SYMPLECTIC CURVATURE FLOW

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Abstract. We introduce a parabolic flow of almost Kähler structures, providing a natural extension of Kähler Ricci flow onto symplectic manifolds. We exhibit this flow as one of a family of parabolic flows of almost Hermitian structures, generalizing our previous work on parabolic flows of Hermitian metrics. We end with a discussion of some problems related to the symplectic curvature flow.

1. Introduction

The purpose of this paper is to introduce a geometric evolution equation on symplectic manifolds. More specifically, let (M^{2n}, ω) denote a compact smooth manifold with closed, nondegenerate 2-form ω. Any such ω admits compatible almost complex structures. Below we will define a coupled degnerate parabolic system of equations for a compatible pair (ω, J) preserving the symplectic condition for ω. If the initial almost complex structure is in fact integrable, then the resulting one-parameter family of complex structures is fixed, i.e. $J(t) = J(0)$, and the family of Kähler forms $\omega(t)$ is a solution to Kähler Ricci flow. This parabolic system is furthermore a special instance of a general family of parabolic flows of almost Hermitian structures. We begin by describing this more general setup, then proceed to define the flow of almost Kähler structures.

Let (M^{2n}, ω, J) be an almost Hermitian manifold. Let ∇ denote the Chern connection associated to (ω, J), which is the unique connection satisfying

$$\nabla\omega \equiv 0, \qquad \nabla J \equiv 0, \qquad T^{1,1} \equiv 0$$

where $T^{1,1}$ refers to the $(1,1)$ component of the torsion of ∇ thought of as a section of $\Lambda^2 \otimes TM$. Let Ω denote the $(4,0)$-curvature tensor associated to this connection, and let

$$S_{ij} = \omega^{kl}\Omega_{klij}.$$

Furthermore, let Q denote a $(1,1)$ form which is a quadratic expression in the torsion T of ∇. Let

$$\mathcal{K}_j^i = \omega^{kl}\nabla_k N_{lj}^i.$$

where N denotes the Nijenhuis tensor associated to J. Also, let $\mathcal{H}$ denote a generic quadratic expression in the torsion which is an endomorphism of the tangent bundle which skew-commutes with J. Finally, let

$$H = \frac{1}{2}\left[\omega(\mathcal{K} - \mathcal{H}, J) + \omega(J, \mathcal{K} - \mathcal{H})\right].$$

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These definitions are spelled out in greater detail in the rest of the paper. Consider the initial value problem

$$\begin{aligned}\frac{\partial}{\partial t}\omega =& -2S + Q + H\\ \frac{\partial}{\partial t}J =& -\mathcal{K} + \mathcal{H}\\ \omega(0) =& \omega_0\\ J(0) =& J_0.\end{aligned} \tag{1.1}$$

It follows from Lemma 3.2 that this system of equations preserves compatibility of the pair (ω, J). Moreover this is a degenerate parabolic system of equations for (ω, J), with degeneracy arising from the action of the diffeomorphism group. In section 3 we prove the general short-time existence of solutions of (1.1), a generalization of Theorem 1.1 of [11].

Theorem 1.1. *Let (M^{2n}, ω_0, J_0) be an almost-Hermitian manifold. There exists $\epsilon > 0$ so that a unique solution to (1.1) with initial condition (ω_0, J_0) exists on $[0, \epsilon)$. Moreover, $g(t)$ is compatible with $J(t)$ for all $t \in [0, \epsilon)$. If J_0 is integrable, then $J(t) = J_0$ for all $t \in [0, \epsilon)$. Furthermore, if J_0 is integrable and g_0 is Kähler, then $g(t)$ is Kähler for all $t \in [0, \epsilon)$ and $g(t)$ solves the Kähler-Ricci flow with initial condition g_0.*

Remark 1.2. It is important to note that equation (1.1) is defining a *family* of equations. Indeed, the choice of Q, and further lower order terms which may be included in the definition of $\mathcal{K}$ and F, are *arbitrary* in the definition of (1.1) and the proof of Theorem 1.1.

Remark 1.3. When J_0 is integrable, the one-parameter family of metrics $\omega(t)$ is a solution to *Hermitian curvature flow*, as defined in [11]. Again, the torsion term Q can be arbitrary for the result of Theorem 1.1.

Remark 1.4. As will be clear from Proposition 4.5, it is possible to define a parabolic flow of metrics compatible with any given almost complex structure. Specifically, given (M^{2n}, J) an almost complex manifold, one can set

$$\frac{\partial}{\partial t}\omega = -2S + Q \tag{1.2}$$

where again Q is a $(1,1)$ form which is a quadratic expression in the torsion. This viewpoint was considered recently by Vezzoni [17]. When J is integrable, this is precisely the family of equations introduced in [11]. If one is interested in understanding metrics compatible with a given almost complex structure, (1.2) could be a useful tool.

We now proceed to define the flow of almost Kähler structures.

Definition 1.5. An almost Hermitian manifold (M^{2n}, ω, J) is *almost Kähler* if

$$d\omega = 0.$$

This condition is a very natural extension of Kähler geometry, and one may consult [1] for a nice fairly recent survey of results on these structures. Due to its connection with symplectic geometry, almost Kähler structures have become a central area of mathematics (see for instance [5], [9]).

An almost Kähler structure has a canonical Hermitian connection ∇ (which coincides with the Chern connection) with curvature Ω. Furthermore, let

$$P_{ij} = \omega^{kl}\Omega_{ijkl}.$$

Also let

$$\mathcal{R}_i^j = J_i^k \operatorname{Rc}_k^j - \operatorname{Rc}_i^k J_k^j.$$

Consider the initial value problem

$$\begin{aligned}
\frac{\partial}{\partial t}\omega =& -2P \\
\frac{\partial}{\partial t}J =& -2g^{-1}\left[P^{(2,0)+(0,2)}\right] + \mathcal{R} \\
\omega(0) =& \,\omega_0 \\
J(0) =& \,J_0.
\end{aligned} \tag{1.3}$$

Theorem 1.6. *Let (M^{2n}, ω_0, J_0) be an almost Kähler manifold. There exists $\epsilon > 0$ so that a unique solution to (1.3) with intial condition (ω_0, J_0) exists on $[0, \epsilon)$. Moreover, the pair $(\omega(t), J(t))$ is a solution to an equation of the type (1.1), where Q and $\mathcal{H}$ are defined in (5.13) and (5.14) respectively, and H is defined so that (4.1) holds. In particular, this instance of equation (1.1) preserves the almost Kähler condition.*

Remark 1.7. In [8] a certain geometric evolution equation was studied on symplectic manifolds. There the perspective taken is that the symplectic structure ω is fixed, and then one studies the gradient flow of the functional

$$\mathcal{F}(J) := \int_M |DJ|^2 \, dV$$

where the metric defining the quantities above is that associated to J via ω. The proof of short time existence of this flow is already technical, due to certain local obstructions in prescribing the skew-symmetric part of the Ricci tensor. Our approach here is different, as we allow both ω and J to change. This seems to have certain advantages, since for instance the diffeomorphism action is the only obstruction to parabolicity.

Here is an outline of the rest of the paper. In section 2 we review some basic aspects of almost Hermitian geometry, and recall the Chern connection. Section 3 contains basic calculations on variations of almost Hermitian structures. In sections 4 and 5 we give the proofs of Theorems 1.1 and 1.6. In section 6, we give a discussion of some special properties of the limiting metrics of (1.3). We end in section 7 by posing a number of problems related to symplectic curvature flow. In a forthcoming paper, we will study the curvature evolution equations of our flow for almost Kähler manifolds and derive some consequences of them.

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2. Background on Almost Hermitian Geometry

In this section we review some basic material about almost Hermitian geometry and various associated connections. Let (M^{2n}, J) be an almost complex manifold. This means that J is an endomorphism of TM satisfying

$$J^2 = -\operatorname{Id}.$$

By the famous theorem of Newlander-Nirenberg [10], the almost complex structure J is *integrable*, i.e. one can find local complex coordinates at each point, if and only if the Nijenhuis tensor vanishes. Following the convention of [7], the Nijenhuis tensor is

$$4N_J(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY]. \tag{2.1}$$

As in the case of complex manifold, the almost-complex structure J induces a decomposition of the space of differential forms on M via the eigenspace decomposition on TM. In particular we will write

$$\Lambda^r(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}.$$

The operator d acts on Λ^r, but in general one does not have $d\Lambda^{p,q} \subset \Lambda^{p+1,q} \oplus \Lambda^{p,q+1}$, due to the potential lack of integrability of J.

