Perturbations of Banach algebras and amenability

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ABSTRACT: In this paper we prove that if (A, π) is an amenable Banach algebra and if ρ is another Banach algebra multiplication on A such that $\|\rho - \pi\| < \frac{1}{11}$, then (A, ρ) is also amenable.

1 Introductions

Let A to be a Banach algebra and X an A-bimodule that is a Banach space. We say that X is a Banach A-bimodule if there exists constant $C > 0$ such that

$$\|a.x\| \leq C\|a\|\|x\|,$$
$$\|x.a\| \leq C\|a\|\|x\| \qquad (a \in A, x \in X).$$

If X is a Banach A-bimodule, then X^* is a Banach A-bimodule for the actions defined by

$$\langle a.f, x\rangle = \langle f, x.a\rangle$$
$$\langle f.a, x\rangle = \langle f, a.x\rangle \quad (a \in A, f \in X^*, x \in X).$$

The Banach A-bimodule X^* defined in this way is said to be a dual Banach A-bimodule.

A linear mapping D from A into X is a derivation if

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in A).$$

For $x \in X$, the mapping $ad_x : A \longrightarrow X$ defined by $ad_x(a) = a.x - x.a$ is a continuous derivation. The derivation D is inner if there exists $x \in X$ such that $D = ad_x$.
A is said to be amenable if for every Banach A-bimodule X , any continuous derivation from A into the dual Banach A-bimodule X^* is inner. This notion has been introduced in [2] and has been studied extensively.

Let A be an Banach algebra. A^{op} is another Banach algebra which is the same as A as Banach spaces but the product of A^{op} is the reverse of the product of A i.e.

$$a \circ b = ba \qquad (a, b \in A),$$

where $\circ$ denotes the multiplication of A^{op}.

The so-called multiplication map, denoted by π, $\pi : A\hat{\otimes}A^{op} \longrightarrow A$ is specified by

$$\pi(a \otimes b) = ab \qquad (a, b \in A)$$

By the difference between the two multiplications π and ρ on a Banach algebra A, we mean the norm of $\pi - \rho$ as an operator from $A\hat{\otimes}A^{op}$ to A. In [3] Johnson proved that if (A, π) is amenable, then there exists an $\epsilon > 0$ such that if ρ is another Banach algebra multiplication on A such that on $\|\pi - \rho\| < \epsilon$, then (A, ρ) is also amenable. But that ϵ here depends on the structure of the Banach algebra A. In this paper we give a partially different proof for that theorem and we prove the following result:

If (A, π) is an amenable Banach algebra, then (A, ρ) is also amenable for every Banach algebra multiplication ρ on A such that $\|\pi - \rho\| < \frac{1}{11}$.

2 Perturbations of Banach algebras

Before going to the mail theorem, we bring two lemmas from [3] that are used in our proof.

For two closed subspaces Y and Z of a Banach space X,their Hausdorff distance is defined by

$$d(Y, Z) = \max\{\sup\{d(y, Z) : \|y\| \leq 1\}, \sup\{d(z, Y) : \|z\| \leq 1\}\}$$

Lemma 2.1. *Let Y and Z be closed subspaces of a Banach space X. Suppose that there is a projection P of X onto Y with $\|P\| < d(Y, Z)^{-1} - 1$. Then P maps Z one to one onto Y and the inverse α of $P|_Z$ satisfies ($d = d(Y, Z)$)*

$$\|\alpha\| \leq (1 + d)(1 - \|P\|d)^{-1}$$
$$\|\alpha(y) - y\| \leq ((1 + d)(1 - \|P\|d)^{-1} - 1)\|y\|$$
$$\|P(z) - z\| \leq d(1 + \|P\|)\|z\|$$

Proof: See [3, Lemma 5.2].

