# Acyclicity of complexes of flat modules

Mitsuyasu Hashimoto

Graduate School of Mathematics, Nagoya University
Chikusa-ku, Nagoya 464–8602 JAPAN
hasimoto@math.nagoya-u.ac.jp

Dedicated to Professor Masayoshi Nagata on his eightieth birthday

**Abstract**

Let $R$ be a noetherian commutative ring, and

$$\mathbb{F} : \cdots \to F_2 \to F_1 \to F_0 \to 0$$

a complex of flat $R$-modules. We prove that if $\kappa(\mathfrak{p}) \otimes_R \mathbb{F}$ is acyclic for every $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{F}$ is acyclic, and $H_0(\mathbb{F})$ is $R$-flat. It follows that if $\mathbb{F}$ is a (possibly unbounded) complex of flat $R$-modules and $\kappa(\mathfrak{p}) \otimes_R \mathbb{F}$ is exact for every $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{G} \otimes^\bullet_R \mathbb{F}$ is exact for every $R$-complex $\mathbb{G}$. If, moreover, $\mathbb{F}$ is a complex of projective $R$-modules, then it is null-homotopic (follows from Neeman's theorem).

## 1. Introduction

Throughout this paper, $R$ denotes a noetherian commutative ring. The symbol $\otimes$ without any subscript means $\otimes_R$. For $\mathfrak{p} \in \operatorname{Spec} R$, let $-(\mathfrak{p})$ denote the functor $\kappa(\mathfrak{p}) \otimes -$, where $\kappa(\mathfrak{p})$ is the field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. An $R$-complex of the form

$$\mathbb{F} : \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

is said to be *acyclic* if $H_i(\mathbb{F}) = 0$ for every $i > 0$.

In this paper, we prove:

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**Theorem 1.** *Let*

$$\mathbb{F} : \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

*be a complex of R-flat modules. If* $\mathbb{F}(\mathfrak{p})$ *is acyclic for every* $\mathfrak{p} \in \operatorname{Spec} R$*, then* $\mathbb{F}$ *is acyclic, and* $H_0(\mathbb{F})$ *is R-flat. In particular,* $M \otimes \mathbb{F}$ *is acyclic for every R-module M.*

It has been known that, for an $R$-linear map of $R$-flat modules $\varphi : F_1 \to F_0$, if $\varphi(\mathfrak{p})$ is injective for every $\mathfrak{p} \in \operatorname{Spec} R$, then $\varphi$ is injective and $\operatorname{Coker} \varphi$ is $R$-flat (see [1, Lemma 4.2], [2, Lemma I.2.1.4] and Corollary 6). This is the special case of the theorem where $F_i = 0$ for every $i \geq 2$. The new proof of the theorem is simpler than the proofs of the special case in [1] and [2].

By the theorem, it follows immediately that if $\mathbb{F}$ is an (unbounded) complex of $R$-flat modules and $\mathbb{F}(\mathfrak{p})$ is exact for every $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{F}$ is *K-flat* (to be defined below) and exact. Combining this and Neeman's result, we can also prove that an (unbounded) complex $\mathbb{P}$ of $R$-projective modules is null-homotopic if $\mathbb{P}(\mathfrak{p})$ is exact for every $\mathfrak{p} \in \operatorname{Spec} R$.

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## 2. Main results

We give a proof of Theorem 1.

*Proof of Theorem 1.* It suffices to prove that $R/I \otimes \mathbb{F}$ is acyclic for every ideal $I$ of $R$. Indeed, if so, then considering the case that $I = 0$, we have that $\mathbb{F}$ is acyclic so that it is a flat resolution of $H_0(\mathbb{F})$. Since $R/I \otimes \mathbb{F}$ is acyclic for every ideal $I$, we have that $\operatorname{Tor}_i^R(R/I, H_0(\mathbb{F})) = 0$ for every $i > 0$. Thus $H_0(\mathbb{F})$ is $R$-flat. So $\operatorname{Tor}_i^R(M, H_0(\mathbb{F})) = 0$ for every $i > 0$, and the last assertion of the theorem follows.

