A BRYLINSKI FILTRATION FOR AFFINE KAC-MOODY ALGEBRAS

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Abstract. Braverman and Finkelberg have recently proposed a conjectural analogue of the geometric Satake isomorphism for untwisted affine Kac-Moody groups. As part of their model, they conjecture that (at dominant weights) Lusztig's q-analog of weight multiplicity is equal to the Poincare series of the principal nilpotent filtration of the weight space, as occurs in the finite-dimensional case. We show that the conjectured equality holds for all affine Kac-Moody algebras if the principal nilpotent filtration is replaced by the principal Heisenberg filtration. The main body of the proof is a Lie algebra cohomology vanishing result. We also give an example to show that the Poincare series of the principal nilpotent filtration is not always equal to the q-analog of weight multiplicity. Finally, we give some partial results for indefinite Kac-Moody algebras.

1. Introduction

Let $\mathcal{L}(\lambda)$ be an integrable highest-weight representation of a symmetrizable Kac-Moody algebra $\mathfrak{g}$. The Kostant partition functions $K(\beta; q)$ are defined for weights β by

$$\sum_{\beta} K(\beta; q)e^{\beta} = \prod_{\alpha\in\Delta^+} (1 - qe^{\alpha})^{-\operatorname{mult}\alpha},$$

where Δ^+ is the set of positive roots and $\operatorname{mult}\alpha = \dim \mathfrak{g}_\alpha$. The q-character of a weight space $\mathcal{L}(\lambda)_\mu$ is the function

$$m^\lambda_\mu(q) = \sum_{w\in W} \epsilon(w)K(w * \lambda - \mu; q), \tag{1}$$

where W is the Weyl group of $\mathfrak{g}$, ϵ is the usual sign representation of W, and $w * \lambda = w(\lambda + \rho) - \rho$ is the shifted action of W. The name "q-character" is used because $m^\lambda_\mu(1) = \dim \mathcal{L}(\lambda)_\mu$.

When $\mathfrak{g}$ is finite-dimensional it is well-known that the q-analogs $m^\lambda_\mu(q)$ are equal to Kostka-Foulkes polynomials, which express the characters of highest-weight representations in terms of Hall-Littlewood polynomials [8], and are Kazhdan-Lusztig polynomials for the affine Weyl group [10]. When μ is dominant the coefficients of $m^\lambda_\mu(q)$ are non-negative. There is an explanation for this phenonemon, first conjectured by Lusztig [10]: the weight space $\mathcal{L}(\lambda)_\mu$ has an increasing filtration ${}^eF^*$ such that $m^\lambda_\mu(q)$ is equal to the Poincare polynomial

$${}^eP^\lambda_\mu(q) = \sum_{i\geq 0} q^i \dim {}^eF^i\mathcal{L}(\lambda)_\mu \;/\; {}^eF^{i-1}\mathcal{L}(\lambda)_\mu \tag{2}$$

of the associated graded space. This identity was first proved by Brylinski for μ regular or $\mathfrak{g}$ of classical type; the filtration ${}^eF^*$ is known as the *Brylinski* or

Brylinski-Kostant filtration, and is defined by

$$^{e}F^{i}(\mathcal{L}(\lambda)_{\mu}) = \{v \in \mathcal{L}(\lambda)_{\mu} : e^{i+1}v = 0\},$$

where e is a principal nilpotent. Brylinki's proof was extended to all dominant weights by Broer [2]. More recently Joseph, Letzter, and Zelikson gave a purely algebraic proof of the identity $m_{\mu}^{\lambda} = {}^{e}P_{\mu}^{\lambda}$, and determined ${}^{e}P_{\mu}^{\lambda}$ for μ non-dominant [7]. The q-analogs of weight multiplicity of an arbitrary symmetrizable Kac-Moody have been studied by Viswanath [14]; he shows that $m_{\mu}^{\lambda}(q)$ are Kostka-Foulkes polynomials for generalized Hall-Littlewood polynomials, and determines $m_{\mu}^{\lambda}(q)$ at some simple μ for an untwisted affine Kac-Moody.

The point of this paper is to extend Brylinski's result to affine (ie. indecomposable of affine type) Kac-Moody algebras. We show that, as in the finite-dimensional case, there is a filtration on $\mathcal{L}(\lambda)_{\mu}$ such that when μ is dominant, $m_{\mu}^{\lambda}(q)$ is equal to the Poincare series of the associated graded space. Unlike the finite-dimensional case, the principal nilpotent is not sufficient to define the filtration in the affine case; instead, we use the positive part of the principal Heisenberg (this form of Brylinski's identity was first conjectured by Teleman). Brylinski's original proof of the identity $m_{\mu}^{\lambda} = {}^{e}P_{\mu}^{\lambda}$ uses a cohomology vanishing result for the flag variety. Our proof is based on the same idea, but uses the Lie algebra cohomology approach of [5]. In particular we prove a vanishing result for Lie algebra cohomology by calculating the Laplacian with respect to a Kahler metric. Although we concentrate on the affine case for simplicity, our results generalize easily to the case when $\mathfrak{g}$ is a direct sum of algebras of finite or affine type. There are two difficulties in extending this result to indefinite symmetrizable Kac-Moody algebras: there does not seem to be a simple analogue of the Brylinski filtration, and the cohomology vanishing result does not extend for all dominant weights μ. We can overcome these difficulties by replacing the Brylinski filtration with an intermediate filtration, and by requiring that the root $\lambda - \mu$ has affine support. Thus we get some partial non-negativity results for the coefficients of $m_{\mu}^{\lambda}(q)$ even when $\mathfrak{g}$ is of indefinite type.

The primary motivation for this paper is a recent conjecture of Braverman and Finkelberg. Recall that when $\mathfrak{g}$ is finite-dimensional, the geometric Satake isomorphism is an equivalence between the representation category of any group G associated to $\mathfrak{g}$, and the category of equivariant perverse sheaves on the loop Grassmannian $\mathrm{Gr} = G^{\vee}((z))/G^{\vee}[[z]]$ of the Langlands dual group $G^{\vee}$. The loop Grassmannian Gr is an ind-variety, realized as an increasing disjoint union of Schubert varieties Gr^{λ} parametrized by weights of G. Under the equivalence, a highest-weight representation $\mathcal{L}(\lambda)$ is sent to the intersection cohomology complex IC^{λ} of $\overline{\mathrm{Gr}^{\lambda}}$. In addition to conjecturing the equality $m_{\mu}^{\lambda} = {}^{e}P_{\mu}^{\lambda}$, Lusztig showed in [10] that $m_{\mu}^{\lambda}(q)$ is equal (after a degree shift) to the generating function $\mathrm{IC}_{\mu}^{\lambda}(q)$ for the dimensions of the stalk of the complex $\mathrm{IC}_{\mu}^{\lambda}$ at a point in $\mathrm{Gr}^{\mu} \subset \overline{\mathrm{Gr}^{\lambda}}$. A direct isomorphism between the stalks $\mathrm{IC}_{\mu}^{\lambda}$ and the graded spaces $\mathrm{gr}\,\mathcal{L}(\lambda)_{\mu}$ appears in the geometric Satake isomorphism [4] [11], leading to another proof that $m_{\mu}^{\lambda} = {}^{e}P_{\mu}^{\lambda}$ (see [4] in particular). Braverman and Finkelberg have proposed a conjectural analogue of the geometric Satake isomorphism for affine Kac-Moody groups [1]. Their conjecture relates representations of $\mathfrak{g}$ to perverse sheaves on an analogue of the loop Grassmannian for $\mathfrak{g}^{\vee}$ when $\mathfrak{g}^{\vee}$ is an untwisted affine Kac-Moody. Their model leads them to conjecture that $m_{\mu}^{\lambda}(q) = {}^{e}P_{\mu}^{\lambda}$ in the affine case, with both related to the intersection cohomology stalks as in the finite-dimensional case.[1] Since we will demonstrate by example that $m^\lambda_\mu(q)$ is not necessarily equal to ${}^eP^\lambda_\mu$, our paper gives a correction of Braverman and Finkelberg's conjecture.

