**Notes on amenability**

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ABSTRACT: We show that for a Banach algebra $A$ with a bounded approximate identity, the amenability of $A\widehat{\otimes}A$, the amenability of $A\widehat{\otimes}A^{op}$ and the amenability of $A$ are equivalent. Also if $A$ is a closed ideal in a commutative Banach algebra $B$, then the weak amenability of $A\widehat{\otimes}B$ implies the weak amenability of $A$.



# 1 Introductions and Preliminaries

Let $A$ to be a Banach algebra and $X$ an $A$-bimodule that is a Banach space. We say that $X$ is a Banach $A$-bimodule if there exists constant $C > 0$ such that

$$\|a.x\| \leq C\|a\|\|x\|,$$
$$\|x.a\| \leq C\|a\|\|x\| \qquad (a \in A, x \in X).$$

If $X$ is a Banach $A$-bimodule, then $X^*$ is a Banach $A$-bimodule for the actions defined by

$$\langle a.f, x\rangle = \langle f, x.a\rangle$$
$$\langle f.a, x\rangle = \langle f, a.x\rangle \quad (a \in A, f \in X^*, x \in X).$$

The Banach $A$-bimodule $X^*$ defined in this way is said to be a dual Banach $A$-bimodule.

A linear mapping $D$ from $A$ into $X$ is a derivation if

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in A).$$

For $x \in X$, the mapping $ad_x : A \longrightarrow X$ defined by $ad_x(a) = a.x - x.a$ is a continuous derivation. The derivation $D$ is inner if there exists $x \in X$ such that $D = ad_x$.
$A$ is said to be amenable if for every Banach $A$-bimodule $X$ , any continuous derivation from $A$ into the dual Banach $A$-bimodule $X^*$ is inner. This notion has been introduced in [4] and has been studied extensively since.
The Banach algebra $A$ is said to be weakly amenable if any continuous derivation from $A$ into the dual Banach $A$-bimodule $A^*$ is inner. This notion was first introduced in [1] for the commutative case and then in [6] for the general case. Every

Amenable Banach algebra has a bounded approximate identity [ 8, Proposition 2.21]. Also if $A$ and $B$ are two amenable Banach algebras then also is $A\hat{\otimes}B$ [ 4, Proposition 5.4].
The question is whether the converse is true or not. The only work on this question is done by B.E. Johnson in the following:

**Proposition 1.1.** *Suppose that $A$ is a Banach algebra and $B$ is another Banach algebra such that there exists $b_0 \in B$ with $b_0 \notin \overline{\mathrm{Lin}\{bb_0 - b_0b : b \in B\}}$. If $A\widehat{\otimes}B$ is amenable then $A$ is amenable.*

**Proof:** See [5, Proposition 3.5] □

But still the question remains open for general $A$ and $B$ even for the case $A = B$.

In section 2, we prove that the amenability of $A\hat{\otimes}A$ implies amenability of $A$ in the case that $A$ has a bounded approximate identity. Indeed we show that for a Banach algebra $A$ with a bounded approximate identity the following are equivalent:

(i) $A$ is amenable;

(ii) $A\widehat{\otimes}A$ is amenable;

(iii) $A\widehat{\otimes}A^{op}$ is amenable (Where as usual $A^{op}$ is the Banach algebra obtaining by reversing the product of $A$).

Since having a bounded approximate identity is a necessary condition for amenability, we can not omit the condition that $A$ has bounded approximate identity unless we can prove that amenability of $A\hat{\otimes}A$ necessitates having a bounded approximate identity for $A$.

In section 3 we investigate the question but for weak amenability instead of amenability. We prove that if $B$ is a commutative Banach algebra and $A$ is a closed ideal in $B$ , then the weak amenability of $A\hat{\otimes}B$ implies the weak amenability of $A$.