Moving to the metric geometry, let ω be an almost-Hermitian metric on M, i.e. ω is a real $(1,1)$ form satisfying the compatibility condition

$$\omega(\cdot,\cdot) = \omega(J\cdot, J\cdot).$$

Associated to this pair is the Riemannian metric

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot).$$

Next we consider connections associated to almost Hermitian manifolds. A very thorough discussion of these connections can be found in [7]. A connection ∇ on TM is called *Hermitian* if

$$\nabla\omega \equiv 0, \qquad \nabla J \equiv 0.$$

These two conditions alone do not suffice to determine a unique connection in general. Indeed, there is freedom yet of $\psi \in \Lambda^3(\mathbb{R}) \cap \Lambda^{2,1} \oplus \Lambda^{1,2}$ and $B \in \Lambda^{1,1} \otimes TM$ satisfying a certain Bianchi identity (see [7] Proposition 2). Certain members of this family are chosen according to certain desirable properties of the torsion. Frequently, one chooses the Chern connection.

Definition 2.1. Given (M^{2n}, ω, J) an almost-Hermitian manifold, the *Chern connection associated to* (ω, J) is the unique connection ∇ satisfying

$$\begin{aligned} \nabla\omega &\equiv 0 \\ \nabla J &\equiv 0 \\ T^{1,1} &\equiv 0 \end{aligned}$$

where T denotes the torsion tensor of ∇ and $T^{1,1}$ is the projection of the vector-valued torsion two-form onto the space of $(1,1)$-forms.

As a final important remark we observe that there is a *canonical* Hermitian connection on almost Kähler manifolds. This connection has the simple form

$$\nabla_X Y = D_X Y - \frac{1}{2} J(D_X J)(Y).$$

3. Variations of Almost Hermitian Structures

Lemma 3.1. *Let (M^{2n}, J) be a complex manifold and suppose $J(t)$ is a one-parameter family of endomorphisms of TM such that $J(0) = J$. Then $J(t)$ is a one-parameter family of almost-complex structures if and only if*

$$J\left(\frac{\partial}{\partial t}J\right) + \left(\frac{\partial}{\partial t}J\right)J = 0.$$

Proof. Since by assumption $J(t)$ is an almost complex structure, we have $J(t)^2 = -\operatorname{Id}$. Thus

$$0 = \frac{\partial}{\partial t} J^2 = JK + KJ.$$

$\square$

Lemma 3.2. *Let $(M^{2n}, \omega(t), J(t))$ be a one-parameter family of Kähler forms and almost-complex structures. Specifically write*

$$\frac{\partial}{\partial t}\omega = \phi + \psi$$
$$\frac{\partial}{\partial t}J = K$$

where $\psi \in \Lambda^{(2,0)+(0,2)}$ and $\phi \in \Lambda^{1,1}$. Then $\omega(t)$ is compatible with $J(t)$ if and only if

$$\psi = \frac{1}{2}\left[\omega(K, J) + \omega(J, K)\right]. \tag{3.1}$$

Proof. We directly compute

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left(\omega(J\cdot, J\cdot) - \omega(\cdot, \cdot)\right) \\ &= (\phi + \psi)(J\cdot, J\cdot) + \omega(K\cdot, J\cdot) + \omega(J\cdot, K\cdot) - (\phi + \psi)(\cdot, \cdot) \\ &= \psi(J\cdot, J\cdot) - \psi(\cdot, \cdot) + \omega(K\cdot, J\cdot) + \omega(J\cdot, K\cdot). \end{aligned}$$

Since $\psi \in \Lambda^{(2,0)+(0,2)}$ the above equation is equivalent to

$$2\psi = \omega(K, J) + \omega(J, K)$$

as required. $\square$

Remark 3.3. Fix a point $p \in M$ and choose some local coordinates. Certainly the above equation for ψ holds if

$$K_a^b = g^{bc}\psi_{ac}.$$

In paricular we have

$$\begin{aligned} \omega(K, J)_{ab} + \omega(J, K)_{ab} &= \omega_{cd}K_a^c J_b^d + \omega_{cd}J_a^c K_b^d \\ &= \omega_{cd}g^{cp}\psi_{ap}J_b^d + \omega_{cd}J_a^c g^{dp}\psi_{bp} \\ &= g^{cp}g_{cb}\psi_{ap} + \omega_{cd}J_a^c g^{dp}\psi_{bp} \\ &= \psi_{ab} - \psi_{ba} \\ &= 2\psi_{ab}. \end{aligned}$$

Thus (3.1) holds. Observe however that K is not determined by ψ alone. Indeed any skew-Hermitian tensor can be added to K and (3.1) will still hold.

Lemma 3.4. *Let (M^{2n}, J) be an almost-complex manifold and let X be a vector field on M. Then*

$$JL_X J + L_X JJ = 0. \tag{3.2}$$

Furthermore, if ω is compatible with J, we have

$$(L_X\omega)^{(2,0)+(0,2)} = \frac{1}{2}\left(\omega(L_X J\cdot, J\cdot) + \omega(J\cdot, L_X J\cdot)\right).$$

Proof. By [3] pg. 86, we have the formula for $L_X J$:

$$\left(L_X J\right)(Y) = [X, JY] - J[X, Y].$$

Given this, the second equation follows by direct calculation. The final equation obviously must hold since it is just the linearized compatibility condition (3.1) and the action of a diffeomorphism preserves compatibility, but we just as well compute

$$\begin{aligned} 0 &= L_X\left(\omega(\cdot,\cdot) - \omega(J\cdot, J\cdot)\right) \\ &= (L_X\omega)(\cdot,\cdot) - \left(L_X\omega\right)(J\cdot, J\cdot) - \omega(L_X J\cdot, J\cdot) - \omega(J\cdot, L_X J\cdot). \end{aligned}$$

Rearranging the above formula gives the result. □

Lemma 3.5. *Let (M^{2n}, J) be an almost complex manifold and let X be a vector field on M. Then*

$$\left(\mathcal{L}_X J\right)_k^l = J_p^l \partial_k X^p - J_k^p \partial_p X^l + \mathcal{O}(X).$$

Proof. Choose local coordinate vector fields e^k. Then using Lemma 3.4 we see

$$\begin{aligned} \left(\mathcal{L}_X J\right)_k^l e^k =& - \left(J e^k\right)^p \partial_p X^l + J_p^l \left[e^k \partial_k X^p\right] + \mathcal{O}(X) \\ =& - J_k^p \partial_p X^l + J_p^l \partial_k X^p + \mathcal{O}(X). \end{aligned}$$

as required. □

4. Parabolic Flows of Almost Hermitian structures

In this section we prove Theorem 1.1. Let us recall some definitions from the introduction used in (1.1). In particular, let (M^{2n}, ω, J) be an almost Hermitian manifold and let ∇ denote the associated Chern connection (see Definition 2.1). Let Ω denote the $(4,0)$ curvature tensor associated to ∇, and consider

$$S_{ij} = \omega^{kl}\Omega_{klij}.$$

Furthermore, let N denote the Nijenhuis tensor associated to J, and let

$$\mathcal{K}_j^i = \omega^{kl}\nabla_k N_{lj}^i.$$

Let Q denote a $(1,1)$ tensor which is quadratic expression in the torsion of ∇, and let $\mathcal{H}$ denote a J-skew endomorphism of the tangent bundle which again is quadratic in the torsion of ∇. Let

$$H = \frac{1}{2}\left[\omega(\mathcal{K} - \mathcal{H}, J) + \omega(J, \mathcal{K} - \mathcal{H})\right]. \tag{4.1}$$

Consider the initial value problem

$$\begin{aligned} \frac{\partial}{\partial t}\omega =& - 2S + Q + H \\ \frac{\partial}{\partial t}J =& - \mathcal{K} + \mathcal{H} \\ \omega(0) =& \ \omega_0 \\ J(0) =& \ J_0. \end{aligned} \tag{4.2}$$

The main goal of this section is to prove the short time existence for this flow as described in Theorem 1.1. We begin by checking that indeed equation (1.1) defines a one-parameter family of almost complex structures.