Lemma 2.2. *Let X_1 and X_2 be Banach spaces and $S, T \in B(X_1, X_2)$ and let S be onto. Suppose that there exists $K > 0$ such that for all $y \in X_2$, there is $x \in X_1$ with $\|x\| \leq K\|y\|$ and $S(x) = y$. If $K\|S - T\| < 1$, then T will also be onto and for each $y \in X_2$, there exists $x \in X_1$ such that $\|x\| \leq K(1 - K\epsilon)^{-1}\|y\|$ and $T(x) = y$, where $\epsilon = \|S - T\|$.*

Proof: It is a special case of [3, Lemma 6.1].

In next theorem and note we denote all multiplications induced by π by a sign of π for example in order to show the product of a and b induced by π, we use $a_\pi b$, We have the same way to show them for ρ. **Note:** If $\pi^{\#}$ and $\rho^{\#}$ are the products respectively induced by π and ρ on $A^{\#}$ ($A^{\#}$ is the unitization of A) then we have

$$\|(\pi^{\#}-\rho^{\#})((a,\alpha)\otimes(b,\beta))\| = \|a_\pi b - a_\rho b\| \leq \|\pi-\rho\|\|a\|\|b\| \quad (a,b\in A).$$

And hence

$$\|(\pi^{\#}-\rho^{\#})((a,\alpha)\otimes(b,\beta))\| \leq \|\pi-\rho\|\|(a,\alpha)\|\|(b,\beta)\|$$

Thus we have

$$\|\pi^{\#}-\rho^{\#}\| \leq \|\pi-\rho\|.$$

Theorem 2.3. *Let (A,π) be an amenable Banach algebra. If ρ is another Banach algebra multiplication on A such that $\|\pi-\rho\| < \frac{1}{11}$, then (A,ρ) is also amenable.*

Proof:*By* the note above, we can assume that A has and identity 1 for both multiplications π and ρ. Let $j: A \longrightarrow A\widehat{\bigotimes}A$ be defined by $j(a) = a\otimes 1$.
Then $\|j\| \leq 1$ and $\pi j = Id_A$. So $\pi^{**}j^{**} = Id_{A^{**}}$. It can be easily checked that $P = \mathrm{Id}_{(A\widehat{\bigotimes}A)^{**}} - j^{**}\pi^{**}$ is a projection onto $\ker\pi^{**}$ with norm at most 2.
By Lemma 2.2, and letting $X_1 = (A\widehat{\bigotimes}A)^{**}$and $X_2 = A^{**}$, $S_1 = \pi^{**}$,$T_1 = \rho^{**}$,by $K = 1$ (since$\|\mathrm{j}^{**}\| \leq 1$), we get that for $\|S_1 - T_1\| = \epsilon < 1$, ρ^{**} will be onto and for every $F \in \ker\pi^{**}$, there is $B \in (A\widehat{\bigotimes}A)^{**}$ such that $\rho^{**}(B) = \rho^{**}(F)$ and

$$\|B\| \leq (1-\epsilon)^{-1}\|\rho^{**}(F)\| = (1-\epsilon)^{-1}\|\rho^{**}(F) - \pi^{**}(F)\| \leq (1-\epsilon)^{-1}\epsilon\|F\|$$

So $F - B \in \ker\rho^{**}$ and $\|F-(F-B)\| = \|B\| \leq \epsilon(1-\epsilon)^{-1}\|F\|$. So that

$$\sup\{d(F,\ker\rho^{**}) : \mathrm{F}\in\ker\pi^{**}\mathrm{and}\|\mathrm{F}\| \leq 1\} \leq \epsilon(1-\epsilon)^{-1}.$$

And similarly by changing the role of S_1 and T_1, we will obtain

$$\sup\{d(F,ker\pi^{**}) : F\in\ker\rho^{**}\mathrm{and}\|\mathrm{F}\| \leq 1\} \leq \epsilon(1-\epsilon)^{-1}$$

Hence

$$d := d(\ker\pi^{**},\ker\rho^{**}) \leq \epsilon(1-\epsilon)^{-1}.$$

So if $\epsilon < \frac{1}{4}$, then

$$\|P\| \leq 2 < (\epsilon(1-\epsilon)^{-1})^{-1} - 1 \leq d(\ker\pi^{**},\ker\rho^{**})^{-1} - 1.$$