## 2 The amenability results

In this section we try to answer the question whether amenability $A\hat{\otimes}B$ implies the amenability of $A$ and $B$ or not. We mainly concentrate on the special case where $A = B$. However, we will also obtain some results about the case where $A$ is not necessarily equal to $B$.
First we start with a simple result:

**Theorem 2.1.** *Suppose that $A$ and $B$ are Banach algebras and $B$ has a non- zero character. If $A\widehat{\otimes}B$ is amenable, then $A$ is also amenable.*

**Proof:** Let $\varphi$ be a non- zero character in $B$ and define the unique mapping $\theta : A\widehat{\otimes}B \longrightarrow A$ acting on elementary tensors by

$$\theta(a \otimes b) = \varphi(b)a \quad (a \in A, b \in B).$$

We show that $\theta$ is an algebra homomorphism (obviously $\theta$ is continuous). Since $\theta$ is linear, it is enough to check this for elementary tensors. To see this we have

$$\theta((a \otimes b)(c \otimes d)) = \theta((ac \otimes bd) = \varphi(bd)ac.$$

On the other hand

$$\theta((a \otimes b))\theta((c \otimes d)) = \varphi(b)a\varphi(d)c = \varphi(bd)ac.$$

So

$$\theta((a \otimes b)(c \otimes d)) = \theta((a \otimes b))\theta((c \otimes d)).$$

And since $\varphi$ is non - zero, $\theta$ is surjective and hence $A$ is amenable. □

Throughout the following we let $\pi : A\widehat{\otimes}A^{op} \longrightarrow A$ be the so-called product map; mapping specified by acting on elementary tensors by $\pi(a \otimes b) = ab \quad (a, b \in A)$ and we let $K = \ker\pi$.
The Banach algebra $A$ can be made into a left $A\widehat{\otimes}A^{op}$-module by the module multiplication specified by

$$(a \otimes b).c = acb \qquad (a, b, c \in A).$$

**Theorem 2.2.** *Suppose that $A\widehat{\otimes}A^{op}$ is amenable and $A$ has a bounded approximate identity. Then $A$ is amenable.*

**Proof:** Since $A$ has a bounded approximate identity, the short exact sequence $(\prod^{op})^* : 0 \longrightarrow A^* \xrightarrow{\pi^*} (A\widehat{\otimes}A^{op})^* \xrightarrow{\imath^*} K^* \longrightarrow 0$ is an admissible short exact sequence of right $A\widehat{\otimes}A^{op}$-modules. ($\imath$ is the inclusion map).
Since $A^*$ is a dual $A\widehat{\otimes}A^{op}$-module, from [2, Theorem 2.3] , $(\prod^{op})^*$ splits and since $A\widehat{\otimes}A^{op}$ has a bounded approximate identity and $\pi$ is onto , [2, Theorem 3.5] implies that $K$ has a bounded right approximate identity. Now since $A$ has a bounded approximate identity, from [2, Theorem 3,10] $A$ is amenable. □

Theorem 2.3 has been the motivation for us to consider the question of under which conditions on the tensor products, $A$ has a bounded approximate identity. The following is one of them. Before going to next Theorem, we need a Lemma.

**Lemma 2.3.** *Let $A$ to be a Banach algebra with a two-sided bounded approximate identity and $X$ a Banach $A$-bimodule on which $A$ acts trivially on one side. Then for every continuous derivation $D$ from $A$ into $X$, there exists a bounded net $(\zeta_i)_i$ in $X$ such that $D(a) = \lim_i a.\zeta_i - \zeta_i.a \quad (a \in A)$.*