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1.2. **Organization.** The definition of the Brylinski filtration and the statements of the main results for affine Kac-Moody algebras are given in Section 2. Proofs follow in Sections 3 and 4. Partial results for indefinite Kac-Moody algebras are given in Section 5.

1.3. **Notation and terminology.** Throughout, $\mathfrak{g}$ will refer to a symmetrizable Kac-Moody algebra. For standard notation and terminology, we mostly follow [9]. We assume a fixed presentation of $\mathfrak{g}$, from which we get a choice of Cartan $\mathfrak{h}$, simple roots $\{\alpha_i\}$, simple coroots $\{\alpha_i^\vee\}$, and Chevalley generators $\{e_i, f_i\}$. We can then grade $\mathfrak{g}$ via the principal grading, ie. by assigning degree 1 to each e_i and degree -1 to each f_i. By choosing a real form $\mathfrak{h}_\mathbb{R}$ of $\mathfrak{h}$ we get an anti-linear Cartan involution $x \mapsto \overline{x}$, defined as the anti-linear involution sending $e_i \mapsto -f_i$ for all i and $h \mapsto -h$ for all $h \in \mathfrak{h}_\mathbb{R}$. As usual $\mathfrak{g}$ has the triangular decompostion $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the standard nilpotent $\bigoplus_{n>0} \mathfrak{g}_n$. The standard Borel is the subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Associated to $\mathfrak{n}$ and $\mathfrak{b}$ are the pro-algebras $\hat{\mathfrak{n}} = \lim_\leftarrow \mathfrak{n}/\mathfrak{n}_k$ and $\hat{\mathfrak{b}} = \lim_\leftarrow \mathfrak{b}/\mathfrak{n}_k$, where $\mathfrak{n}_k = \bigoplus_{n>k} \mathfrak{g}_n$.

2. The Brylinski filtration for affine Kac-Moody algebras

A principal nilpotent (with respect to a given presentation) of a symmetrizable Kac-Moody algebra is an element $e \in \mathfrak{g}_1$ of the form $e = \sum c_i e_i$, where $c_i \in \mathbb{C} \setminus \{0\}$ for all simple roots e_i. If $\mathfrak{g}$ is affine it is well-known that the algebras $\mathfrak{s}_e = \{x \in \mathfrak{g} : [x, e] \in Z(\mathfrak{g})\}$ are Heisenberg algebras, and these algebras are called principal Heisenberg subalgebras.

Definition 2.1. *Let $\mathcal{L}(\lambda)$ be a highest-weight module of an affine Kac-Moody algebra $\mathfrak{g}$. Define the* Brylinski filtration *with respect to the principal Heisenberg $\mathfrak{s}$ by*

$$ {}^\mathfrak{s}F^i\mathcal{L}(\lambda)_\mu = \{v \in \mathcal{L}(\lambda)_\mu : x^{i+1}v = 0 \text{ for all } x \in \mathfrak{s} \cap \mathfrak{n}\}. $$

Let ${}^\mathfrak{s}P^\lambda_\mu(q)$ be the Poincare series of the associated graded space of $\mathcal{L}(\lambda)_\mu$.

Note that the principal nilpotents form a single H-orbit, so the filtration ${}^\mathfrak{s}F^*$ is independent of the choice of principal Heisenberg.

Recall that a weight μ is real-valued if $\mu(h) \in \mathbb{R}$ for all $h \in \mathfrak{h}_\mathbb{R}$, and dominant if $\mu(\alpha_i^\vee) \geq 0$ for all simple coroots $\alpha_i^\vee$,

Theorem 2.2. *Let $\mathcal{L}(\lambda)$ be an integrable highest weight representation of an affine Kac-Moody algebra $\mathfrak{g}$, where λ is a real-valued dominant weight. If μ is a dominant weight of $\mathcal{L}(\lambda)$ then $P^\lambda_\mu(q) = m^\lambda_\mu(q)$.*

[1]There seems to be a typo in [1]: root multiplicities are omitted in the definition of the Kostant partition functions.

The dual $\hat{\mathfrak{n}}^*$ of a pro-algebra will refer to the continuous dual. If V is a $\hat{\mathfrak{b}}$-module then $H^*_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; V)$ will denote the relative continuous cohomology of $(\hat{\mathfrak{b}}, \mathfrak{h})$. The proof of Theorem 2.2 depends on

Theorem 2.3. *Let $\mathcal{L}(\lambda)$ be an integrable highest weight representation of an affine Kac-Moody algebra $\mathfrak{g}$, where λ is a real-valued dominant weight. Let $V = \mathcal{L}(\lambda) \otimes S^*\hat{\mathfrak{n}}^* \otimes \mathbb{C}_{-\mu}$, where μ is a dominant weight of $\mathcal{L}(\lambda)$. Then $H^q_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; V) = 0$ for $q > 0$, and in addition there is a graded isomorphism $\operatorname{gr} \mathcal{L}(\lambda)_\mu \cong H^0_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; V)$, where the latter space is graded by symmetric degree.*

Proof of Theorem 2.2 from Theorem 2.3. Let $V^p = \mathcal{L}(\lambda) \otimes S^p\hat{\mathfrak{n}}^* \otimes \mathbb{C}_{-\mu}$. By Theorem 2.3, $P^\lambda_\mu(q) = \sum_{p\geq 0} \dim H^0_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; V^p)q^p = \sum \chi(\hat{\mathfrak{b}}, \mathfrak{h}; V^p)q^p$, where χ is the Euler characteristic (the second equality follows from cohomology vanishing). Since $\hat{\mathfrak{n}}^*$ has finite-dimensional weight spaces and all weights belong to the negative root cone, $\bigwedge^* \hat{\mathfrak{n}}^* \otimes \mathcal{L}(\lambda) \otimes S^p\hat{\mathfrak{n}}^*$ has finite-dimensional weight spaces. Thus we can write

$$\begin{aligned}\sum_{p\geq 0} \chi(\hat{\mathfrak{b}}, \mathfrak{h}; V^p)q^p &= \sum_{p,k\geq 0} (-1)^k q^p \dim \left(\bigwedge^k \hat{\mathfrak{n}}^* \otimes V^p \right)^{\mathfrak{h}} \\ &= [e^\mu] \operatorname{ch} \mathcal{L}(\lambda) \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{\operatorname{mult}\alpha} (1 - qe^{-\alpha})^{-\operatorname{mult}\alpha}.\end{aligned}$$

Applying the Weyl-Kac character formula

$$\operatorname{ch} \mathcal{L}(\lambda) = \sum_{w\in W} \epsilon(w) e^{w*\lambda} \cdot \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{-\operatorname{mult}\alpha}$$

we get the result. □

The proof of Theorem 2.3 will be given in Sections 3 and 4. If $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ is a direct sum of indecomposables of finite and affine type, the conclusions of Theorems 2.2 and 2.3 remain true with $\mathfrak{s}$ replaced by a direct sum of principal nilpotents (for the finite components) and principal Heisenbergs (for the affine components).

2.1. **Examples.** We now give some elementary examples to show that ${}^{\mathfrak{s}}F$ is different from ${}^eF$. Consider $\widehat{\mathfrak{sl}_2}$, the affine Kac-Moody algebra realized as $\mathfrak{sl}_2[z^{\pm 1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where c is a central element, and d is the derivation $\frac{\partial}{\partial z}$. Let $\{H, E, F\}$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_2$, and take principal nilpotent $e = E + Fz$. The principal Heisenberg $\mathfrak{s}$ is spanned by the elements ez^n, $n \in \mathbb{Z}$, along with c.

The Cartan subalgebra of $\widehat{\mathfrak{sl}_2}$ is $\operatorname{span}\{H, c, d\}$. Denote a weight $\alpha H^* + hc^* + nd^*$ by (α, h, n). The weight $\lambda = (\alpha, h, n)$ is dominant if $0 \leq \alpha \leq h$, and the corresponding irreducible highest-weight representation $L(\lambda)$ can be realized as the quotient of the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ by the $U(\mathfrak{g})$-submodule generated by $F^{\alpha+1} \otimes 1$ and $(Ez^{-1})^{h-\alpha+1} \otimes 1$. Let

$$w = (Fz^{-1})(Ez^{-1})v,$$

where v is the highest weight vector in $L(c^*)$. Note that w is a weight vector of weight $(0, 1, -2)$. It is easy to check, using the defining relations for $L(c^*)$, that $e^2 w = 0$, while $(ez)ew = 3v$, so $w \in {}^eF^2$ but not in ${}^{\mathfrak{s}}F^2$.