Assume the contrary, and let $I$ be maximal among the ideals $J$ such that $R/J \otimes \mathbb{F}$ is not acyclic. Then replacing $R$ by $R/I$ and $\mathbb{F}$ by $R/I \otimes \mathbb{F}$, we may assume that $R/I \otimes \mathbb{F}$ is acyclic for every nonzero ideal $I$ of $R$, but $\mathbb{F}$ itself is not acyclic.

Assume that $R$ is not a domain. There exists a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = R$$

such that for each $i$, $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} R$. Since each $\mathfrak{p}_i$ is a nonzero ideal, $R/\mathfrak{p}_i \otimes \mathbb{F}$ is acyclic. So $M_i \otimes \mathbb{F}$ is acyclic for every $i$. In particular, $\mathbb{F} \cong M_r \otimes \mathbb{F}$ is acyclic, and this is a contradiction. So $R$ must be a domain.

For each $x \in R \setminus 0$, there is an exact sequence

$$0 \to \mathbb{F} \xrightarrow{x} \mathbb{F} \to R/Rx \otimes \mathbb{F} \to 0.$$

Since $R/Rx \otimes \mathbb{F}$ is acyclic, we have that $x : H_i(\mathbb{F}) \to H_i(\mathbb{F})$ is an isomorphism for every $i > 0$. In particular, $H_i(\mathbb{F})$ is a $K$-vector space, where $K = \kappa(0)$ is the field of fractions of $R$. So

$$H_i(\mathbb{F}) \cong K \otimes H_i(\mathbb{F}) \cong H_i(K \otimes \mathbb{F}) = H_i(\mathbb{F}(0)) = 0 \qquad (i > 0),$$

and this is a contradiction. □

Let $A$ be a ring. A complex $\mathbb{F}$ of left $A$-modules is said to be *$K$-flat* if the tensor product $\mathbb{G} \otimes^\bullet_A \mathbb{F}$ is exact for every exact complex $\mathbb{G}$ of right $A$-modules, see [4, Definition 5.1].

For a chain complex

$$\mathbb{H} : \cdots \to H_{i+1} \xrightarrow{d_{i+1}} H_i \xrightarrow{d_i} H_{i-1} \to \cdots$$

of left or right $A$-modules, we denote the complex

$$\cdots \to H_{i+1} \to \operatorname{Ker} d_i \to 0$$

by $\tau_{\geq i}\mathbb{H}$ or $\tau^{\leq -i}\mathbb{H}$. Since $\mathbb{G} \cong \varinjlim \tau^{\leq n}\mathbb{G}$, $\mathbb{F}$ is $K$-flat if and only if $\mathbb{G} \otimes^\bullet_A \mathbb{F}$ is exact for every exact complex $\mathbb{G}$ of right $A$-modules bounded above (i.e., $\mathbb{G}_{-i} = \mathbb{G}^i = 0$ for $i \gg 0$). A complex $\mathbb{F}$ of flat left $A$-modules is $K$-flat if it is bounded above, as can be seen easily from the spectral sequence argument. A null-homotopic complex $\mathbb{F}$ is $K$-flat, since $\mathbb{G} \otimes^\bullet_A \mathbb{F}$ is null-homotopic for every complex $\mathbb{G}$.

**Lemma 2.** *Let $A$ be a ring, and*

$$\mathbb{F} : \cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \to \cdots$$

*a complex of flat left $A$-modules. Then the following are equivalent.*

(1) *$M \otimes_A \mathbb{F}$ is exact for every right $A$-module $M$.*

(2) $\mathbb{F}$ *is exact, and* $\operatorname{Im} d_i$ *is flat for every* $i$.