Lemma 4.1. *Let (M^{2n}, J) be an almost-Hermitian manifold. Then, viewing the Nijenhuis tensor N as a section of $\Lambda^1 \otimes \mathrm{End}(TM)$,*

$$JN + NJ = 0.$$

Proof. We can derive this by direct calculation using the definition of the Nijenhuis tensor (2.1). First

$$\begin{aligned} -4JN &= J\left([X,Y] + J\left([JX,Y] + [X,JY]\right) - [JX,JY]\right) \\ &= J[X,Y] - [JX,Y] - [X,JY] - J[JX,JY]. \end{aligned}$$

Next

$$\begin{aligned} -4NJ &= [X,JY] + J\left([JX,JY] + [X,JJY]\right) - [JX,JJY] \\ &= [X,JY] + J[JX,JY] - J[X,Y] + [JX,Y]. \end{aligned}$$

The result follows adding these two calculations together. □

Lemma 4.2. *Let (M^{2n}, ω, J) be an almost Hermitian manifold. Then*

$$J\mathcal{K} + \mathcal{K}J = 0.$$

Proof. We may write the result of Lemma 4.1 in coordinates as

$$J_k^m N_{jm}^l + N_{jk}^m J_m^l = 0$$

We differentiate this using the Chern connection. Since J is parallel we see

$$0 = J_k^m \nabla_i N_{jm}^l + \nabla_i N_{jk}^m J_m^l$$

we can now take the required contraction of indices using ω to yield the statement of the lemma. □

Definition 4.3. Let (M^{2n}, ω, J) be an almost Hermitian manifold. Let $\overline{\nabla}$ denote some fixed connection on TM. Define a vector field

$$X^p = X(\omega, J, \overline{\nabla})^p = \omega^{kl} \overline{\nabla}_k J_l^p. \tag{4.3}$$

Proposition 4.4. *Let (M^{2n}, ω, J) be an almost-Hermitian manifold and let $\overline{\nabla}$ denote some fixed connection on TM. The map*

$$J \to \mathcal{K} - L_{X(\omega, J, \overline{\nabla})} J$$

is a second order elliptic operator.

Proof. We recall a coordinate formula for the Nijenhuis tensor.

$$N_{jk}^i = J_j^p \partial_p J_k^i - J_k^p \partial_p J_j^i - J_p^i \partial_j J_k^p + J_p^i \partial_k J_j^p.$$

It follows that

$$\begin{aligned} \mathcal{K}_j^i &= \omega^{kl} \nabla_k N_{lj}^i \\ &= \omega^{kl} \left(J_l^q \partial_k \partial_q J_j^i - J_j^q \partial_k \partial_q J_l^i - J_q^i \partial_k \partial_l J_j^q + J_q^i \partial_k \partial_j J_l^q \right) + \mathcal{O}(\partial J, \partial \omega) \\ &= \; - g^{qk} \partial_k \partial_q J_j^i - \omega^{kl} \left(J_j^q \partial_k \partial_q J_l^i + J_q^i \partial_k \partial_l J_j^q - J_q^i \partial_k \partial_j J_l^q \right) + \mathcal{O}(\partial J, \partial \omega) \end{aligned} \tag{4.4}$$

where the notation $\mathcal{O}(\partial J, \partial \omega)$ means an expression which only depends on at most first derivatives of J and ω (possibly in a nonlinear fashion). In particular, Chern connection terms are of this form. Note that the matrix ω is skew-symmetric, but coordinate derivatives are symmetric, therefore the middle term in the parentheses in the last line vanishes. Also, using (4.3) and Lemma 3.5 we express

$$\left[L_{X(\omega,J,\overline{\nabla})}J\right]_j^i = \omega^{kl}\left(J_q^i\partial_j\partial_k J_l^q - J_j^q\partial_q\partial_k J_l^i\right) + \mathcal{O}(\partial J, \partial\omega)$$

Combining these two calculations yields

$$\left[\mathcal{K} - L_{X(\omega,J,\overline{\nabla})}J\right]_j^i = -g^{kl}\partial_k\partial_l J_j^i + \mathcal{O}(\partial J, \partial\omega).$$

The claim follows immediately. □

Proposition 4.5. *Let* (M^{2n}, g, J) *be an almost-Hermitian manifold. The map*

$$\omega \to S(\omega)$$

is a second order elliptic operator.

Proof. Fix a point $p \in M$, and choose a local frame of $(1,0)$ vector fields $\{e_i\}$ such that $g_{i\bar{j}}(p) = \delta_{ij}$. In this frame we compute using metric compatibility of ∇,

$$\begin{aligned}
S_{k\bar{l}} &= \omega^{i\bar{j}}\Omega_{i\bar{j}k\bar{l}}\\
&= \omega^{i\bar{j}}\left\langle \nabla_i\nabla_{\bar{j}}e_k - \nabla_{\bar{j}}\nabla_i e_k - \nabla_{[e_i,e_{\bar{j}}]}e_k, e_{\bar{l}}\right\rangle\\
&= \omega^{i\bar{j}}\left(e_i\left\langle\nabla_{\bar{j}}e_k, e_{\bar{l}}\right\rangle - \left\langle\nabla_{\bar{j}}e_k, \nabla_i e_{\bar{l}}\right\rangle - e_{\bar{j}}\left\langle\nabla_i e_k, e_{\bar{l}}\right\rangle + \left\langle\nabla_i e_k, \nabla_{\bar{j}}e_{\bar{l}}\right\rangle\right) + \mathcal{O}(\partial\omega, \partial J)\\
&= \omega^{i\bar{j}}\left(e_i\left\langle\nabla_{\bar{j}}e_k, e_{\bar{l}}\right\rangle - e_{\bar{j}}e_i\left\langle e_k, e_{\bar{l}}\right\rangle + e_{\bar{j}}\left\langle e_k, \nabla_i e_{\bar{l}}\right\rangle\right) + \mathcal{O}(\partial\omega, \partial J).
\end{aligned}$$

Now using J compatibility of the connection and the fact that the torsion T has no $(1,1)$-component we see that

$$\begin{aligned}
\left\langle\nabla_{\bar{j}}e_k, e_{\bar{l}}\right\rangle &= \left\langle\nabla_{\bar{j}}e_k - \nabla_k e_{\bar{j}}, e_{\bar{l}}\right\rangle\\
&= \left\langle T_{\bar{j}k} + [e_{\bar{j}}, e_k], e_{\bar{l}}\right\rangle\\
&= \mathcal{O}(\omega, \partial J).
\end{aligned}$$

The last line follows since the only nonvanishing term is determined by the Nijenhuis tensor. Likewise one concludes that $\left\langle e_k, \nabla_{e_i}e_{\bar{l}}\right\rangle = \mathcal{O}(\omega, \partial J)$. Therefore

$$\begin{aligned}
S_{k\bar{l}} &= -g^{i\bar{j}}e_{\bar{j}}e_i\omega_{k\bar{l}} + \mathcal{O}(\partial\omega, \partial^2 J)\\
&= -\frac{1}{2}g^{ab}\partial_a\partial_b\omega_{k\bar{l}} + \mathcal{O}(\partial\omega, \partial^2 J).
\end{aligned}$$

The result follows. □

We can now give the proof of Theorem 1.1.

Proof. First we show existence. Fix $\overline{\nabla}$ any connection on M and let X be defined as in Definition 4.3. Consider the following gauge-fixed version of equation (4.2)

$$\begin{aligned}
\frac{\partial}{\partial t}\omega &= -2S + Q + H + L_{X(g,J)}\omega = \mathcal{D}_1(\omega, J)\\
\frac{\partial}{\partial t}J &= -\mathcal{K} + \mathcal{H} + L_{X(g,J)}J = \mathcal{D}_2(\omega, J).
\end{aligned} \tag{4.5}$$

First of all, it follows from Lemmas 3.4 and 4.2 that the evolution equation for J in fact defines a family of almost complex structures. Furthermore, by (4.1), Lemma 3.4, and Lemma 3.2

it follows that the pair $(\omega(t), J(t))$ remains compatible along a solution to (4.5). We observe that by definition the vector field $X(g, J)$ can be expressed completely in terms of first derivatives of J and therefore $\left(L_{X(\omega,J)}\omega\right)_{ij}$ is a first order operator in ω. Use $\mathcal{L}_\omega, \mathcal{L}_J$ to denote linearization in the ω and J variables respectively. It follows from Proposition 4.5 that

$$\begin{aligned}\sigma\left[\widehat{\mathcal{L}_\omega\mathcal{D}_1}\right](h)_{ij} &= \sigma\left[\widehat{\mathcal{L}(-2S)}\right](h)_{ij}\\ &= |\xi|^2 h_{ij}.\end{aligned}$$