The same idea can be used to calculate Poincare series. For the above example, where $\lambda = (0, 1, 0)$ and $\mu = (0, 1, -2)$, we have $\dim \mathcal{L}(\lambda)_\mu = 2$. The Poincare series for ${}^{e}F$ is $q+q^4$, while the Poincare series for ${}^{\mathfrak{s}}F$ is $m^\lambda_\mu(q) = q^2+q^4$. For an example with a dominant regular weight, let $\lambda = (0,3,0)$ and $\mu = (2,3,-3)$. The Poincare series of ${}^{e}F$ is $q + 2q^2 + q^3 + q^5$, while $m^\lambda_\mu(q) = q + q^2 + 2q^3 + q^5$.

3. Reduction to cohomology vanishing

In this section we introduce an equivalent filtration to the Brylinski filtration, which will allow us to reduce Theorem 2.3 to a cohomology vanishing statment. The line of argument is inspired by [3] and [5]. As usual, $\mathfrak{g}$ will be an arbitrary symmetrizable Kac-Moody algebra except where stated.

Associated to $\mathfrak{g}$ is a Kac-Moody group $\mathcal{G}$. The standard Borel subgroup $\mathcal{B}$ of $\mathcal{G}$ is a solvable pro-group with Lie algebra $\hat{\mathfrak{b}}$. The standard unipotent subgroup $\mathcal{U} \subset \mathcal{B}$ is a unipotent pro-group with Lie algebra $\hat{\mathfrak{n}}$. The Borel $\mathcal{B}$ also contains a torus H corresponding to $\mathfrak{h}$. Defining the new filtration requires two lemmas.

Lemma 3.1. *There are pro-algebraic morphisms $\mathcal{U} \cong \mathcal{B}/H \cong \hat{\mathfrak{n}}$ giving $\mathcal{U}$ the structure of an affine pro-variety with an affine $\mathcal{B}$-action.*

Proof. Pick $\delta \in \mathfrak{h}$ acting on $\mathfrak{g}_n$ as multiplication by n, and define $\pi : \mathcal{B} \to \hat{\mathfrak{n}}$ by $\mathrm{Ad}(b)\delta = \delta + \pi(b)$. Then the composition $\mathcal{U} \hookrightarrow \mathcal{B} \to \mathcal{B}/H \to \hat{\mathfrak{n}}$ is an isomorphism. $\hat{\mathfrak{n}}$ is naturally an affine pro-variety by the identification of $\mathfrak{n}/\mathfrak{n}_k$ with $\bigoplus_{n=1}^k \mathfrak{g}_n$, while $\mathcal{B}/H$ has a left-translation action of $\mathcal{B}$. If $b_1, b_2 \in \mathcal{B}$ then $\mathrm{Ad}(b_1 b_2)\delta = \mathrm{Ad}(b_1)(\delta + \pi(b_2)) = \delta + \pi(b_1) + \mathrm{Ad}(b_1)\pi(b_2)$, so $\pi(b_1 b_2) = \mathrm{Ad}(b_1)\pi(b_2) + \pi(b_1)$ and the resulting action of $\mathcal{B}$ on $\hat{\mathfrak{n}}$ is affine. $\square$

Lemma 3.2. *Let V be a pro-representation of $\mathcal{B}$. Then evaluation at the identity gives an isomorphism $(V \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}} \to V^H$.*

Proof. Any element $v \in V^H$ extends to a $\mathcal{B}$-invariant function $\mathcal{U} \to V$ by $[b] \mapsto bv$. $\square$

If V is a pro-representation of $\mathcal{B}$ then V^H can be filtered via polynomial degree on $\mathbb{C}[\mathcal{U}]$. If μ is a weight of $\mathfrak{g}$ then extending μ by zero on $\mathcal{U}$ makes $\mathbb{C}_{-\mu}$ into a pro-representation of $\mathcal{B}$. The reason for introducing a new filtration is the following lemma, which reduces the proof of Theorem 2.3 to a vanishing result.

Lemma 3.3. *Let $W = \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu}$, and filter $\mathcal{L}(\lambda)_\mu = W^H$ via the isomorphism $W^H \cong (W \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}}$. If $H^1_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^*\hat{\mathfrak{n}}^*) = 0$ then $H^0_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^*\hat{\mathfrak{n}}^*) \cong \mathrm{gr}\, \mathcal{L}(\lambda)_\mu$.*

Proof. Let $\mathcal{F}^p$ be the subset of $\mathbb{C}[\mathcal{U}]$ of polynomials of degree at most p. Then $\mathrm{gr}\, \mathbb{C}[\mathcal{U}] = S^*\hat{\mathfrak{n}}^*$ as $\mathcal{B}$-modules, so there are short exact sequences

$$0 \to W \otimes \mathcal{F}^{p-1} \to W \otimes \mathcal{F}^p \to W \otimes S^p\hat{\mathfrak{n}}^* \to 0$$

of $\mathcal{B}$-modules for all p. The corresponding long exact sequence in Lie algebra cohomology is

$$H^i_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes \mathcal{F}^{p-1}) \to H^i_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes \mathcal{F}^p) \to H^i_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^p\hat{\mathfrak{n}}^*) \to H^{i+1}_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes \mathcal{F}^{p-1}).$$

Since $H^i_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^p\hat{\mathfrak{n}}^*) = 0$ for $i = 1$, the inclusion $W \otimes \mathcal{F}^{p-1} \hookrightarrow W \otimes \mathcal{F}^p$ induces a surjection in degree one cohomology for all p. Since $\mathcal{F}^{-1} = 0$, $H^1_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes \mathcal{F}^p) = 0$ for all p. The long exact sequence in degree $i = 0$ gives an isomorphism

$H^0_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^p \hat{\mathfrak{n}}^*) \cong (W \otimes \mathcal{F}^p)^{\mathfrak{b}} / (W \otimes \mathcal{F}^{p-1})^{\mathfrak{b}}$. This latter quotient is the graded space of $(W \otimes \mathbb{C}[U])^{\mathfrak{b}}$ as required. □

Now we show that the new filtration is equal to the Brylinski filtration when $\mathfrak{g}$ is affine.

Proposition 3.4. *Let $\mathcal{L}(\lambda)$ be an integrable highest-weight representation of an an affine Kac-Moody $\mathfrak{g}$. Then the Brylinski filtration on a weight space $\mathcal{L}(\lambda)_\mu$ agrees with the filtration of $\mathcal{L}(\lambda)_\mu \cong (\mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu} \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}}$ by polynomial degree.*

The proof of Proposition 3.4 requires two lemmas.

Lemma 3.5. *If $\mathfrak{g}$ is affine and $\mathfrak{s}$ is a principal Heisenberg then $\mathrm{Ad}(\mathcal{B})(\mathfrak{s} \cap \mathfrak{n})$ is dense in $\hat{\mathfrak{n}}$.*

Proof. The principal nilpotents form a dense orbit, so it is only necessary to prove this fact for a single principal nilpotent. We claim that there is a principal nilpotent such that $f = -\overline{e} \in \mathfrak{s}_e$, so that in particular $[e, f] \in Z(\mathfrak{g})$. Indeed, let A be the generalized Cartan matrix defining $\mathfrak{g}$, ie. $A_{ij} = \alpha_j(\alpha_i^\vee)$. Since $\mathfrak{g}$ is affine there is a vector $c > 0$, unique up to a scalar multiple, such that $A^t c = 0$. If we pick $e = \sum \sqrt{c_i} e_i$ then $[e, f] = \sum c_i \alpha_i^\vee$, and $\alpha_j([e, f]) = \sum c_i A_{ij} = (A^t c)_j = 0$ for all simple roots α_j.