(3) *For every complex* $\mathbb{G}$ *of right* $A$*-modules,* $\mathbb{G} \otimes_A^\bullet \mathbb{F}$ *is exact.*

(4) $\mathbb{F}$ *is* $K$*-flat and exact.*

*Proof.* (1) $\Rightarrow$ (2). Obviously, $\mathbb{F} \cong A \otimes_A \mathbb{F}$ is exact. Thus

$$\mathbb{F}' : \cdots \to F_{i+1} \to F_i \to 0$$

is a flat resolution of $\operatorname{Im} d_i$, where $F_{n+i}$ has the homological degree $n$ in $\mathbb{F}'$. For every $i \in \mathbb{Z}$,

$$\operatorname{Tor}_1^A(M, \operatorname{Im} d_i) \cong H_1(M \otimes_A \mathbb{F}') \cong H_{i+1}(M \otimes_A \mathbb{F}) = 0$$

for every right $A$-module $M$. Thus $\operatorname{Im} d_i$ is $A$-flat.

(2) $\Rightarrow$ (1). For every $i \in \mathbb{Z}$,

$$H_{i+1}(M \otimes_A \mathbb{F}) \cong H_1(M \otimes_A \mathbb{F}') \cong \operatorname{Tor}_1^A(M, \operatorname{Im} d_i) = 0,$$

where $\mathbb{F}'$ is as above.

(1), (2) $\Rightarrow$ (3). Since $\mathbb{G} \cong \varinjlim \tau^{\leq n}\mathbb{G}$, we may assume that $\mathbb{G}$ is bounded above. Since $\mathbb{F} \cong \varinjlim \tau^{\leq n}\mathbb{F}$ and $\tau^{\leq n}\mathbb{F}$ satisfies (2) (and hence (1)), we may assume that $\mathbb{F}$ is also bounded above. Then by an easy spectral sequence argument, $\mathbb{G} \otimes_A^\bullet \mathbb{F}$ is exact.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1). Let $\mathbb{P}$ be a projective resolution of $M$. Since $\mathbb{P}$ is a bounded above complex of flat left $A^{\mathrm{op}}$-modules and $\mathbb{F}$ is an exact complex of right $A^{\mathrm{op}}$-modules, $\mathbb{P} \otimes_A^\bullet \mathbb{F}$ is exact. Let $\mathbb{Q}$ be the mapping cone of $\mathbb{P} \to M$. Then $\mathbb{Q} \otimes_A^\bullet \mathbb{F}$ is also exact, since $\mathbb{Q}$ is exact and $\mathbb{F}$ is $K$-flat. By the exact sequence of homology groups

$$H_i(\mathbb{P} \otimes_A^\bullet \mathbb{F}) \to H_i(M \otimes_A \mathbb{F}) \to H_i(\mathbb{Q} \otimes_A^\bullet \mathbb{F}),$$

we have that $M \otimes_A \mathbb{F}$ is also exact. $\square$

In [4, Proposition 5.7], (4) $\Rightarrow$ (3) above is proved essentially.

**Corollary 3.** *Let*

$$\mathbb{F} : \cdots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \to \cdots$$

*be a (possibly unbounded) complex of flat* $R$*-modules. If* $\mathbb{F}(\mathfrak{p})$ *is exact for every* $\mathfrak{p} \in \operatorname{Spec} R$*, then* $\mathbb{F}$ *is* $K$*-flat and exact.*

*Proof.* By Lemma 2, it suffices to show that for every $n \in \mathbb{Z}$ and every $R$-module $M$, $H_n(M \otimes \mathbb{F}) = 0$. But this is trivial by Theorem 1 applied to the complex

$$\cdots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \to 0.$$

□

The following was proved by A. Neeman [3, Corollary 6.10].

**Theorem 4.** *Let $A$ be a ring, and $\mathbb{P}$ a complex of projective left $A$-modules. If $\mathbb{P}$ is $K$-flat and exact, then $\mathbb{P}$ is null-homotopic.*

By Corollary 3 and Theorem 4, we have

**Corollary 5.** *Let $\mathbb{P}$ be a complex of $R$-projective modules. If $\mathbb{P}(\mathfrak{p})$ is exact for every $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{P}$ is null-homotopic.*

The following also follows.