And hence by Lemma 2.1, there exists a linear homeomorphism α from kerπ^{**} onto kerρ^{**} such that

$$\|\alpha\| \leq (1-3\epsilon)^{-1}, \|\alpha^{-1}\| \leq \|P\| \leq 2$$
$$\|F-\alpha(F)\| \leq 3\epsilon(1-3\epsilon)^{-1}\|F\| \quad (F \in \mathrm{ker}\pi^{**})$$
$$\|F-\alpha^{-1}(F)\| \leq 3\epsilon(1-\epsilon)^{-1}\|F\| \quad (F \in \mathrm{ker}\rho^{**}).$$

Suppose that $F \in (A\widehat{\otimes}A)$ is an elementary tensor say $b \otimes c$ for $b, c \in A$. Then for $a \in A$, we have

$$\begin{aligned}\|a._\pi F - a._\rho F\| &= \|a.(b\otimes c) - a._\rho(b\otimes c)\| \\ &= \|ab\otimes c - a_\rho b\otimes c\| = \|(a_\rho b - ab)\|\|c\| \\ &\leq \|\rho - \pi\|\|a\otimes b\|\|c\| \\ &\leq \epsilon\|a\|\|b\|\|c\| = \epsilon\|a\|\|F\|.\end{aligned}$$

So that

$$\|a._\pi F - a._\rho F\| \leq \epsilon\|a\|\|F\| \quad (a \in A, F \in A\widehat{\otimes}A).$$

And by using Goldsteine's Theorem, we have

$$\|a._\pi F - a._\rho F\| \leq \epsilon\|F\| \quad (F \in (A\widehat{\otimes}A)^{**}) \qquad (\dagger)$$

Similarly

$$\|F._\pi a - F.\rho a\| \leq \epsilon\|a\|\|F\| \quad (a \in A, F \in (A\widehat{\otimes}A)^{**}).$$

Now consider the derivation $D : A \longrightarrow \mathrm{ker}\pi^{**} (\cong (\mathrm{ker}\pi)^{**})$ by $D(a) = a\otimes 1 - 1\otimes a$, then amenability of (A, π) implies the existence of an element $\xi \in \mathrm{ker}\pi^{**}$ such that

$$a\otimes 1 - 1\otimes a = a._\pi\xi - \xi._\pi a \quad (a \in A).$$

Let $\delta = \alpha(\xi) \in \mathrm{ker}\rho^{**}$. Then we have

$$\begin{aligned}\|a._\pi\xi - a._\rho\delta\| &= \|a._\pi\xi - a._\rho(\alpha(\xi))\| \\ &\leq \|a._\pi\xi - a._\pi(\alpha(\xi))\| + \|a._\pi(\alpha(\xi)) - a._\rho(\alpha(\xi))\| \\ &\leq 3\epsilon(1-3\epsilon)^{-1}\|a\|\|\xi\| + \epsilon(1-3\epsilon)^{-1}\|a\|\|\xi\|. \quad \text{(By properties of } \alpha \text{ and } (\dagger))\end{aligned}$$

And similarly

$$\|\xi._\pi a - \delta._\rho a\| \leq 4\epsilon(1-3\epsilon)^{-1}\|a\|\|\xi\|.$$

So that

$$\begin{aligned}\|a\otimes 1 - 1\otimes a - (a._\rho\delta - \delta._\rho a)\| &= \|a._\pi\xi - \xi._\pi a - (a._\rho\delta - \delta._\rho a)\| \\ &\leq \|a._\pi\xi - a._\rho\delta\| + \|\xi._\pi a - \delta._\rho a\| \\ &\leq 8\epsilon(1-3\epsilon)^{-1}\|a\|.\end{aligned}$$

So

$$\|a \otimes 1 - 1 \otimes a - (a.\rho\delta - \delta._\rho a)\| \leq O(\epsilon)\|a\| \qquad (a \in A). \qquad (\ddagger)$$