Now we show that $\mathfrak{n} = (\mathfrak{s}_e \cap \mathfrak{n}) + [\mathfrak{b}, e]$. In degree one we have $[\mathfrak{h}, e] = \mathfrak{g}_1$. For higher degrees, let $\{,\}$ denote the standard non-degenerate contragradient Hermitian form on $\mathfrak{g}$ which is positive definite on $\mathfrak{n}$. An element $x \in \mathfrak{n}$ is orthogonal to $[\mathfrak{b}, e]$ if and only if $0 = \{[e, z], x\} = \{z, [f, x]\}$ for all $z \in \mathfrak{b}$, or in other words if and only if $x \in C_{\mathfrak{g}}(f)$. Suppose $x \in \mathfrak{g}_n$, $n \geq 2$ belongs to $[\mathfrak{b}, e]^\perp$. Using the fact that $[e, f] \in Z(\mathfrak{g})$ we get that $\{[e, x], [e, x]\} = \{[f, x], [f, x]\} = 0$, and conclude that $x \in \mathfrak{s}_e$.

$(\mathfrak{s} \cap \mathfrak{n}) + [\mathfrak{b}, e] = \mathfrak{n}$ implies that $\mathcal{B} \times (\mathfrak{s} \cap \mathfrak{n}) \to \hat{\mathfrak{n}}$ is a submersion in a neighbourhood of $(\mathbb{1}, e)$. Since $\mathcal{B}$ acts algebraically on $\mathfrak{s} \cap \mathfrak{n} \subset \hat{\mathfrak{b}}$, the subset $\mathcal{B}(\mathfrak{s} \cap \mathfrak{n})$ is dense in $\hat{\mathfrak{n}}$. □

Lemma 3.6. *Let $\mathcal{L}(\lambda)$ be an integrable highest-weight module. Considered as a $\mathcal{B}$-module, $\mathcal{L}(\lambda)$ is a submodule of $\mathbb{C}[\mathcal{U}] \otimes \mathbb{C}_\lambda$.*

Proof. This statement would follow immediately from a Borel-Weil theorem for the thick flag variety of a Kac-Moody group. As we are not aware of a formal statement of the Borel-Weil theorem in this context, we recover the result from the dual of the quotient map $M_{low}(-\lambda) \to \mathcal{L}_{low}(-\lambda)$, where $M_{low}(-\lambda) = U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} C_{-\lambda}$ is a lowest weight Verma module, and $\mathcal{L}_{low}(-\lambda)$ is the irreducible representation with lowest weight $-\lambda$. Both these spaces are $\mathfrak{g}$-modules with finite gradings induced by the principal grading of $\mathfrak{g}$. Let $M_{low}(-\lambda)^*$ and $\mathcal{L}(-\lambda)^*$ denote the finitely-supported duals, consisting of linear functions which are supported on a finite number of components.

Using the fact that $M_{low}(-\lambda)$ is a free $U(\mathfrak{n})$-module, we can identity $M_{low}(-\lambda)$ with $S^*\mathfrak{n} \otimes \mathbb{C}_{-\lambda}$ where $S^*\mathfrak{n}$ has the $\mathfrak{b}$-action $(y, x) \mapsto [y, \delta] \circ x + \mathrm{ad}(y)x$, and δ is defined as in Lemma 3.1 as an element of $\mathfrak{h}$ which acts on $\mathfrak{g}_n$ as multiplication by n. The finitely supported dual of $M_{low}(-\lambda)$ can be identified with $S^*\hat{\mathfrak{n}}^* \otimes \mathbb{C}_\lambda$ where $\mathfrak{b}$ acts on $S^*\hat{\mathfrak{n}}^*$ by $(y, f) \mapsto \mathrm{ad}^t(y)f + \iota([\delta, y])f$. It is not hard to check that this action integrates to the $\mathcal{B}$-action coming from identifying $S^*\hat{\mathfrak{n}}^*$ with $\mathbb{C}[\mathcal{U}]$. Since the quotient map preserves the principal grading, the dual of the surjection

$M_{low}(-\lambda) \to \mathcal{L}_{low}(-\lambda)$ is an inclusion $\mathcal{L}(\lambda) = \mathcal{L}_{low}(-\lambda)^* \hookrightarrow M_{low}(-\lambda)^* = \mathbb{C}[\mathcal{U}] \otimes \mathbb{C}_\lambda$ as required. □

Proof of Proposition 3.4. Let $V = \mathbb{C}_\beta \otimes \mathbb{C}[\mathcal{U}]$, where $\beta = \lambda - \mu$. By the last lemma, we can prove the Proposition with $\mathcal{L}(\lambda)_\mu$ replaced by V^H, where the filtration on V^H is defined by $V^H \cong (V \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}}$. An element f of this latter set can be identified with a $\mathcal{B}$-invariant function $\mathcal{U} \times \mathcal{U} \to \mathbb{C}_\beta$. The polynomial degree on the second factor is the maximum t-degree of $f(u, tx)$ as u ranges across $\mathcal{U}$ and x ranges across $\hat{\mathfrak{n}} \cong \mathcal{U}$. Suppose this maximum is achieved at (u_0, x_0). Since $\mathcal{B}(\mathfrak{s} \cap \mathfrak{n})$ is dense in $\hat{\mathfrak{n}}$, we can assume that $x_0 = \mathrm{Ad}(b)s$ for $b \in \mathcal{B}$ and $s \in \mathfrak{s} \cap \mathfrak{n}$. Now $\mathfrak{s} \cap \mathfrak{n}$ is abelian and graded, so the graded components of s commute with each other. This allows us to find $\tilde{s} \in \mathfrak{s} \cap \mathfrak{n}$ such that $\pi(e^{t\tilde{s}}) = ts$. Since the degree of $f(u_0, \cdot)$ is achieved on the line $\mathrm{Ad}(b)\pi(e^{t\tilde{s}})$, it is also achieved on the parallel line $\mathrm{Ad}(b)\pi(e^{t\tilde{s}}) + \pi(b) = \pi(be^{t\tilde{s}})$. Thus the polynomial degree of f is equal to the t-degree of $f(u_0, b\pi(e^{t\tilde{s}})) = \beta(b) f(b^{-1}u_0, \pi(e^{t\tilde{s}}))$. Since $\beta(b)$ is a non-zero scalar, we conclude that there is $u \in \mathcal{U}$ and $s \in \mathfrak{s} \cap \mathfrak{n}$ such that the degree of f is equal to the t-degree of $f(u, \pi(e^{ts}))$. Conversely if $s \in \mathfrak{s} \cap \mathfrak{n}$ then $\pi(e^{ts})$ is a line in $\hat{\mathfrak{n}}$, so the degree of f is equal to the t-degree of $f(u, \pi(e^{ts}))$ as u ranges across $\mathcal{U}$ and s ranges across $\mathfrak{s} \cap \mathfrak{n}$.

Given $f \in (\mathbb{C}_\beta \otimes \mathbb{C}[\mathcal{U}] \otimes \mathbb{C}[\mathcal{U}])$ let $\tilde{f} \in \mathbb{C}_\beta \otimes \mathbb{C}[\mathcal{U}]$ be the restriction to $\mathcal{U} \times \{\mathbb{1}\}$. The t-degree of $f(u, \pi(e^{ts}))$ is equal to the t-degree of $(e^{-ts}\tilde{f})(u)$. Since

$$e^{-ts}\tilde{f} = \sum_{n \geq 0} \frac{(-1)^n t^n}{n!} s^n \tilde{f},$$

the degree of f is clearly equal to the smallest n such that $s^{n+1}\tilde{f} = 0$ for all $s \in \mathfrak{s} \cap \mathfrak{n}$. □

The proof of Proposition 3.4 works just as well with $\mathfrak{s} \cap \mathfrak{n}$ replaced by any graded abelian subalgebra $\mathfrak{a}$ of $\hat{\mathfrak{n}}$ such that $\mathrm{Ad}(\mathcal{B})\,\mathfrak{a}$ is dense in $\hat{\mathfrak{n}}$. For example, in the finite-dimensional case we could take $\mathfrak{a} = \mathbb{C}e$. If $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ is a direct sum of indecomposables of finite or affine type then we can take $\mathfrak{a} = \bigoplus \mathfrak{a}_i$, where $\mathfrak{a}_i$ is either the positive part of the principal Heisenberg, or the positive nilpotent, depending on whether $\mathfrak{g}_i$ is affine or finite.