Furthermore, from Proposition 4.4 we conclude that

$$\sigma\left[\widehat{\mathcal{L}_J\mathcal{D}_2}\right](K)_i^j = |\xi|^2 K_i^j$$

We also need to check the linearization of $\mathcal{D}_2$ in the variable ω. Since by construction we have that $\mathcal{D}_2$ only depends on first derivatives of ω, we conclude

$$\sigma\left[\widehat{\mathcal{L}_\omega\mathcal{D}_2}\right](h)_i^j = 0.$$

We note that second derivative terms of J appear in the evolution of ω, therefore these terms appear in the full linearized operator. Collecting these observations we conclude that the overall symbol is upper-triangular. In particular it takes the form

$$\sigma\left[\widehat{\mathcal{L}\mathcal{D}}\right](h, K) = \begin{pmatrix} I & * \\ 0 & I\end{pmatrix}\begin{pmatrix} h \\ K\end{pmatrix}$$

It follows that (4.5) is a strictly parabolic system of equations, and therefore short-time existence follows from standard theory. Now we want to pull back our solution by the family of diffeomorphisms generated by X. Specifically let ϕ_t be a one-parameter family of diffeomorphisms of M defined by the ODE

$$\begin{aligned}\frac{\partial}{\partial t}\phi_t &= -X(\omega(t), J(t), \overline{\nabla})\\ \phi_0 &= \mathrm{id}_M\,.\end{aligned} \tag{4.6}$$

It follows that

$$\begin{aligned}\frac{\partial}{\partial t}\left(\phi_t^*\omega(t)\right) &= \frac{\partial}{\partial s}|_{s=0}\left(\phi_{t+s}^*\omega(t+s)\right)\\ &= \phi_t^*\left(\frac{\partial}{\partial t}\omega(t)\right) + \frac{\partial}{\partial s}|_{s=0}\left(\phi_{t+s}^*\omega(t)\right)\\ &= \phi_t^*\left(-S + Q + F + L_{X(\omega(t),J(t))}\omega\right)\\ &\quad + \frac{\partial}{\partial s}|_{s=0}\left[\left(\phi_t^{-1}\circ\phi_{t+s}\right)^*\phi_t^*\omega_t\right]\\ &= (-S + Q + F)\left(\phi_t^*(\omega), \phi_t^*(J)\right)\\ &\quad + \phi_t^*\left(L_{X(\omega(t),J(t))}\right) - L_{\left(\phi_t^{-1}\right)_* X(\omega(t),J(t))}\left(\phi_t^*\omega(t)\right)\\ &= (-S + Q + F)\left(\phi_t^*(\omega), \phi_t^*(J)\right).\end{aligned}$$

Likewise we may compute

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\phi_t^* J(t)\right) =&\ \frac{\partial}{\partial s}|_{s=0}\left(\phi_{t+s}^* J(t+s)\right)\\
=&\ \phi_t^*\left(\frac{\partial}{\partial t}J(t)\right)+\frac{\partial}{\partial s}|_{s=0}\left(\phi_{t+s}^* J(t)\right)\\
=&\ \phi_t^*\left(-\mathcal{K}(\omega,J)+L_{X(\omega(t),J(t))}\right)\\
&+\frac{\partial}{\partial s}|_{s=0}\left[\left(\phi_t^{-1}\circ\phi_{t+s}\right)^*\phi_t^* J_t\right]\\
=&\ -\mathcal{K}(\phi_t^*\omega(t),\phi_t^* J(t))+\phi_t^* L_{X(\omega(t),J(t))}\\
&-L_{(\phi_t^{-1})_* X(\omega(t),J(t))}\left(\phi_t^* J(t)\right)\\
=&\ -\mathcal{K}(\phi_t^*\omega(t),\phi_t^* J(t)).
\end{aligned}$$

Therefore $(\phi_t^*\omega(t),\phi_t^*(J(t)))$ is a solution to (4.2).

Next we show uniqueness. As in the proof of uniqueness for Ricci flow, we will show that the diffeomorphism ODE (4.6), when written with respect to the changing metric, is in fact a parabolic equation for ϕ. What is more, as we now show, our choice of X is essentially equivalent to that used for Ricci flow short-time existence. Let $\Gamma_C,\Gamma,\overline{\Gamma}$ denote the connection coefficients of the Chern , Levi-Civita, and background connections respectively. Consider the following calculation:

$$\begin{aligned}
X^p &= \omega^{kl}\overline{\nabla}_k J_l^p\\
&= \omega^{kl}\partial_k J_l^p+\mathcal{O}(\omega,J)\\
&= \omega^{kl}\left(\nabla_k J_l^p+\left(\Gamma_C\right)_{kl}^q J_q^p-\left(\Gamma_C\right)_{kq}^p J_l^q\right)+\mathcal{O}(\omega,J)\\
&= \omega^{kl}\left(\Gamma_C\right)_{kl}^q J_q^p+g^{kq}\left(\Gamma_C\right)_{kq}^p+\mathcal{O}(\omega,J).
\end{aligned}\tag{4.7}$$

The first term is the contraction of the Chern connection coefficient with a skew-symmetric one-form, and hence vanishes. Specifically we compute

$$\begin{aligned}
\omega^{kl}\left(\Gamma_C\right)_{kl}^q J_q^p &= \frac{1}{2}\omega^{kl}\left(\left(\Gamma_C\right)_{kl}^q-\left(\Gamma_C\right)_{lk}^q\right)J_q^p\\
&= \frac{1}{2}\omega^{kl}T_{kl}^q J_q^p\\
&= 0
\end{aligned}\tag{4.8}$$

since the torsion of ∇ has no $(1,1)$ component. Next observe that since g is symmetric, the contraction $g^{kq}\Gamma_{kq}^p$ does note involve the torsion of the connection. In particular we conclude

$$g^{kq}\left(\Gamma_C\right)_{kq}^p=g^{kq}\Gamma_{kq}^p$$

In particular, combining these calculations we may conclude that

$$X^p=g^{kl}\left[\Gamma_{kl}^p-\overline{\Gamma}_{kl}^p\right]+\mathcal{O}(\omega,J).\tag{4.9}$$

In particular we have shown that, up to lower order terms, the vector field we used in our short-time existence proof is the same as that used for Ricci flow. So, set $\widetilde{g}=\phi_t^* g(t)$, $\widetilde{J}=\phi_t^* J(t)$. It follows (see [4] pg. 89) that one may rewrite the solution to (4.6) as

$$\begin{aligned}
\frac{\partial}{\partial t}\phi_t =&\ \Delta_{\widetilde{g}(t),g_0}\phi_i(t)+\mathcal{O}\left(\partial\phi\right)\\
\phi_0 =&\ \mathrm{id}_M
\end{aligned}\tag{4.10}$$

where $\Delta_{\widetilde{g}(t),g_0}$ is the harmonic map Laplacian taken with respect to the metrics $\widetilde{g}(t)$ and g_0.

We now proceed with the proof of uniqueness. Let $\widetilde{g}_1(t), \widetilde{g}_2(t)$ be two solutions to (4.2) with $\widetilde{g}_1(0) = \widetilde{g}_2(0) = g_0$. Let $\phi_i(t)$ be solutions to (4.10) with respect to $\widetilde{g}_i$, which exists in general because (4.10) is strictly parabolic and M is compact. Now, pushing forward by these diffeomorphisms we observe that $g_i(t) := (\phi_i(t))_* \widetilde{g}_i(t)$ are both solutions of (4.5). Since $g_1(0) = g_2(0)$ and solutions to (4.5) are unique, it follows that $g_1(t) = g_2(t)$ as long as these metrics are defined. But now one observes that $\phi_1(t)$ and $\phi_2(t)$ are both solutions to the same ODE (4.6) with the same initial condition, and are therefore equal. It follows that $\widetilde{g}_1(t) = \widetilde{g}_2(t)$ as long as they are both defined and the result follows. $\square$

5. Symplectic Curvature Flow

5.1. **Definition and Short Time Existence.** In this section we will motivate and investigate the equation (1.3). Let (M^{2n}, ω, J) be an almost-Kähler manifold, which as defined in section 2 is an almost Hermitian manifold satisfying $d\omega = 0$. Also recall the remark at the end of section 2 that on an almost Kähler manifold one has a canonical connection ∇ satisfying

$$\nabla_X Y = D_X Y - \frac{1}{2} J (D_X J)(Y)$$

where D is the Levi-Civita connection of g. To facilitate the discussion let us record some useful curvature quantities in almost Kähler geometry. First, let Ric denote the usual Ricci curvature of the Levi-Civita connection, and let Ric^J denote the J-invariant part of the Ricci tensor of g, i.e.