**Proof:** Since we can embed $X$ into $X^{**}$ through the canonical injection, we can consider $D$ as a continuous derivation into the dual module $X^{**}$. Also since the action of $A$ on one side of $X$ is trivial, action of $A$ on other side of $X^*$ is trivial. Therefore $D$ is inner. Hence there exists $\xi \in X^{**}$ such that

$$D(a) = a.\xi - \xi.a \qquad (a \in A).$$

Now by Goldstein's Theorem, there is a bounded net $(\tau_j)_{j \in J}$ in $X$ converging to $\xi$ in weak$^*$ topology of $X^{**}$. Thus

$$D(a) = a.\xi - \xi.a = \mathrm{wk}^* - \lim_{\mathrm{j}} \mathrm{a}.\tau_\mathrm{j} - \tau_\mathrm{j}.\mathrm{a} \qquad (\mathrm{a} \in \mathrm{A}),$$

and hence

$$D(a) = \mathrm{wk} - \lim_{\mathrm{j}} \mathrm{a}.\tau_\mathrm{j} - \tau_\mathrm{j}.\mathrm{a} \qquad (\mathrm{a} \in \mathrm{A}).$$

Let $\Delta = \{a_1, a_2, ..., a_n\}$ be a finite subset of $A$. Then in $\bigoplus_{i=1}^n X$, we have

$$(D(a_1), ..., D(a_n)) \in weak - cl(co(\{(a_1.\tau_j - \tau_j.a_1, ..., a_n.\tau_j - \tau_j.a_n) : j \in J\}))$$

Therefore by Mazur's Theorem

$$(D(a_1), ..., D(a_n)) \in \mathrm{norm} - \mathrm{cl}(\mathrm{co}(\{(\mathrm{a}_1.\tau_\mathrm{j} - \tau_\mathrm{j}.\mathrm{a}_1, ..., \mathrm{a}_\mathrm{n}.\tau_\mathrm{j} - \tau_\mathrm{j}.\mathrm{a}_\mathrm{n})) : \mathrm{j} \in \mathrm{J}\})$$

And hence for $\epsilon > 0$, there exists $\zeta_{\Delta,\epsilon} \in \mathrm{co}(\{\tau_\mathrm{j} : \mathrm{h} \in \mathrm{J}\})$ such that

$$\|D(a_i) - (a_i.\zeta_{\Delta,\epsilon} - \zeta_{\Delta,\epsilon}.a_i)\| < \epsilon \qquad (a_i \in \Delta)$$

So by ordering the set of the finite subsets of $A$ by inclusion and positive real numbers by decreasing order , the net $(\zeta_{\Delta,\epsilon})$ is the desired net. $\square$

**Theorem 2.4.** *Suppose that $A\widehat{\otimes}A^{op}$ has a bounded approximate identity and each one of the topologies on $A$ defined by the family of seminorms $\rho_a : b \mapsto \|ab\|$ and $\gamma_a : b \mapsto \|ba\|$ is stronger than weak topology on $A$. Then $A$ has a (two-sided) bound approximate identity.*

**Proof**: Suppose that $A\widehat{\otimes}A^{op}$ has a bounded approximate identity. we consider $A$ as an $A\widehat{\otimes}A^{op}$-bimodule by actions specified by:

$$(a \otimes b) \bullet c = acb$$
$$c \bullet (a \otimes b) = 0 \quad (a, b, c \in A)$$

It can be easily seen that $A$ is a Banach $A\widehat{\otimes}A^{op}$-bimodule by the actions above . Now we define a derivation $D : A\widehat{\otimes}A^{op} \longrightarrow A$ by acting on elementary tensors as $D(a \otimes b) = ab \quad (a, b \in A)$. $D$ is obviously continuous and also $D$ is a derivation since

$$D((a \otimes b) \cdot (c \otimes d)) = D(ac \otimes db) = acdb$$

($\cdot$ is the product in $A\widehat{\otimes}A^{op}$ ). On the other hand:

$$(a \otimes b) \bullet D(c \otimes d) + D(a \otimes b) \bullet (c \otimes d) = (a \otimes b) \bullet cd = acdb$$