4. Cohomology vanishing

4.1. **Nakano's identity and the Laplacian.** We need some tools to prove the necessary cohomology vanishing result. Throughout this section $\mathfrak{g}$ will be an arbitrary symmetrizable Kac-Moody algebra. (V, π) will be a $\hat{\mathfrak{b}}$-module such that $\pi|_{\mathfrak{g}_0}$ extends to an action of $\overline{\mathfrak{b}}$ (also denoted by π). Note that since $\mathfrak{n} = \mathfrak{g}/\overline{\mathfrak{b}}$, $\hat{\mathfrak{n}}^*$ is both a $\hat{\mathfrak{b}}$-module and a $\overline{\mathfrak{b}}$-module. $\overline{\mathfrak{n}} = \mathfrak{g}/\,\mathfrak{b}$ has the same property.

Definition 4.1. *The* semi-infinite chain complex $(C^{*,*}(V), \bar{\partial}, D)$ *is the bicomplex*

$$C^{-p,q}(V) = \left(\bigwedge^q \hat{\mathfrak{n}}^* \otimes \bigwedge^p \overline{\mathfrak{n}} \otimes V \right)^{\mathfrak{g}_0}.$$

with differentials $\bar{\partial}$ and D, where the former is the Lie algebra cohomology differential of $\hat{\mathfrak{n}}$ with coefficients in $\bigwedge^* \overline{\mathfrak{n}} \otimes V$, and the latter is the Lie algebra homology differential of $\overline{\mathfrak{n}}$ with coefficients in $\bigwedge^* \hat{\mathfrak{n}}^* \otimes V$, both restricted to $\mathfrak{g}_0$-invariants.

To make the definition of $\bar{\partial}$ and D more explicit, identify $C^{*,*}(V)$ with $\bigwedge^*(\hat{\mathfrak{n}}^* \oplus \bar{\mathfrak{n}}) \otimes V$. Then the Clifford algebra of $\mathfrak{n} \oplus \bar{\mathfrak{n}} \oplus \hat{\mathfrak{n}}^* \oplus \hat{\bar{\mathfrak{n}}}^*$ with the dual pairing acts on $C^{*,*}(V)$, where $\hat{\mathfrak{n}}^*$ and $\bar{\mathfrak{n}}$ act by exterior multiplication, and $\mathfrak{n}$ and $\hat{\bar{\mathfrak{n}}}^*$ act by interior multiplication. Pick a homogeneous basis $\{z_i\}_{i\geq 1}$ for $\mathfrak{n}$, let $\{z^i\}$ denote the dual basis, and let $z_{-i} = \overline{z_i}$. Then

$$\bar{\partial} = \sum_{k\geq 1} \epsilon(z^k) \left(\frac{1}{2} \operatorname{ad}^t_n(z_k) + \operatorname{ad}_{\overline{\mathfrak{n}}}(z_k) + \pi(z_k) \right),$$

where ϵ is exterior multiplication, while

$$D = \sum_{k\geq 1} \left(\frac{1}{2} \operatorname{ad}_{\overline{\mathfrak{n}}}(z_{-k}) + \operatorname{ad}^t_{\mathfrak{n}}(z_{-k}) + \pi(z_{-k}) \right) \iota(z^{-k}),$$

where ι is interior multiplication.

The semi-infinite cocycle is defined by $\gamma|_{\mathfrak{g}_m \times \mathfrak{g}_n} = 0$ if $m + n \neq 0$ and by

$$\gamma(x, y) = \sum_{0\leq n<k} \operatorname{tr}_{\mathfrak{g}_n}(\operatorname{ad}(x)\operatorname{ad}(y))$$

for $x \in \mathfrak{g}_k$, $y \in \mathfrak{g}_{-k}$, $k \geq 0$. Since $\mathfrak{h} = \mathfrak{g}_0$ is abelian, $(x, y) = -\gamma(x, \overline{y})$ defines a Hermitian form on $\mathfrak{n}$.

Lemma 4.2. *Let $\langle , \rangle$ be the symmetric invariant form on $\mathfrak{n}$ (real-valued on a real-form of $\mathfrak{g}$) such that $\{\cdot, \cdot\} = -\langle \cdot, \bar{\cdot} \rangle$ is contragradient and positive-definite on $\mathfrak{n}$. Then the Hermitian form $(\cdot, \cdot) = -\gamma(\cdot, \bar{\cdot})$ agrees with the form defined by*

$$(x, y) = 2\langle \rho, \alpha \rangle \{x, y\}, x \in \mathfrak{g}_\alpha.$$

Proof. Suppose $x, y \in \mathfrak{g}_\alpha$. If $\{u_i\}$ and $\{u^i\}$ are dual bases of $\mathfrak{h}$ with respect to $\langle , \rangle$ then

$$\begin{aligned} \operatorname{tr}_{\mathfrak{g}_0}(\operatorname{ad}(x)\operatorname{ad}(\overline{y})) &= \sum_i \langle u_i, [x, [\overline{y}, u^i]] \rangle \\ &= \langle x, \overline{y} \rangle \langle \alpha, \alpha \rangle. \end{aligned}$$

Next, let $\{e^i_\beta\}$ and $\{e^i_{-\beta}\}$ be dual bases of $\mathfrak{g}_\beta$ and $\mathfrak{g}_{-\beta}$ with respect to $\langle , \rangle$. Let $\rho \in \mathfrak{h}^*$ be such that $\rho(\alpha_i^\vee) = 1$ for all coroots $\alpha_i^\vee$. Then

$$\gamma(x, y) = \langle x, \overline{y} \rangle \langle \alpha, \alpha \rangle + \sum_{\beta\in\Delta^+} \sum_i \langle e^i_{-\beta}, [x, [\overline{y}, e^i_\beta]_-] \rangle,$$

where x_- is the projection of $x \in \mathfrak{g}$ to $\overline{\mathfrak{n}}$ using the triangular decomposition. Rearranging $\langle e^i_{-\beta}, [x, [\overline{y}, e^i_\beta]_-] \rangle = \langle x, [e^i_{-\beta}, [e^i_\beta, \overline{y}]_-] \rangle$ and applying Lemma 2.3.11 of [9], we get that $\gamma(x, \overline{y}) = 2\langle \rho, \alpha \rangle \langle x, \overline{y} \rangle$. □

The result of Lemma 4.2 is that $(,)$ defines a $\mathfrak{g}_0$-contragradient Kahler metric on $\mathfrak{n}$. Suppose V has a positive-definite Hermitian form contragradient with respect to π. Using the Kahler metric on $\mathfrak{n}$, we can give $C^{*,*}(V)$ a positive-definite Hermitian form by defining $(\overline{x}, \overline{y}) = \overline{(x, y)}$ for $x, y \in \mathfrak{n}$. Let $\overline{\square} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ be the $\bar{\partial}$-Laplacian, and $\square = DD^* + D^*D$ be the D-Laplacian. Then a version of Nakano's identity holds:

Proposition 4.3 (Nakano's identity [12] [13])**.** *The $\bar{\partial}$-Laplacian $\overline{\square}$ and the D-Laplacian $\square$ are related by*

$$\overline{\square} = \square + \deg + \operatorname{Curv},$$

where deg *acts on* $C^{p,q}(V)$ *as multiplication by* $p+q$, *and*

$$\mathrm{Curv} = -\sum_{i,j\geq 1} \epsilon(z^i)\iota(z_j)\left([\pi(z_i),\pi(z_{-j})] - \pi([z_i,z_{-j}])\right),$$

on $C^{0,q}(V)$ *for* $\{z_i\}$ *a homogeneous basis of* $\mathfrak{n}$ *orthonormal in* $(,)$.