$$\mathrm{Ric}^J = \frac{1}{2} \left[\mathrm{Ric}(\cdot,\cdot) + \mathrm{Ric}(J\cdot, J\cdot) \right].$$

Furthermore set

$$\rho(\cdot,\cdot) = \mathrm{Ric}^J(J\cdot,\cdot).$$

Note $\rho \in \Lambda^{1,1}$. Next set

$$\rho^* = R(\omega)$$

i.e., the Levi-Civita curvature operator acting on the Kähler form ω. One can see [1] for more information on these quantities. Returning to the canonical connection, ∇ induces a Hermitian connection on the anticanonical bundle, and we denote the curvature form of this connection by P. Alternatively, if Ω denotes the curvature of ∇, one has

$$P_{ij} = \omega^{kl} \Omega_{ijkl}.$$

By the general Chern-Weil theory, P is a closed form and $P \in \pi c_1(M, J)$. Let us record some lemmas relating these different curvature tensors.

Lemma 5.1. *Let (M^{2n}, ω, J) be an almost Kähler manifold. Then*

$$\rho^* - \rho = \frac{1}{2} D^* D \omega. \tag{5.1}$$

Proof. By the Weitzenböck formula for 2-forms ([3] pg. 53) applied to ω we conclude

$$\Delta_d \omega - D^* D\omega = \mathrm{Ric}(\omega\cdot,\cdot) - \mathrm{Ric}(\cdot,\omega\cdot) - 2R(\omega).$$

Since $d\omega = d^*\omega = 0$ we conclude

$$2R(\omega) + \left[\mathrm{Ric}(\cdot, J\cdot) - \mathrm{Ric}(J\cdot,\cdot) \right] = D^* D\omega.$$

The Ricci curvature terms simplify to -2ρ, and the result follows. □

Furthermore (see [1]) one has the relation

$$P = \rho^* - \frac{1}{2}N^1, \tag{5.2}$$

where

$$N^1(X,Y) = \left\langle D_{JX}\omega, D_Y\omega \right\rangle.$$

As a consequence of (5.1) and (5.2) we conclude

$$P = \rho + \frac{1}{2}\left(D^*D\omega - N^1\right). \tag{5.3}$$

Lemma 5.2. *Let (M^{2n}, ω, J) be an almost Kähler manifold. Then*

$$P^{(2,0)+(0,2)} = \frac{1}{2}\left[D^*D\omega - N^2\right] \tag{5.4}$$

where

$$N^2_{ab} = g^{ij}\omega_{pq}D_iJ_a^pD_jJ_b^q. \tag{5.5}$$

Proof. Starting from (5.3), we note that both ρ and $\mathcal{N}$ are $(1,1)$ forms, so it remains to determine the $(2,0)+(0,2)$ component of $D^*D\omega$. We do this by computing the $(1,1)$ component, which we will compute in local coordinates.

$$\begin{aligned}
-\left(D^*D\omega\right)^{1,1}_{ab} =&\ -\frac{1}{2}\left[\left(D^*D\omega\right)(J,J) + D^*D\omega\right]_{ab}\\
=&\ \frac{1}{2}g^{ij}\left[(D_iD_j\omega_{pq})J_a^pJ_b^q + D_iD_j\omega_{ab}\right]\\
=&\ \frac{1}{2}g^{ij}\left[D_iD_j\left(\omega_{pq}J_a^pJ_b^q\right) + D_iD_j\omega_{ab}\right.\\
&\left. -\omega_{pq}\left((D_iD_jJ_a^p)J_b^q + D_iJ_a^pD_jJ_b^q + D_jJ_a^pD_iJ_b^q + J_a^pD_iD_jJ_b^q\right)\right].
\end{aligned}$$

Using compatibility of ω with J,

$$D_iD_j\left(\omega_{pq}J_a^pJ_b^q\right) = D_iD_j\omega_{ab}.$$

Also, we have that

$$\begin{aligned}
-\omega_{pq}(D_iD_jJ_a^p)J_b^q =&\ -g_{pb}D_iD_jJ_a^p\\
=&\ -D_iD_j\left(g_{pb}J_a^p\right)\\
=&\ -D_iD_j\left(\omega_{ab}\right).
\end{aligned}$$

Next we compute

$$\begin{aligned}
-\omega_{pq}J_a^pD_iD_jJ_b^q =&\ g_{aq}D_iD_jJ_b^q\\
=&\ -D_iD_j\omega_{ab}.
\end{aligned}$$

It follows that

$$\left(D^*D\omega\right)^{1,1}_{ab} = g^{ij}\omega_{pq}D_iJ_a^pD_jJ_b^q.$$

The lemma follows. □

With Lemma 3.2 in mind our motivation is clear: one would like to define a flow of symplectic structures $\frac{\partial}{\partial t}\omega = -P$ as in the case of Kähler Ricci flow. However, P is not a $(1,1)$-form, so ω would not stay compatible with J, and then the definitions fall apart. Thus one is naturally led to allowing J to flow as well. Lemma 3.2 suggests *part, but not all* of what should appear in the evolution equation for J. Indeed, imagining ω as fixed, then specifying a variation for J is equivalent to specifying one for g, and Lemma 3.2 is only saying that the $(2,0)+(0,2)$ portions must match up. There is still freedom to choose the $(1,1)$ part, and by examining the relevant equations one is able to make an intelligent choice for this portion. It is a small miracle that in the almost Kähler setting there is a very natural choice of evolution for J which ends up being parabolic. Using the above discussion, let us now rewrite equation (1.3) more explicitly. In particular, we set

$$\mathcal{N}_i^j = g^{jk} N_{ik}^2. \tag{5.6}$$

Also, as in the introduction, set

$$\mathcal{R}_i^j = J_i^k \operatorname{Rc}_k^j - \operatorname{Rc}_i^k J_k^j \tag{5.7}$$

where the index on the Ricci tensor has been raised with respect to the associated metric. Then one sees that equation (1.3) is equivalent to

$$\begin{aligned} \frac{\partial}{\partial t}\omega =& -2P \\ \frac{\partial}{\partial t}J =& -D^*DJ + \mathcal{N} + \mathcal{R} \\ \omega(0) =& \ \omega_0 \\ J(0) =& \ J_0. \end{aligned} \tag{5.8}$$

Now we give the proof of Theorem 1.6.

Proof. Observe that an endomorphism of the form $g^{-1}\psi$ where $\psi \in \Lambda^{(2,0)+(0,2)}$ automatically satisfies $KJ + JK = 0$. Also we may compute

$$\begin{aligned} J_a^b \left(\operatorname{Rc}_b^d J_d^c - \operatorname{Rc}_d^c J_b^d\right) + \left(\operatorname{Rc}_a^d J_d^b - \operatorname{Rc}_d^b J_a^d\right) J_b^c =& \operatorname{Rc}_b^d J_a^b J_d^c + \operatorname{Rc}_a^c - \operatorname{Rc}_a^c - \operatorname{Rc}_d^b J_a^d J_b^c \\ =& \ 0. \end{aligned}$$

