}_{\mathrm{Grass}\,(1,\mathrm{Mat}\,(m,n-1))} \Delta^0(m,n-1) = 2n-m-1.
\end{aligned}$$

□

Proof of Theorem 3.2. (i) If $m \ge 2n$, then $\dim Y_{d,m} < \dim M_d$, hence $Y_{F,m}$ is empty for general $F \in \mathrm{Sym}^d V$.
(ii) Let $\Psi_{d,m} : \mathcal{Y}_{d,m} \to \mathcal{M}_d$ be the morphism of the schemes over $\operatorname{Spec} \mathbf{Z}$ whose fiber at $\operatorname{Spec} K \to \operatorname{Spec} \mathbf{Z}$ is the morphism $\psi_{d,m} : Y_{d,m} \to M_d$ for any field K. If $m \le 2n-1$ and m is prime to the characteristic of K, then $\operatorname{codim}_{Y_{d,m}} W_{d,m} \ge 1$ by Proposition 3.3, hence $\psi_{d,m} : Y_{d,m} \to M_d$ is dominant. Therefore $\Psi_{d,m} : \mathcal{Y}_{d,m} \to \mathcal{M}_d$ is a dominant morphism for $m \le 2n-1$. Since $\Psi_{d,m}$ is a proper morphism, $\Psi_{d,m}$ is surjective, hence $Y_{F,m}$ is non-empty for any field K and for any $F \in \mathrm{Sym}^d V$.
(iii) By [4, Proposition 10.4], $Y_{F,m}$ is smooth of dimension $2n-m-1$ for $[F] \in$

$M_d \setminus \psi_{d,m}(W_{d,m})$. Hence we will show that $\psi_{d,m}(W_{d,m}) \subsetneqq M_d$ is a proper Zariski closed subset. It is well-known that the hypersurface X_F is smooth for a general $F \in \operatorname{Sym}^d V$. Then $Y_{F,1}$ and $Y_{F,2}$ are smooth, hence $\psi_{d,2}(W_{d,2}) \subsetneqq M_d$ is a proper Zariski closed subset. If $3 \leq m \leq 2n-1$ and m is prime to the characteristic of K, then $\dim W^0_{d,m} < \dim M_d$ by Proposition 3.3, hence $\psi_{d,m}(W_{d,m}) = \psi_{d,2}(W_{d,2}) \cup \psi_{d,m}(W^0_{d,m}) \subsetneqq M_d$ is a proper Zariski closed subset.
(iv) We assume $(n,m) \neq (2,2)$. If $m \leq 2n-2$ and m is prime to the characteristic of K, then $\operatorname{codim}_{Y_{d,m}} W_{d,m} \geq 2$ by Proposition 3.3. Using the same argument as the proof of [6, Chapter V. (4.3.3)], the general fiber of $\Psi_{d,m} : \mathcal{Y}_{d,m} \to \mathcal{M}_d$ is connected for $m \leq 2n-2$. By Zariski's Main Theorem, $Y_{F,m}$ is connected for any field K and for any $F \in \operatorname{Sym}^d V$. We assume $(n,m) = (2,2)$. We denote by $X_d \to M_d$ the universal family of curves of degree d in $\mathbf{P}^2$. Then the natural projection $\phi : Y_{d,2} \to X_d$ is a birational projective morphism. By Zariski's Main Theorem, any fiber of ϕ is connected. Since X_F is connected, $Y_{F,2}$ is connected. □

Theorem 3.5. *Assume that $n \geq 2$ and the characteristic of K is not 3. For a cubic form $F \in \operatorname{Sym}^3 V$, the variety $Y_{F,3}$ is smooth of dimension $2n-4$ if and only if X_F is a smooth hypersurface in $\mathbf{P}^n$.*

Proof. We assume that X_F is not a smooth hypersurface. For $p \in \operatorname{Sing} X_F(\bar{K})$, there is a line L in $\mathbf{P}^n_{\bar{K}}$ such that $(p, L) \in Y_{F,3}(\bar{K})$. Then $([F], p, L) \in W_{3,3}(\bar{K})$, hence $Y_{F,3}$ is not smooth of dimension $2n-4$. Conversely, we assume that $Y_{F,3}$ is not smooth of dimension $2n-4$. There is a pair $(p, L) \in Y_{F,3}(\bar{K})$ such that $([F], p, L) \in W_{3,3}(\bar{K})$. If $a_3 \neq 0$, then $p \in \operatorname{Sing} X_F(\bar{K})$. If $a_3 = 0$ and $p \notin \operatorname{Sing} X_F(\bar{K})$ then

$$\operatorname{rank} \begin{pmatrix} a_{0,2} & \cdots & a_{0,n} & 0 & \cdots & 0 \\ a_{1,2} & \cdots & a_{1,n} & a_{0,2} & \cdots & a_{0,n} \\ a_{2,2} & \cdots & a_{2,n} & a_{1,2} & \cdots & a_{1,n} \end{pmatrix} < 3.$$

There exist $\alpha, \beta \in \bar{K}$ such that $a_{2,j} = \alpha a_{0,j} + \beta a_{1,j}$ and $a_{1,j} = \beta a_{0,j}$ for $2 \leq j \leq n$. Let $s \in \bar{K}$ be satisfying

$$s^2 + \beta s + (\alpha + \beta^2) = 0.$$

Then $F(s, 1, 0, \ldots, 0) = 0$ and $\frac{\partial F}{\partial x_j}(s, 1, 0, \ldots, 0) = 0$ for $0 \leq j \leq n$, hence X_F is not a smooth hypersurface. □

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