**Corollary 6** ([1, Lemma 4.2], [2, Lemma I.2.1.4])**.** *Let $\varphi : F_1 \to F_0$ be an $R$-linear map between $R$-flat modules. Then the following are equivalent.*

**1** *$\varphi$ is injective and* $\operatorname{Coker} \varphi$ *is $R$-flat.*

**2** *$\varphi$ is pure.*

**3** *$\varphi(\mathfrak{p})$ is injective for every $\mathfrak{p} \in \operatorname{Spec} R$.*

*Proof.* **1⇒2⇒3** is obvious. **3⇒1** is a special case of Theorem 1. □

**Corollary 7** ([2, Corollary I.2.1.6])**.** *Let $F$ be a flat $R$-module. If $F(\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{Spec} R$, then $F = 0$.*

*Proof.* Consider the zero map $F \to 0$, and apply Corollary 6. We have that this map is injective, and hence $F = 0$. □

If $M$ is a finitely generated $R$-module and $M(\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ of $R$, then $M = 0$. This is a consequence of Nakayama's lemma.

**Corollary 8.** *Let $\varphi : F_1 \to F_0$ be an $R$-linear map between $R$-flat modules. If $\varphi(\mathfrak{p})$ is an isomorphism for every $\mathfrak{p} \in \operatorname{Spec} R$, then $\varphi$ is an isomorphism.*

*Proof.* By Corollary 6, $\varphi$ is injective and $C := \operatorname{Coker}\varphi$ is $R$-flat. Since $C(\mathfrak{p}) \cong \operatorname{Coker}(\varphi(\mathfrak{p})) = 0$ for every $\mathfrak{p} \in \operatorname{Spec} R$, we have that $C = 0$ by Corollary 7. □

**Corollary 9.** *Let $M$ be an $R$-module. If $\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), M) = 0$ for every $i > 0$ and every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, then $M$ is $R$-flat. If $\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), M) = 0$ for every $i \geq 0$ and every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, then $M = 0$.*

*Proof.* For the first assertion, Let $\mathbb{F}$ be a projective resolution of $M$, and apply Theorem 1. The second assertion follows from the first assertion and Corollary 7. □

**Corollary 10.** *Let $M$ be an $R$-module. If $\operatorname{Ext}_R^i(M, \kappa(\mathfrak{p})) = 0$ for every $i > 0$ (resp. $i \geq 0$) and every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, then $M$ is $R$-flat (resp. $M = 0$).*

*Proof.* This is trivial by Corollary 9 and the fact

$$\operatorname{Ext}_R^i(M, \kappa(\mathfrak{p})) \cong \operatorname{Hom}_{\kappa(\mathfrak{p})}(\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), M), \kappa(\mathfrak{p})).$$

□

## 3. Some examples

**Example 11.** There is an acyclic projective complex

$$\mathbb{P} : \cdots \to P_1 \to P_0 \to 0$$

over a noetherian commutative ring $R$ such that $H_0(\mathbb{P})$ is $R$-flat and $h_0(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H_0(\mathbb{P}(\mathfrak{p}))$ is finite and constant, but $H_0(\mathbb{P})$ is neither $R$-finite nor $R$-projective.

*Proof.* Set $R = \mathbb{Z}$, $M = \sum_{p \text{ prime}} (1/p)\mathbb{Z} \subset \mathbb{Q}$, and $\mathbb{P}$ to be a projective resolution of $M$. Then $M$ is $R$-torsion free, and is $R$-flat. Since $M_{(p)} = (1/p)\mathbb{Z}_{(p)}$, $h_0(\mathfrak{p}) = 1$ for every $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}$. A finitely generated nonzero $\mathbb{Z}$-submodule of $\mathbb{Q}$ must be rank-one free, but $M$ is not a cyclic module, and is not rank-one free. This shows that $M$ is not $R$-finite. As $R$ is a principal ideal domain, every $R$-projective module is free. If $M$ is projective, then it is free of rank $h_0((0)) = 1$. But $M$ is not finitely generated, so $M$ is not projective. □

*Remark* 12. Let $(R, \mathfrak{m})$ be a noetherian *local* ring, $F$ a flat $R$-module, and $c$ a non-negative integer. If $\dim_{\kappa(\mathfrak{p})} F(\mathfrak{p}) = c$ for every $\mathfrak{p} \in \operatorname{Spec} R$, then $F \cong R^c$, see [2, Corollary III.2.1.10].