Thus it follows from Lemma 3.1 that the equation for J defines a flow through almost complex structures. Furthermore recall from Remark 3.3 that an endomorphism $g^{-1}\psi$, where $\psi \in \Lambda^{(2,0)+(0,2)}$ automatically satisfies (3.1). Also let us observe

$$\begin{aligned} (\omega(\mathcal{R}, J) + \omega(J, \mathcal{R}))_{ij} =& \ \omega_{pq}\mathcal{R}_i^p J_j^q + \omega_{pq} J_i^p \mathcal{R}_j^q \\ =& \ \omega_{pq}\left[\operatorname{Rc}_i^m J_m^p - J_i^m \operatorname{Rc}_m^p\right] J_j^q + \omega_{pq} J_i^p \left[\operatorname{Rc}_j^m J_m^q - J_j^m \operatorname{Rc}_m^q\right] \\ =& \ g_{pj}\left[\operatorname{Rc}_i^m J_m^p - J_i^m \operatorname{Rc}_m^p\right] - g_{iq}\left[\operatorname{Rc}_j^m J_m^q - J_j^m \operatorname{Rc}_m^q\right] \\ =& \ g_{rs} J_p^r J_j^s \operatorname{Rc}_i^m J_m^p - g_{pj} J_i^m \operatorname{Rc}_m^p - g_{rs} J_i^r J_q^s \operatorname{Rc}_j^m J_m^q + g_{iq} J_j^m \operatorname{Rc}_m^q \\ =& \ -g_{ms} J_j^s \operatorname{Rc}_i^m - g_{pj} J_i^m \operatorname{Rc}_m^p + g_{rm} J_i^r \operatorname{Rc}_j^m + g_{iq} J_j^m \operatorname{Rc}_m^q \\ =& \ -\operatorname{Rc}_{is} J_j^s - J_i^m \operatorname{Rc}_{mj} + J_i^r \operatorname{Rc}_{jm} + J_j^m \operatorname{Rc}_{mi} \\ =& \ 0 \end{aligned}$$

where the last line follows from symmetry of the Ricci tensor and relabeling indices. Thus it follows from Lemma 3.2 that the equation (1.3) preserves compatibility of (ω, J). Also, since

P is closed, it follows that

$$\frac{\partial}{\partial t} d\omega = d \frac{\partial}{\partial t} \omega = -dP = 0,$$

so $d\omega = 0$ is preserved.

Turning now to the short time existence, let X be defined as in (4.3) and consider the gauge-fixed operators

$$\begin{aligned} \mathcal{D}_1(\omega, J) &= -2P + L_X \omega \\ \mathcal{D}_2(\omega, J) &= -D^* DJ + \mathcal{N} + \mathcal{R} + L_X J. \end{aligned}$$

Consider first the equation for ω, and use the second Bianchi identity we conclude

$$\begin{aligned} [-2P + L_X\omega]_{ij} &= \left[-2\omega^{kl}\Omega_{ijkl}\right] + \mathcal{O}(\partial^2 J, \partial\omega) \\ &= -2\left[\omega^{kl}\Omega_{klij} + \left(\nabla N + N^{*2}\right)_{ij}\right] + \mathcal{O}(\partial^2 J, \partial\omega) \\ &= -2S_{ij} + \mathcal{O}(\partial^2 J, \partial\omega) \end{aligned} \tag{5.9}$$

In the first line we have just observed that the term $\mathcal{L}_X\omega$ only involves terms which have two derivatives of J and at most one derivative of ω, as is clear from the definition (4.3). The second line is an application of the Bianchi identity. The fact that the torsion terms rely only on the Nijenhuis tensor is a consequence of the almost Kähler condition, since in general extra terms arising from the metric contributions to the torsion of the Hermitian connection will appear when applying the Bianchi identity. Indeed, one does not expect in general the operator P to be elliptic, since in essence it relies only on the volume form of ω. It follows from the calculation of Proposition 4.5 that

$$\mathcal{D}_1(\omega, J)_{ab} = g^{ij}\partial_i\partial_j\omega_{ab} + \mathcal{O}(\partial^2 J, \partial\omega).$$

Next we compute a coordinate formula for $\mathcal{D}_2$ in stages. First of all we have

$$\begin{aligned} [-D^*DJ]^l_k &= g^{ij}\partial_i\left[DJ\right]^l_{jk} + \mathcal{O}(\partial J, \partial\omega) \\ &= g^{ij}\partial_i\left[\partial_j J^l_k - \Gamma^p_{jk}J^l_p + \Gamma^l_{jp}J^p_k\right] + \mathcal{O}(\partial J, \partial\omega) \\ &= g^{ij}\left[\partial_i\partial_j J^l_k - \partial_i\Gamma^p_{jk}J^l_p + \partial_i\Gamma^l_{jp}J^p_k\right] + \mathcal{O}(\partial J, \partial\omega) \end{aligned} \tag{5.10}$$

where here Γ denotes the Levi Civita connection. Next, using Lemma 3.5 and (4.9) we compute

$$\begin{aligned} [\mathcal{L}_X J]^l_k &= J^l_p\partial_k\left(g^{ij}\Gamma^p_{ij}\right) - J^p_k\partial_p\left(g^{ij}\Gamma^l_{ij}\right) + \mathcal{O}(\partial J, \partial\omega) \\ &= J^l_p g^{ij}\partial_k\Gamma^p_{ij} - J^p_k g^{ij}\partial_p\Gamma^l_{ij} + \mathcal{O}(\partial J, \partial\omega). \end{aligned}$$

Combining the above calculations we observe

$$\begin{aligned} \mathcal{D}_2(\omega, J)^l_k &= g^{ij}\left[\partial_i\partial_j J^l_k - \partial_i\Gamma^p_{jk}J^l_p + \partial_i\Gamma^l_{jp}J^p_k\right] + J^l_p g^{ij}\partial_k\Gamma^p_{ij} - J^p_k g^{ij}\partial_p\Gamma^l_{ij} \\ &\quad + J^p_k \operatorname{Rc}^l_p - \operatorname{Rc}^p_k J^l_p + \mathcal{O}(\partial J, \partial\omega) \\ &= g^{ij}\partial_i\partial_j J^l_k + g^{ij}J^p_k\left[\partial_i\Gamma^l_{jp} - \partial_p\Gamma^l_{ij} + \operatorname{Rm}^l_{pij}\right] \\ &\quad + g^{ij}J^l_p\left[\partial_k\Gamma^p_{ij} - \partial_i\Gamma^p_{jk} - \operatorname{Rm}^p_{kij}\right] + \mathcal{O}(\partial J, \partial\omega) \\ &= g^{ij}\partial_i\partial_j J^l_k + \mathcal{O}(\partial J, \partial\omega). \end{aligned} \tag{5.11}$$

Now the proof of short time existence follows well understood lines. Let $\mathcal{L}$ denote linearization, and let ψ, K denote variation vectors for ω and J respectively. From (5.9) and Proposition 4.5 we conclude that

$$\begin{aligned}\sigma\left[\widehat{\mathcal{L}\mathcal{D}_1}\right](\psi)_{ij} =&\ \sigma\left[\widehat{\mathcal{L}(-S)}\right](\psi)_{ij}\\ =&\ \left|\xi\right|^2\psi_{ij}.\end{aligned}$$

Next it follows from (5.11) that

$$\sigma\left[\widehat{\mathcal{L}\mathcal{D}_2}\right](K)_a^b = \left|\xi\right|^2 K_a^b,$$

and also

$$\sigma\left[\widehat{\mathcal{L}\mathcal{D}_2}\right](\psi)_a^b = 0.$$

We conclude

$$\sigma\left[\widehat{\mathcal{L}\mathcal{D}}\right](\psi, K) = \begin{pmatrix} I & * \\ 0 & I\end{pmatrix}\begin{pmatrix}\psi \\ K\end{pmatrix}.$$

Thus the symbol of the linearization of $\mathcal{D}$ is upper triangular with strictly positive diagonal entries, and is hence positive definite. The claim of short time existence and uniqueness now follows as in the proof of Theorem 1.1. The claim that (1.3) is of the family of equations in (1.1) is left to the next subsection. □

5.2. **Equivalent Formulations.** While the expression (5.8) certainly makes a number of its properties transparent, we would like to derive some other forms of this equation which will be relavent for other purposes. Specifically, we first want to relate (5.8) to (1.1).

Proposition 5.3. *Let $(M^{2n}, \omega(t), J(t))$ be a one-parameter family of almost Kähler structures solving (1.3). Then the family $(\omega(t), J(t))$ is a solution to*

$$\begin{aligned}\frac{\partial}{\partial t}\omega =&\ -S + Q^1 + H\\ \frac{\partial}{\partial t}J =&\ -\mathcal{K} + \mathcal{H}^1\end{aligned}\tag{5.12}$$

where Q^1 and $\mathcal{H}^1$ are defined in (5.13) and (5.14) respectively, and H is defined so that (4.1) holds. In particular, (5.12) is a degenerate parabolic equation for almost Hermitian pairs (ω, J) which preserves the almost Kähler condition.