4.2. **Laplacian calculation for symmetrizable Kac-Moody algebras.** Given an operator T on $\hat{\mathfrak{n}}^*$, let $d_R(T)$ and $d_L(T)$ denote the operators on $\bigwedge^* \hat{\mathfrak{n}}^* \otimes S^*\hat{\mathfrak{n}}^*$ defined by

$$\alpha_1 \wedge \ldots \wedge \alpha_k \otimes \beta \mapsto \sum_{i=1}^{k} (-1)^i \alpha_1 \wedge \ldots \check{\alpha}_i \ldots \wedge \alpha_k \otimes T(\alpha_i)\circ\beta$$

and

$$\alpha \otimes \beta_1 \circ \ldots \circ \beta_l \mapsto \sum_{i=1}^{l} T(\beta_i) \wedge \alpha \otimes \beta_1 \circ \ldots \circ \check{\beta}_i \circ \ldots \circ \beta_l$$

respectively. Define an operator J on $\hat{\mathfrak{n}}^*$ by $f \mapsto f/2\langle\rho,\alpha\rangle$ if $f \in \mathfrak{g}^*_\alpha$. As in the last section, let $\langle,\rangle$ be a real-valued symmetric invariant bilinear form such that $\{,\} = -\langle\cdot,\bar{\cdot}\rangle$ is contragradient and positive-definite on $\mathfrak{n}$.

Proposition 4.4. *Extend the contragradient Hermitian form* $\{,\}$ *on* $\mathfrak{n}$ *to* $V = S^*\hat{\mathfrak{n}}^*$. *On* $C^{0,q}(V)$,

$$\mathrm{Curv}_V = \sum_{s\geq 0} d_L(\mathrm{ad}^t(y'_s))d_R(\mathrm{ad}^t(y_s)J) - \deg,$$

where $\{y_s\}$ *is a homogeneous basis for* $\mathfrak{b}$ *and* $\{y'_s\}$ *is a basis for* $\overline{\mathfrak{b}}$ *dual with respect to* $\langle,\rangle$.

Proof. Let $V' = S^*\overline{\mathfrak{n}}$, and let π denote the actions of $\mathfrak{b}$ and $\overline{\mathfrak{b}}$ on V'. From Proposition 4.3 we see that $\mathrm{Curv}_{V'}$ is a second-order differential operator, and thus is determined by its action on $\hat{\mathfrak{n}}^* \otimes \overline{\mathfrak{n}}$. We claim that if $f \in \hat{\mathfrak{n}}^*$ and $w \in \overline{\mathfrak{n}}$ then

$$\mathrm{Curv}_{V'}(f\otimes w) = \sum_{s\geq 0} \mathrm{ad}^t_{\mathfrak{n}}(w)y^s \otimes \mathrm{ad}_{\overline{\mathfrak{n}}}(y_s)\phi^{-1}(f),$$

where $\phi : \overline{\mathfrak{n}} \to \hat{\mathfrak{n}}^*$ is the isomorphism induced by the Kahler metric, and $\{y_s\}$ is any homogeneous basis of $\mathfrak{b}$. To prove this claim, let $\{z_i\}$ be orthonormal with respect to the Kahler metric, and think about $f = z^k$, $w = z_{-l}$. Observe that

$$\pi(z)w = \sum_{i<0} z^i([z,w])z_i.$$

Using this expression, we get that if $z_{-j} \in \mathfrak{g}_{-m}$ then

$$([\pi(z_i),\pi(z_{-j})] - \pi([z_i,z_{-j}]))w = \sum_{-m\leq n<0}\ \sum_{z_{-k}\in\mathfrak{g}_n} z^{-k}([z_{-j},[z_i,w]])z_{-k}.$$

We can then remove the reference to m and write

$$([\pi(z_i),\pi(z_{-j})] - \pi([z_i,z_{-j}]))w = \sum_{k>0}\sum_{s\geq 0} z^{-k}([z_{-j},y_s])y^s([z_i,w])z_{-k}.$$

Now $\iota(z^{-j})$ is zero on $(0,q)$-forms so

$$\mathrm{Curv}_{V'}(z^k \otimes z_{-l}) = -\sum_{i>0} z^i \left([\pi(z_i), \pi(z_{-k})] - \pi([z_i, z_{-k}])\right) z_{-l}$$
$$= -\sum_{i,j>0}\sum_{s\geq 0} z^i z^{-j}([z_{-k}, y_s]) y^s([z_i, z_{-l}]) z_{-j}$$

.

By summing over $z_i \in \mathfrak{g}_n$ for fixed n, it is possible to move the z_{-l} action from z_i to z^i. The last expression becomes

$$-\sum_{s\geq 0}\sum_{j>0} (\mathrm{ad}^t(z_{-l}) y^s) z^{-j}([z_{-k}, y_s]) z_{-j} = \sum_{s\geq 0} (\mathrm{ad}^t(z_{-l}) y^s) \pi(y_s)(z_{-k}).$$

The proof of the claim is finished by noting that $z_{-k} = \phi^{-1}(z^k)$.

Next, the contragradient metric $\{,\}$ gives an isomorphism $\psi : \overline{\mathfrak{n}} \to \hat{\mathfrak{n}}^*$ of $\mathfrak{b}$ and $\overline{\mathfrak{b}}$-modules. $J = \psi\phi^{-1}$, while $\mathrm{ad}^t(w) y^s = \mathrm{ad}^t(y'_s)\psi(w)$ where $\{y'_s\}$ is the dual basis to $\{y_s\}$. Identifying V with V' via ψ gives

$$\mathrm{Curv}_V(f \otimes g) = \sum_{s\geq 0} \mathrm{ad}^t(y'_s) g \otimes \mathrm{ad}^t(y_s) J f.$$

Given $S, T \in \mathrm{End}(\hat{\mathfrak{n}}^*)$, define a second-order operator $\mathrm{Switch}(S,T)$ on $\bigwedge^* \hat{\mathfrak{n}}^* \otimes S^*\hat{\mathfrak{n}}^*$ by $f \otimes g \mapsto Tg \otimes Sf$. Then $\mathrm{Switch}(S,T) = d_L(T) d_R(S) - (TS)^\wedge$, where $(TS)^\wedge$ is the extension of TS to $\bigwedge^* \hat{\mathfrak{n}}^*$ as a derivation. We have shown that

$$\mathrm{Curv}_V = \sum_{s\geq 0} \mathrm{Switch}(\mathrm{ad}^t(y_s)J, \mathrm{ad}^t(y'_s)) = \sum_{s\geq 0} d_L(\mathrm{ad}^t(y'_s)) d_R(\mathrm{ad}^t(y_s)J) - (TJ)^\wedge,$$

where $T = \sum_{s\geq 0} \mathrm{ad}^t(y'_s)\,\mathrm{ad}^t(y_s)$. It is not hard to see that that $(T\psi(y))(x) = -\gamma(x,y)$ for $x \in \mathfrak{n}$, $y \in \overline{\mathfrak{n}}$, so $T = J^{-1}$ by Lemma 4.2. □

Note that $d_R(TJ) = d_L(T^*)$, where T^* is the adjoint of $T \in \mathrm{End}(\hat{\mathfrak{n}}^*)$ in the contragradient metric. The map J appears because the Kahler metric is used on $\bigwedge^* \hat{\mathfrak{n}}^*$ while the contragradient metric is used on $S^*\hat{\mathfrak{n}}^*$. Since the isomorphism ψ appearing in the proof is an isometry, $\mathrm{ad}^t(x)^* = -\mathrm{ad}(\overline{x})^*$ in the contragradient metric.