*Remark* 13. Let

$$\mathbb{P} : 0 \to P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} \cdots$$

be a complex of $R$-flat modules such that $P^0$ is $R$-projective. Assume that $\mathbb{P}(\mathfrak{p})$ is acyclic (i.e., $H^i(\mathbb{P}(\mathfrak{p})) = 0$ for every $i > 0$) and $h^0_{\mathbb{P}}(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H^0(\mathbb{P}(\mathfrak{p}))$ is finite for every $\mathfrak{p} \in \operatorname{Spec} R$. If $h^0_{\mathbb{P}}$ is a locally constant function on $\operatorname{Spec} R$, then $H^0(\mathbb{P})$ is $R$-finite $R$-projective, $H^i(M \otimes \mathbb{P}) = 0$ $(i > 0)$, and the canonical map $M \otimes H^0(\mathbb{P}) \to H^0(M \otimes \mathbb{P})$ is an isomorphism for every $R$-module $M$, see [2, Proposition III.2.1.14]. If, moreover, $\mathbb{P}$ is a complex of $R$-projective modules, then $\operatorname{Im} d^i$ is $R$-projective for every $i \geq 0$, as can be seen easily from Theorem 4.

**Example 14.** Let $M$ be an $R$-module. Even if $M(\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{Spec} R$, $M$ may not be zero. Even if $\operatorname{Tor}_1^R(\kappa(\mathfrak{p}), M) = 0$ for every $\mathfrak{p} \in \operatorname{Spec} R$, $M$ may not be $R$-flat.

Indeed, let $(R, \mathfrak{m}, k)$ be a $d$-dimensional regular local ring, and $E$ the injective hull of $k$. Then

$$\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), E) \cong \begin{cases} k & \text{for } i = d \text{ and } \mathfrak{p} = \mathfrak{m} \\ 0 & \text{otherwise} \end{cases}.$$

$E$ is not $R$-flat unless $d = 0$.

*Proof.* Since $\operatorname{supp} E = \{\mathfrak{m}\}$, $\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), E) = 0$ unless $\mathfrak{p} = \mathfrak{m}$.

Let $\boldsymbol{x} = (x_1, \ldots, x_d)$ be a regular system of parameters of $R$, and $\mathbb{K}$ the Koszul complex $K(\boldsymbol{x}; R)$, which is a minimal free resolution of $k$. Note that $\mathbb{K}$ is self-dual. That is, $\mathbb{K}^* \cong \mathbb{K}[-d]$, where $\mathbb{K}^* = \operatorname{Hom}^\bullet_R(\mathbb{K}, R)$, and $\mathbb{K}[-d]^n = \mathbb{K}^{n-d}$. So

$$\begin{aligned}\operatorname{Tor}_i^R(k, E) &\cong H^{-i}(\mathbb{K} \otimes E) \cong H^{-i}(\mathbb{K}^{**} \otimes E) \cong H^{-i}(\operatorname{Hom}^\bullet_R(\mathbb{K}[-d], E)) \\ &\cong H^{-i}(\operatorname{Hom}^\bullet_R(k[-d], E)) \cong \begin{cases} k & (i = d) \\ 0 & (i \neq d) \end{cases}.\end{aligned}$$

□

**Example 15.** There is a complex $\mathbb{P}$ of projective modules over a noetherian commutative ring $R$ such that for each $\mathfrak{m} \in \operatorname{Max}(R)$, $\mathbb{P}(\mathfrak{m})$ is exact, but $\mathbb{P}$ is not exact, where $\operatorname{Max}(R)$ denotes the set of maximal ideals of $R$.

*Proof.* Let $R$ be a DVR with its field of fractions $K$, and $\mathbb{P}$ a projective resolution of $K$. $\square$

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