Proof. Let (ω, J) be an almost Kähler structure, and let Ω denote the curvature of the Chern connection. Let $\{e_i\}$ denote a local orthonormal frame for $T^{1,0}(M)$. First recall the Bianchi identity for a connection ∇:

$$\Sigma_{X,Y,Z}\left[\Omega(X,Y)Z - T(T(X,Y),Z) - \nabla_X T(Y,Z)\right] = 0$$

For our almost Kähler structure the torsion T is completely determined by the Nijenhuis tensor, which is a $(0,2)$ form with values in $(1,0)$ vectors. Using this we compute an expression for the $(1,1)$ part of P.

$$\begin{aligned}
P(e_j,\overline{e}_k) &= \Omega(e_j,\overline{e}_k,e_i,\overline{e}_i)\\
&= \Omega(e_i,\overline{e}_k,e_j,\overline{e}_i) + \langle N(N(e_i,e_j),\overline{e}_k),\overline{e}_i\rangle + \langle \nabla_{\overline{e}_k} N(e_i,e_j),\overline{e}_i\rangle\\
&= -\Omega(e_i,\overline{e}_k,\overline{e}_i,e_j) + \langle N(N(e_i,e_j),\overline{e}_k),\overline{e}_i\rangle + \langle \nabla_{\overline{e}_k} N(e_i,e_j),\overline{e}_i\rangle\\
&= S(e_j,\overline{e}_k) - \langle N(N(\overline{e}_i,\overline{e}_k),e_i),e_j\rangle - \langle \nabla_{e_i} N(\overline{e}_k,\overline{e}_i),e_j\rangle\\
&\quad + \langle N(N(e_i,e_j),\overline{e}_k),\overline{e}_i\rangle + \langle \nabla_{\overline{e}_k} N(e_i,e_j),\overline{e}_i\rangle .
\end{aligned}$$

But since N takes values in $(1,0)$ vectors and ∇ is a Hermitian connection, it follows that

$$\langle \nabla e_i N(\overline{e}_k,\overline{e}_i),e_j\rangle = \langle \nabla_{\overline{e}_k} N(e_i,e_j),\overline{e}_i\rangle = 0.$$

It follows that

$$P^{1,1} = S + Q^1$$

where

$$Q^1_{i\overline{j}} = \omega^{k\overline{l}}\left(g_{m\overline{l}} N^{\overline{p}}_{ki} N^{m}_{\overline{p}\overline{j}} - g_{m\overline{j}} N^{p}_{\overline{l}\overline{j}} N^{\overline{m}}_{pk}\right). \tag{5.13}$$

Next we examine the evolution equation for J. Choose normal coordinates for the associated metric at a point p. Then, including the precise lower order terms in (4.4), we see that

$$\begin{aligned}
\omega^{kl}\nabla_k N^i_{lj} =& -g^{kl}\partial_k\partial_l J^i_j + \omega^{kl}\left(J^i_q\partial_k\partial_j J^q_l - J^q_j\partial_k\partial_q J^i_l\right)\\
&+ \omega^{kl}\left(D_k J^p_l D_p J^i_j - D_k J^p_j D_p J^i_l - D_k J^i_p D_l J^p_j + D_k J^i_p D_j J^i_l\right)\\
&+ \frac{1}{2}\omega^{kl}\left(N^i_{kp}N^p_{lj} - N^p_{kl}N^i_{pj} - N^p_{kj}N^i_{lp}\right).
\end{aligned}$$

Furthermore, by a calculation similar to (5.11), we can compute in normal coordinates at p,

$$(-D^*DJ + \mathcal{N} + \mathcal{R})^i_j = g^{kl}\partial_k\partial_l J^i_j + g^{kl}J^p_j\partial_p\Gamma^i_{kl} - g^{kl}J^i_p\partial_j\Gamma^p_{kl} + \mathcal{N}^i_j.$$

Furthermore, calculating as in (4.7), (4.8), again using the normal coordinates,

$$\begin{aligned}
g^{kl}J^p_j\partial_p\Gamma^i_{kl} &= J^p_j\partial_p\left(g^{kl}\Gamma^i_{kl}\right)\\
&= J^p_j\partial_p\left(g^{kl}(\Gamma_C)^i_{kl} + \omega^{kl}(\Gamma_C)^q_{kl}J^i_q\right)\\
&= J^p_j\partial_p\left(\omega^{kl}\partial_k J^i_l - \nabla_k J^i_l\right)\\
&= J^p_j\omega^{kl}\partial_p\partial_k J^i_l - J^p_j\omega^{kr}J^l_q D_p J^q_r D_k J^i_l.
\end{aligned}$$

Likewise

$$g^{kl}J^i_p\partial_j\Gamma^p_{kl} = \omega^{kl}J^i_p\partial_j\partial_k J^p_l - J^i_p\omega^{kr}J^l_q D_j J^q_r D_k J^p_l.$$

Combining these calculations yields

$$(-D^*DJ + \mathcal{N} + \mathcal{R}) = -\mathcal{K} + \mathcal{H}^1$$

where

$$\begin{aligned}
\left(\mathcal{H}^1\right)^i_j =& \omega^{kl}\left(D_k J^p_l D_p J^i_j - D_k J^p_j D_p J^i_l - D_k J^i_p D_l J^p_j + D_k J^i_p D_j J^i_l\right)\\
&+ \frac{1}{2}\omega^{kl}\left(N^i_{kp}N^p_{lj} - N^p_{kl}N^i_{pj} - N^p_{kj}N^i_{lp}\right)\\
&- J^p_j\omega^{kr}J^l_q D_p J^q_r D_k J^i_l + J^i_p\omega^{kr}J^l_q D_j J^q_r D_k J^p_l + g^{ik}g^{pq}\omega_{rs}D_p J^r_j D_q J^s_k.
\end{aligned} \tag{5.14}$$

Finally, it is clear by construction that if we define H so that (4.1) holds, it must equal $-2P^{2,0+0,2}$. It follows that a solution to (1.3) is a solution to (5.12), and the proposition follows. □

Next we want to derive the evolution equation for the associated Riemannian metric.

Proposition 5.4. *Let $(M^{2n}, \omega(t), J(t))$ be a one-parameter family of almost Kähler structures solving (1.3). Then the associated Riemannian metric $g(t)$ satisfies*

$$\frac{\partial}{\partial t} g = -2\operatorname{Ric} + N^1(\cdot, J\cdot) - N^2(\cdot, J\cdot).$$

Proof. We begin with a general calculation using the notation of Lemma 3.2. Specifically, we have

$$\begin{aligned}\frac{\partial}{\partial t} g(\cdot,\cdot) &= \frac{\partial}{\partial t}\left[\omega(\cdot, J\cdot)\right]\\ &= \left[\phi(\cdot, J\cdot) + \psi(\cdot, J\cdot) + \omega(\cdot, K\cdot)\right].\end{aligned}$$

Let us compute these three terms separately. First of all it follows from (5.3) and (5.4) that

$$\begin{aligned}P^{1,1}(\cdot, J\cdot) &= \left(\rho - \frac{1}{2}N^1 + \frac{1}{2}N^2\right)(\cdot, J\cdot)\\ &= \operatorname{Ric}^J(\cdot,\cdot) - \frac{1}{2}N^1(\cdot, J\cdot) + \frac{1}{2}N^2(\cdot, J\cdot).\end{aligned}$$

Thus

$$\begin{aligned}\phi(\cdot, J\cdot) &= -2P^{1,1}(\cdot, J\cdot)\\ &= -2\operatorname{Ric}^J(\cdot,\cdot) + N^1(\cdot, J\cdot) - N^2(\cdot, J\cdot).\end{aligned}$$

Now observe that

$$\psi(\cdot, J\cdot) = -2P^{2,0+0,2}(\cdot, J\cdot).$$

Next consider

$$\omega(\cdot, K\cdot)_{ij} = \omega_{ik}K_j^k = \omega_{ik}\left(g^{kl}\left(-2P_{jl}^{2,0+0,2}\right) + J_j^l \operatorname{Rc}_l^k - \operatorname{Rc}_j^l J_l^k\right).$$

The first term simplifies to

$$\begin{aligned}-2\omega_{ik}g^{kl}P_{jl}^{2,0+0,2} &= -2J_i^p g_{pk} g^{kl} P_{jl}^{2,0+0,2}\\ &= -2J_i^l P_{jl}^{2,0+0,2}\\ &= 2J_i^l\left(J_j^m J_l^p P_{mp}^{2,0+0,2}\right)\\ &= -2J_j^m P_{mi}^{2,0+0,2}\\ &= 2P^{2,0+0,2}(\cdot, J\cdot)_{ij}.\end{aligned}$$

Next we calculate

$$\begin{aligned}\omega_{ik}\left(\operatorname{Rc}_j^l J_l^k - J_j^l \operatorname{Rc}_l^k\right) &= J_i^p g_{pk}\left(J_j^l \operatorname{Rc}_l^k - \operatorname{Rc}_j^l J_l^k\right)\\ &= \left(J^*\operatorname{Ric} - \operatorname{Rc}\right)_{ij}\\ &= -2\left(\operatorname{Rc} - \operatorname{Ric}^J\right)_{ij}.\end{aligned}$$

Combining the above calculations, the result follows. □

6. The structure of critical metrics

In this section we record some results on the structure of the limiting objects of equations (1.3).