4.3. **Cohomology vanishing for affine Kac-Moody algebras.** If $\mathfrak{g}$ is affine then $\mathfrak{g}$ can be realized as the algebra $(L[z^{\pm 1}] \oplus \mathbb{C}c \oplus \mathbb{C}d)^{\tilde{\sigma}}$, where L is a simple Lie algebra and $\tilde{\sigma}$ is an automorphism of $\mathfrak{g}$ defined by

$$\tilde{\sigma}(c) = c, \tilde{\sigma}(d) = d, \tilde{\sigma}(xz^n) = q^{-n}\sigma(x) z^n, \quad x \in L$$

for σ a diagram automorphism of L of finite order k and q a fixed kth root of unity. The bracket is defined by

$$[xz^m + \gamma_1 c + \beta_1 d, yz^n + \gamma_2 c + \beta_2 d] =$$
$$[x,y] z^{m+n} + \beta_1 n y z^n - \beta_2 m x z^m + \delta_{m,-n} m \langle x, y\rangle c,$$

for $x, y \in L$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \mathbb{C}$, where $\langle , \rangle$ is the basic symmetric invariant bilinear form on L. The diagram automorphism acts diagonalizably on L, so that

$$\mathfrak{g} = \bigoplus_{i=0}^{k-1} L_i z^i \otimes \mathbb{C}[z^{\pm k}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where L_i is the q^i-eigenspace of σ. The eigenspace L_0 is a simple Lie algebra, and there is a Cartan $\overset{\circ}{\mathfrak{h}}\subset L$ compatible with σ such that $\overset{\circ}{\mathfrak{h}}_0=\overset{\circ}{\mathfrak{h}}\cap L_0$ is a Cartan in L_0. The algebra $\mathfrak{h}=\overset{\circ}{\mathfrak{h}}_0\oplus\mathbb{C}c\oplus\mathbb{C}d$ is a Cartan for $\mathfrak{g}$. The eigenspaces L_i are irreducible L_0-modules. Choose a set of simple roots $\alpha_1,\ldots,\alpha_l$ for L_0, and let ψ be either the highest weight of L_1 (if $k>1$), or the highest root of L_0 (if $k=0$). Then $\alpha_0=d^*-\psi,\alpha_1,\ldots,\alpha_l$ is a set of simple roots for $\mathfrak{g}$, and $\alpha_0^\vee=c-\psi^\vee,\alpha_1^\vee,\ldots,\alpha_l^\vee$ is a set of simple coroots. There is a unique real form $\mathfrak{h}_{\mathbb{R}}=\mathrm{span}_{\mathbb{R}}\{\alpha_i^\vee\}\oplus\mathbb{R}d$, and the anti-linear Cartan involution sends $xz^m+\alpha c+\beta d\mapsto\overline{x}z^{-m}-\overline{\alpha}c-\overline{\beta}d$, where $x\mapsto\overline{x}$ is the compact involution of x in L.

The following lemma finishes the proof of Theorem 2.3.

Lemma 4.5. *Let μ be a dominant weight of an integrable highest weight $\mathfrak{g}$-module $\mathcal{L}(\lambda)$, where λ is a real-valued dominant weight and $\mathfrak{g}$ is affine. If μ is dominant then $H^q_{cts}(\hat{\mathfrak{b}},\mathfrak{h};\mathcal{L}(\lambda)\otimes S^*\hat{\mathfrak{n}}^*\otimes\mathbb{C}_{-\mu})=0$ for all $q>0$.*

Proof. The result is trivial if $\lambda=\mu=0$, so assume that λ and μ have positive level.

$S^*\hat{\mathfrak{n}}^*$ has a contragradient positive-definite Hermitian form from $\{,\}$. Since μ is a real-valued weight, $\mathbb{C}_{-\mu}$ has a contragradient positive-definite Hermitian form. Finally, $\mathcal{L}(\lambda)$ has a contragradient positive-definite Hermitian form because λ is a real-valued dominant weight. Putting everything together, $V=\mathcal{L}(\lambda)\otimes S^*\hat{\mathfrak{n}}^*\otimes\mathbb{C}_{-\mu}$ has a contragradient positive-definite Hermitian form.

The cohomology $H^*(\hat{\mathfrak{b}},\mathfrak{h};V)$ can be identified with the kernel of the Laplacian $\overline{\Box}$ on the zero column $C^{0,*}(V)$ of the semi-infinite chain complex. By Nakano's identity, $\overline{\Box}=\Box+\deg+\mathrm{Curv}$. $\Box$ is positive semi-definite by definition. The curvature term splits into a sum $\mathrm{Curv}=\mathrm{Curv}_{\mathcal{L}(\lambda)}+\mathrm{Curv}_{S^*}+\mathrm{Curv}_{\mathbb{C}_{-\mu}}$. Since $\mathcal{L}(\lambda)$ is representation of $\mathfrak{g}$, $\mathrm{Curv}_{\mathcal{L}(\lambda)}$ is zero. Next consider $\mathrm{Curv}_{S^*}+\deg$. We use the realisation of $\mathfrak{g}$ via the loop algebra. The contragradient metric $\{,\}$ induces a positive-definite metric on the loop algebra $\mathfrak{g}'/\mathbb{C}c$, so we can pick a homogeneous basis for $\mathfrak{b}$ consisting of an orthonormal basis $\{y_s\}$ for $\mathfrak{g}'/\mathbb{C}c$, as well as c and d. The dual basis to $\{c,d,y_0,\ldots,y_s,\ldots\}$ is $\{d,c,-\overline{y_0},\ldots,-\overline{y_s},\ldots\}$. Since c is in the centre, we have $\mathrm{ad}^t(c)=0$, so the terms $d_L(\mathrm{ad}^t(c))$ and $d_R(\mathrm{ad}^t(c)J)$ in Curv_{S^*} are zero. Consequently

$$\mathrm{Curv}_{S^*}+\deg=\sum_{s\geq 0}d_L(\mathrm{ad}^t(-\overline{y_s}))d_R(\mathrm{ad}^t(y_s)J)=\sum_{s\geq 0}d_R(\mathrm{ad}^t(y_s)J)^*d_R(\mathrm{ad}^t(y_s)J)$$

is semi-positive. Finally we get that

$$\mathrm{Curv}_{\mathbb{C}_{-\mu}}=-\sum_{\alpha\in\Delta^+}\sum_{i,j}\epsilon(z_\alpha^i)\iota(z_{\alpha,j})\mu([z_{\alpha,i},\overline{z_{\alpha,j}}]),$$

where $z_{\alpha,i}$ runs through a basis for $\mathfrak{g}_\alpha$ orthonormal in the Kahler metric. Now

$$-\mu([z_{\alpha,i},\overline{z_{\alpha,j}}])=\{z_{\alpha,i},z_{\alpha_j}\}\langle\mu,\alpha\rangle,$$

The result is that $\mathrm{Curv}_{\mathbb{C}_{-\mu}}$ is a derivation which multiplies occurences of z_α^j by the non-negative number $2\langle\rho,\alpha\rangle\langle\mu,\alpha\rangle$, and thus is semi-positive.

Now we look more closely at the kernel of $\overline{\Box}$. The operator $\mathrm{Curv}_{\mathbb{C}_{-\mu}}$ is strictly positive on $z^{\beta_1,i_1}\wedge\cdots\wedge z^{\beta_k,i_k}\otimes v$ unless all $\beta_i\in\mathbb{Z}[Y]$, where $Y=\{\alpha_i:\mu(\alpha_i^\vee)=0\}$. Let A_Y be the submatrix of the defining matrix A of $\mathfrak{g}$ with rows and columns indexed by $\{i:\alpha_i\in Y\}$. Recall that the Kac-Moody algebra $\mathfrak{g}(A_Y)$ defined by A_Y embeds in $\mathfrak{g}$. The standard nilpotent of $\mathfrak{g}(A_Y)$ is $\mathfrak{n}_Y=\bigoplus_{\alpha\in\Delta^+\cap\mathbb{Z}[Y]}\mathfrak{g}_\alpha\subset\mathfrak{g}$. Let

$\mathfrak{u}_Y = \bigoplus_{\alpha\in\Delta^+\setminus\mathbb{Z}[Y]} \mathfrak{g}_\alpha$. Since μ has positive level, Y is a strict subset of simple roots, and since $\mathfrak{g}$ is affine, $\mathfrak{g}(A_Y)$ is finite-dimensional. Harmonic cocycles must belong to the kernel of $\mathrm{Curv}_{\mathbb{C}_{-\mu}}$, so any harmonic cocycle ω must be in the $\mathfrak{h}$-invariant part of

$$\bigwedge^* \hat{\mathfrak{n}}_Y^* \otimes S^*\hat{\mathfrak{n}}^* \otimes \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu}.$$