Where $O(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0^+$.
From now on all the multiplications we consider are respect to the multiplication ρ on A. We denote the multiplication in $A\widehat{\otimes}A^{op}$ by $\star_\rho$. Also we show the Arens product on $(A\widehat{\otimes}A^{op})^{**}$ with the same notation. So for elementary tensors,

$$(a \otimes b) \star_\rho (c \otimes d) = ac \otimes db$$

For $R = \sum_i a_i \otimes b_i \in \ker\rho$ we have

$$\begin{aligned} R \star_\rho \delta - R &= \sum_i (a_i \otimes b_i) \star_\rho \delta - \delta \sum_i a_i b_i - \sum_i a_i \otimes b_i + 1 \otimes \sum_i a_i b_i \\ &= \sum_i (a_i ._\rho \delta - \delta._\rho a_i - a_i \otimes 1 + 1 \otimes a_i)._\rho b_i. \end{aligned}$$

So

$$\begin{aligned} \|R \star_\rho \delta - R\| &= \|\sum_i (a_i ._\rho \delta - \delta._\rho a_i - a_i \otimes 1 + 1 \otimes a_i)._\rho b_i\| \\ &\leq \sum_i \|\frac{a_i}{\|a_i\|}._\rho \delta - \delta._\rho \frac{a_i}{\|a_i\|} + \frac{a_i}{\|a_i\|} \otimes 1 + 1 \otimes \frac{a_i}{\|a_i\|}\| \|a_i\| \|b_i\| \\ &\leq \|R\| \sup_{a \in A_1} \|a._\rho \delta - \delta._\rho a - a \otimes 1 + 1 \otimes a\|. \end{aligned}$$

Now if $R \in (\ker\rho)^{**}$, then by Goldsteine's Theorem, there exists a net $(r_i)_i$ with $\|r_i\| \leq \|R\|$, in $\ker\pi$ such that $r_i \longrightarrow_i R \quad$ wk*. Note that since $\ker\rho^{**} \cong (\ker\rho)^{**}$,isometrically, then for notational convenience, we don't disguise between δ as an element in $\ker\rho^{**}$ and its image as an element of $(\ker\rho)^{**}$.
Thus

$$r_i ._\rho \delta - r_i \longrightarrow_i R._\rho \delta - R \quad \text{wk}^*.$$

And hence $\|R._\rho\delta - R\| \leq \sup_i \|r_i ._\rho \delta - r_i\|$. So we have

$$\|R \star_\rho \delta - R\| \leq \|R\| \sup_{a \in A_1} \|a._\rho \delta - \delta._\rho a - a \otimes 1 + 1 \otimes a\| \quad (R \in (\ker\rho)^{**}).$$

And hence by ($\ddagger$), we obtain

$$\|R \star_\rho \delta - R\| \leq O(\epsilon)\|R\| \quad (R \in (\ker\rho)^{**}).$$

If we define $\lambda : (\ker\rho)^{**} \longrightarrow (\ker\rho)^{**}$ by $\lambda(S) = S \star_\rho \delta$, then for $\epsilon < \frac{1}{11}$, $O(\epsilon) = \frac{8\epsilon}{(1-3\epsilon)} < 1$ and hence $\|\lambda - Id_{(\ker\rho)^{**}}\| < 1$ and thus λ will be invertible.

Since λ is surjective, there exists $x \in (\ker\rho)^{**}$ such that $\lambda(x) = \delta$. So $x \star_\rho \delta = \delta$ and therefore for every $y \in (\ker\rho)^{**}$, we have $(y \star_\rho x - y) \star_\rho \delta = 0$ but this means that

$$\lambda(y \star_\rho x - y) = 0 \qquad (y \in (\ker\rho)^{**}).$$

Now by injectivity of λ, we have

$$y \star_\rho x = y \qquad (y \in (\ker\rho)^{**}).$$

Hence x will be a right identity for $(\ker\rho)^{**}$ and hence $\ker\rho$ has a bounded right approximate identity. So from [1,Theorem 3.10] , (A, ρ) is amenable. $\square$

References

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