Definition 6.1. Let (M^{2n}, ω, J) be an almost Kähler manifold. We say that this manifold is *static* if there exists $\lambda \in \mathbb{R}$ such that

$$P = \lambda \omega \tag{6.1}$$

$$D^* D J - \mathcal{N} - \mathcal{R} = 0. \tag{6.2}$$

Let us say a word on the definition of this condition. We want to understand the limiting behavior of equation (1.3), hence the first condition arises for solutions which simply rescale the metric. Observe though that even for solutions which are scaling the metric, one expects J to remain fixed as one cannot scale almost complex structures. Thus the static condition defined above is a natural expression of the expected smooth limit points of (1.3).

Lemma 6.2. *Let (M^{2n}, ω, J) be a static structure. Then*

$$\operatorname{Ric} - J^* \operatorname{Ric} = 0,$$

i.e. the Ricci tensor is J-invariant.

Proof. Equation (6.1) implies that $P^{2,0+0,2} = 0$. Equation (6.2) may be expressed as

$$g^{-1}\left[P^{2,0+0,2} + (\operatorname{Ric} - J^* \operatorname{Ric})\right] = 0,$$

and so the lemma follows. □

Let us show some further structure in dimension 4. Let (M^4, g) be an oriented Riemannian manifold. Since one may decompose $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, the action of the curvature tensor on Λ^2 decomposes accordingly, and is typically written

$$R = \left(\begin{array}{c|c} W^+ + \frac{s}{12} I & \overset{\circ}{\mathrm{Rc}} \\ \hline \overset{\circ}{\mathrm{Rc}} & W^- + \frac{s}{12} I \end{array}\right) \tag{6.3}$$

where $\overset{\circ}{\mathrm{Rc}}$ is a certain action of the traceless Ricci tensor and W^+ and W^- are the self-dual and anti-self-dual Weyl curvatures. If one further has (M^4, ω, J) an almost Hermitian manifold, then one can refine the decomposition of Λ^2 as

$$\Lambda^2 = \left((\omega) \oplus \Lambda^{2,0}\right) \oplus \Lambda_0^{1,1} \tag{6.4}$$

where $\Lambda_0^{1,1}$ are real $(1,1)$ forms orthogonal to ω. Using this further decomposition one yields, adopting notation of [2],

$$R = \left(\begin{array}{c|c||c} a & W_F^+ & R_F \\ \hline W_F^{+*} & W_{00}^+ + \frac{1}{2} b I & R_{00} \\ \hline\hline R_F^* & R_{00}^* & W_{00}^- + \frac{1}{3} c I \end{array}\right) \tag{6.5}$$

where the tensors in this equation are defined by comparing with (6.3) and using the refined decomposition of forms of (6.4). The double bars indicate the original decomposition into self-dual and anti-self-dual forms. Now we recall a curvature calculation in [2] which decomposes the curvature tensor of the *canonical* connection of an almost Kähler manifold according to (6.4).

Proposition 6.3. *([2] Proposition 2)*

$$\Omega = \left(\begin{array}{c|c||c} \frac{s^\nabla}{12} & W_F^+ & R_F - 2C \\ \hline 0 & 0 & 0 \\ \hline\hline R_F^* & R_{00} & W^- + \frac{1}{3}cI \end{array}\right) \tag{6.6}$$

One may consult [2] for the precise definition of C, which is not relevant to us here. All the other tensors are the same as what appears in (6.5). It is important to observe that this matrix acts from the right on two-forms. For instance, the image acting from the right lies entirely in $(1,1)$ forms, as required.

Proposition 6.4. *Let (M^4, ω, J) be a static structure. Then*

$$W_F \equiv 0.$$

Proof. This immediate from (6.6) and the fact that $P = \lambda\omega$. □

Returning to (6.5) it follows that the ω is an eigenvector for the action of W_+. This condition is related to delicate topological estimates of LeBrun [9] related to the Seiberg-Witten equations. It remains to be seen what topological consequences can be derived from (6.1)

7. Remarks and open problems

Recall from [14], [15] we know that the solution to Kähler Ricci flow exists smoothly as long as the associated cohomology class is in the Kähler cone. Therefore it is natural, for purposes of understanding the long time existence and singularity formation of solutions to (7.1), to understand the corresponding cone $\mathcal{C}$ of symplectic forms in $H^2(M, \mathbb{R})$. Note that $\mathcal{C}$ consists of all cohomology classes in $H^2(M, \mathbb{R})$ which can be represented by a symplectic form. Any symplectic form ω admits compatible almost complex structures, and moreover the space of these almost complex structures is contractible. Thus one may define the *canonical class*

$$K = c_1(M, \omega) := c_1(M, J)$$

where J is any almost complex structure compatible with ω and the orientation. It is clear that the homotopy classes of symplectic structures define the same canonical class. Therefore, associated to a solution of (1.3), one has the well-defined *associated ODE in cohomology*

$$\frac{d}{dt}[\omega] = -K. \tag{7.1}$$

It is clear by the definition that given a solution to (1.3), the associated one parameter family of cohomology classes satisfies (7.1). Thus, we have

Lemma 7.1. *Given $(M^{2n}, \omega(t), J(t))$ a solution to (1.3), let*

$$T^* := \sup\{t > 0 | [\omega(t)] = [\omega(0)] - tK \in \mathcal{C}\}.$$

Furthermore, let T denote the maximal existence time of $(\omega(t), J(t))$. Then

$$T \leq T^*$$

It is natural to conjecture: *The maximal existence time for (1.3) with initial $\omega(0)$ is given by T^*.* This is the analogue of the theorem of Tian-Zhang ([14], [15]) mentioned above for Kähler Ricci flow.

If the above $T^* < \infty$, then (1.3) develops finite-time singularity. The second basic problem is to study *the nature of such a singularity.* Is it possible that such a singularity is caused by

J-holomorphic spheres as we see in the case of Kähler manifolds? The case of 4-dimensional symplectic manifolds is of particular interest and may be easier to study. We expect that *either $(\omega(t), J(t))$ collapses to a lower dimensional space or converges to a smooth pair (ω_T, J_T) outside a subvariety as t tends to $T \leq T^*$.* If so, we may do surgery and extend (1.3) across T. In a forthcoming paper, we will study how the curvature of the canonical connection behaves near finite-time singularity. Presumably, the curvature blows up at the singularity. By scaling, one may get ancient solutions for (1.3). A basic problem is to classify all the ancient solutions. In dimension 4, it may be possible to classify.

Another natural problem is to find functionals which are monotonic along (1.3). In particular, *is (1.3) a gradient flow like the Ricci flow and the pluriclosed flow of* [12]? We showed in [13] that the parabolic flow of pluriclosed metrics of [12] is in fact a gradient flow. This was done by exhibiting that after change by a certain diffeomorphism solutions to this flow are equivalent to solutions to the B-field renormalization group flow of string theory. In light of Proposition 5.4, solutions to (1.3) have the metric evolving by the Ricci flow plus certain lower order terms, therefore one expects to be able to add a certain Lagrangian to the Perelman functionals to obtain a gradient flow property for (1.3), as in the B-field renormalization group flow.

Finally, we believe that this new symplectic curvature flow will be useful in studying the topology of symplectic manifolds, particularly in dimension 4. It follows from the results in section 6 that static solutions in dimension 4 are of anti-self-dual type, more precisely, the self-dual part of curvature for the canonical connection is determined by its scalar curvature. This gives a hope to use (1.3) to prove a symplectic version of the Miyaoka-Yau inequality for complex surfaces. Such an inequality for symplectic 4-manifolds has been long speculated. For still further applications, we are led to studying limits of (1.3) as time t tends to ∞ and after appropriate scalings. The limits should include the above static metrics, soliton solutions as well as collapsed metrics which generalize the metrics studied by Song-Tian for elliptic surfaces.

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