As a vector space, this set can be identified with $\Omega^*_{pol}\hat{\mathfrak{n}}_Y \otimes \mathbb{C}[\hat{\mathfrak{u}}_Y] \otimes \mathcal{L}(\lambda)$, where Ω^*_{pol} is the ring of polynomial differential forms and $\hat{\mathfrak{u}}_Y$ is pro-Lie algebra associated to $\mathfrak{u}_Y$. For ω to be in the kernel of $\deg + \mathrm{Curv}_{S^*}$, ω must lie in the kernel of the operators $d_R(\mathrm{ad}^t(y_s)J)$, $s \geq 0$. Since $d_R(\mathrm{ad}^t(c)J) = 0$, we get that $d_R(\mathrm{ad}^t(x)J)\omega = 0$ for every $x \in \mathfrak{b}_Y \subset \mathfrak{g}' \cap \mathfrak{b}$, where $\mathfrak{b}_Y$ is the standard Borel of $\mathfrak{g}(A_Y)$. Let J_Δ^{-1} denote the diagonal extension of J^{-1} to $\bigwedge^* \hat{\mathfrak{n}}^*$. Then $J_\Delta^{-1}\omega$ vanishes under contraction by the vector fields $\mathfrak{n}_Y \to T\mathfrak{n}_Y : x \mapsto (x, [x, y])$, $y \in \mathfrak{b}$. At a point $x \in \mathfrak{n}_Y$, these vector fields span the tangents to $\mathcal{B}_Y$-orbits. $\mathfrak{n}_Y$ is the positive nilpotent of a finite-dimensional Kac-Moody, so $\mathfrak{n}_Y$ has a dense $\mathcal{B}_Y$-orbit and thus ω must be zero. □

The same proof applies with slight modification if $\mathfrak{g}$ is a direct sum of indecomposables of finite or affine type.

5. A Brylinski filtration for indefinite Kac-Moody algebras

In this section $\mathfrak{g}$ will be an arbitrary symmetrizable Kac-Moody algebra. Recall from the proof of Lemma 4.5 that if A is the defining matrix of $\mathfrak{g}$ and Z is a subset of the simple roots then A_Z refers to the submatrix of A with rows and columns indexed by $\{i : \alpha_i \in Z\}$.

Proposition 5.1. *Let $\mathfrak{g}$ be the symmetrizable Kac-Moody algebra defined by the generalized Cartan matrix A, and suppose μ is a dominant weight of an integrable highest weight representation $\mathcal{L}(\lambda)$, where λ is real-valued. Write $\lambda - \mu = \sum k_i\alpha_i$, $k_i \geq 0$, and let $Z = \{\alpha_i : k_i > 0\}$. If A_Z is a direct sum of indecomposables of finite and affine type then $H^q_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; S^*\hat{\mathfrak{n}}^* \otimes \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu}) = 0$ for $q > 0$.*

Recall that the weight space $\mathcal{L}(\lambda)_\mu$ of an integrable highest weight representation is filtered via polynomial degree on the isomorphic space $(\mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu} \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}}$. Let ${}^{deg}P^\lambda_\mu(q)$ be the corresponding Poincare polynomial. Excepting Proposition 3.4, the results of Sections 2 and 3 imply the following corollary:

Corollary 5.2. *If the hypotheses of Proposition 5.1 hold then $m^\lambda_\mu(q) = {}^{deg}P^\lambda_\mu(q)$*

The conclusions of Theorem 2.3 hold similarly, with the Brylinski filtration replaced by the degree filtration.

Proof of Proposition 5.1. . We continue to use the notation of Section 4. For instance, $V = S^*\hat{\mathfrak{n}}^* \otimes \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu}$. Recall that $\overline{\Box} = \Box + \deg + \mathrm{Curv}_V$, and $\mathrm{Curv}_V = \mathrm{Curv}_{\mathcal{L}(\lambda)} + \mathrm{Curv}_{\mathbb{C}_{-\mu}} + \mathrm{Curv}_{S^*}$. The operators $\Box$, $\mathrm{Curv}_{\mathcal{L}(\lambda)}$, and $\mathrm{Curv}_{\mathbb{C}_{-\mu}}$ are positive semi-definite as before, while

$$\deg + \mathrm{Curv}_{S^*} = \sum_{k\geq 1} d_R(\mathrm{ad}^t(x_k)J)^* d_R(\mathrm{ad}^t(x_k)J) + \sum_i d_L(\mathrm{ad}^t(u^i)) d_R(\mathrm{ad}^t(u_i)J),$$

where $\{x_k\}$ is a basis for $\mathfrak{n}$ orthonormal in the contragradient metric, and $\{u_i\}$ and $\{u^i\}$ are dual bases for $\mathfrak{h}$. The first summand in this equation is positive semidefinite, but the second is not if there are roots with $\langle\alpha,\alpha\rangle < 0$. Indeed, writing

$$\sum_i d_L(\mathrm{ad}^t(u^i))d_R(\mathrm{ad}^t(u_i)J) = \sum_i \mathrm{Switch}(\mathrm{ad}^t(u^i)J, \mathrm{ad}^t(u_i)) + \sum_i (\mathrm{ad}^t(u^i)\,\mathrm{ad}^t(u_i)J)^\wedge, \tag{3}$$

we see that the first summand in Equation (3) is the second order operator defined by

$$x \otimes y \mapsto \frac{\langle\alpha,\beta\rangle}{2\langle\rho,\alpha\rangle} y \otimes x, x \in \mathfrak{g}^*_\alpha, y \in \mathfrak{g}^*_\beta,$$

while the second summand in Equation (3) is the derivation of $\bigwedge^* \hat{\mathfrak{n}}^*$ induced by the map

$$x \mapsto \frac{\langle\alpha,\alpha\rangle}{2\langle\rho,\alpha\rangle} x, x \in \mathfrak{g}^*_\alpha$$

on $\hat{\mathfrak{n}}^*$.

Let $\mathfrak{g}(A_Z)$ be the corresponding Kac-Moody subalgebra of $\mathfrak{g}$, and let $\mathfrak{n}_Z$ be the standard nilpotent. $\mathfrak{g}(A_Z)$ has a Cartan subalgebra $\mathfrak{h}_Z \subset \mathfrak{h}$, and the real-valued non-degenerate symmetric invariant form on $\mathfrak{g}$ restricts to such a form on $\mathfrak{g}(A_Z)$. Any $\mathfrak{h}$-invariant element of $\bigwedge^* \hat{\mathfrak{n}}^* \otimes V$ must belong to $\bigwedge^* \hat{\mathfrak{n}}^*_Z \otimes S^*\hat{\mathfrak{n}}^*_Z \otimes \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu}$. We claim that the operator $\sum_i d_L(\mathrm{ad}^t(u^i))d_R(\mathrm{ad}^t(u_i)J)$ on $\bigwedge^* \hat{\mathfrak{n}}^* \otimes S^*\hat{\mathfrak{n}}^*$ restricts on $\bigwedge^* \hat{\mathfrak{n}}^*_Z \otimes S^*\hat{\mathfrak{n}}^*_Z$ to the operator $\sum_i d_L(\mathrm{ad}^t(v^i))d_R(\mathrm{ad}^t(v_i)J)$, where $\{v_i\}$ and $\{v^i\}$ are dual bases of $\mathfrak{h}_Z$. To verify this claim, note that a choice of symmetric invariant form corresponds to a choice of a diagonal matrix D with positive diagonal entries, such that DA is a symmetric matrix. If $x \in \mathfrak{h}^*$ the invariant form satisfies $\langle x, \alpha_i\rangle = D_{ii}x(\alpha_i^\vee)$. The operator in Equation (3) thus depends only on A and D; the claim follows from the observation that the action of the operator on $\bigwedge^* \hat{\mathfrak{n}}^*_Z \otimes S^*\hat{\mathfrak{n}}^*_Z$ depends only on A_Y and D_Y.

Now suppose A_Y is a direct sum of indecomposables of finite and affine type. The operator $\sum_i d_L(\mathrm{ad}^t(v^i))d_R(\mathrm{ad}^t(v_i)J)$ decomposes into a summand for each component, each of which is positive semi-definite as in the proof of Lemma 4.5. We finish as in the proof of Lemma 4.5, but taking $Y = \{\alpha_i \in Z : \mu(\alpha_i) = 0\}$. □

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