Therefore $D \in \mathbb{Z}^1(A\widehat{\otimes}A^{op}, A)$. Now since the right action of $A\widehat{\otimes}A^{op}$ on $A$ is trivial and $A\widehat{\otimes}A^{op}$ has a bounded approximate identity, from Lemma 2.4, there exists a bounded net$(\zeta_i)_i$ in $A$ such that $D(a \otimes b) = \lim_i ad_{\zeta_i}(a \otimes b)$.
So $ab = \lim_i a\zeta_i b \quad (a, b \in A)$. Thus for all $a, b \in A$

$$\lim_i a(b - \zeta_i b) = 0 \qquad \lim_i (b - b\zeta_i)a = 0 \tag{1}$$

If we denote the topology induced by the family of seminorms $\{\rho_a | a \in A\}$ by $\tau$ and the topology induced by the family of seminorms $\{\gamma_a | a \in A\}$ by $\varsigma$, then from (1) we have

$$a\zeta_i \longrightarrow a \quad (in\ \tau\ for\ \ all\ a \in A) \tag{2}$$

$$\zeta_i a \longrightarrow a \quad (in\ \varsigma\ for\ \ all\ a \in A) \tag{3}$$

since we assume both $\tau$ and $\varsigma$ to be stronger than weak topology on $A$, then by (2) and (3), $A$ has a weakly two-sided bounded approximate identity and hence $A$ has a two-sided bounded approximate identity. □

**Theorem 2.5.** *Suppose that $A\widehat{\otimes}A^{op}$ is amenable and that $A$ has the property that each one of the topologies induced on $A$ by the family of seminorms $\{\rho_a | a \in A\}$ where $\rho_a(b) = \|ab\|$ and $\{\gamma_a | a \in A\}$ where $\gamma_a(b) = \|ba\|$, are stronger than the weak topology on $A$. Then $A$ is amenable.*

**Proof:** Firstly by the fact that $A\widehat{\otimes}A^{op}$ necessarily has a two-sided bounded approximate identity and from Theorem 2.5 we have that $A$ has a two-sided bounded approximate identity and then from Theorem 2.3 we have $A$ is amenable. $\square$

In next Theorem we attempt to relate amenability of $A\widehat{\otimes}A$ (in the case that $A$ has a bounded approximate identity) to the amenability of $A\widehat{\otimes}A^{op}$ and then by using the preceding theorems, we attempt to prove the amenability of $A$ when $A\widehat{\otimes}A$ is amenable. Before going to next Theorem, we need a Lemma.

**Lemma 2.6.** *Let $A$ be Banach algebra with a bounded approximate identity such that for any neo-unital Banach $A$-bimodule $X$ and $Y$ a closed submodule of $X$, every $f \in Z_A(Y^*)$ can be extended to a functional $\tilde{f} \in Z_A(X^*)$. Then $A$ is amenable.*

**proof:** As in the proof of [7, Theorem 1], for concluding the amenability of $A$, it is enough to have the property in the Lemma for the Banach $A$-bimodule $L = (A\widehat{\otimes}A)^*\widehat{\otimes}(A\widehat{\otimes}A)$ with the module actions specified by

$$a.(x^* \otimes x) = x^* \otimes a.x,$$
$$(x^* \otimes x).a = x^* \otimes x.a \quad (a \in A, x \in (A\widehat{\otimes}A), x^* \in (A\widehat{\otimes}A)^*).$$

Since $A$ has bounded approximate identity, $X = A\widehat{\otimes}A$ is neo-unital and hence by the above definition of the actions of $A$ on $L$, $L$ is also neo-unital. $\square$

**Theorem 2.7.** *Suppose that $A$ is a Banach algebra with a bounded approximate identity such that $A\widehat{\otimes}A$ is amenable. Then $A\widehat{\otimes}A^{op}$ is also amenable.*

**proof:** Suppose that $X$ is a Banach neo-unital $A\widehat{\otimes}A^{op}$-bimodule and that $\bullet$ denotes the action of $A\widehat{\otimes}A^{op}$ on $X$. We define:

$$(a \otimes b) \circ x = \lim_i (a \otimes e_i) \bullet x \bullet (e_i \otimes b),$$
$$x \circ (a \otimes b) = \lim_i (e_i \otimes b) \bullet x \bullet (a \otimes e_i) \quad (x \in X \text{and } \text{a, b} \in \text{A}).$$

First we note that the above limits exist because by the assumption that $X$ is neo-unital we have:
If $x \in X$ then there exist $y \in X$ and $u, v \in A\widehat{\bigotimes}A^{op}$ such that $x = u \bullet y \bullet v$ and then we have:

$$(a \otimes e_i) \bullet x \bullet (e_i \otimes b) = (a \otimes e_i) \bullet u \bullet y \bullet v \bullet (e_i \otimes b) = ((a \otimes e_i) \star u) \bullet y \bullet (v \star (e_i \otimes b)),$$

where $\star$ denotes the product in $A\widehat{\otimes}A^{op}$. Since $(e_i)_{i\in\Lambda}$ is a bounded approximate identity for $A$, it can be easily seen that $\lim_i (a\otimes e_i)\star u = a.u$ and $\lim_i v\star(e_i\otimes b) = v.b$,

where $a.(e \otimes f) = ae \otimes f$ and $(e \otimes f).b = e \otimes bf$.
So $\lim_i (a \otimes e_i) \bullet x \bullet (e_i \otimes b)$ exists and we can similarly prove the existence of the second limit. Also $\circ$ induces a module action of $A \widehat{\bigotimes} A$ on $X$ . To see the reason, by linearity, it is enough to check the module conditions for elementary tensors.

$$((a \otimes b)(c \otimes d)) \circ x = (ac \otimes bd) \circ x = \lim_i (ac \otimes e_i) \bullet x \bullet (e_i \otimes bd)$$

On the other hand:

$$\begin{aligned}(a \otimes b) \circ ((c \otimes d) \circ x) &= (a \otimes b) \circ (\lim_j (c \otimes e_j) \bullet x \bullet (e_j \otimes d)) \\ &= \lim_i (a \otimes e_i) \bullet (\lim_j (c \otimes e_j) \bullet x \bullet (e_j \otimes d)) \bullet (e_i \otimes b) \\ &= \mathrm{lim_i lim_j} (ac \otimes e_j e_i) \bullet x \bullet (e_j e_i \otimes bd) \\ &= \mathrm{lim_i} (ac \otimes e_i) \bullet x \bullet (e_i \otimes bd).\end{aligned}$$

Hence

$$((a \otimes b)(c \otimes d)) \circ x = (ac \otimes bd) \circ x = (a \otimes b) \circ ((c \otimes d) \circ x).$$

In a similar way we can show that

$$x \circ ((a \otimes b)(c \otimes d)) = (x \circ (a \otimes b))(c \otimes d).$$

Also we have:

$$\begin{aligned}((a \otimes b) \circ x) \circ (c \otimes d) &= \mathrm{lim_i} (e_i \otimes d) \bullet (\mathrm{lim_j} (a \otimes e_j) \bullet x \bullet (e_j \otimes b)) \bullet (c \otimes e_i) \\ &= \mathrm{lim_i lim_j} ((e_i \otimes d) \star (a \otimes e_j)) \bullet x \bullet ((e_j \otimes b) \star (c \otimes e_i)) \\ &= \mathrm{lim_i lim_j} (e_i a \otimes e_j d) \bullet x \bullet (e_j c \otimes e_i b) \\ &= (a \otimes d) \bullet x \bullet (c \otimes b).\end{aligned}$$

On the other hand:

$$\begin{aligned}(a \otimes b) \circ (x \circ (c \otimes d)) &= \mathrm{lim_i lim_j} (a \otimes e_i) \bullet ((e_j \otimes d) \bullet x \bullet (c \otimes e_j)) \bullet (e_i \otimes b) \\ &= \mathrm{lim_i lim_j} ((ae_j \otimes de_i) \bullet x \bullet (ce_i \otimes be_j) \\ &= (a \otimes d) \bullet x \bullet (c \otimes b).\end{aligned}$$

Hence

$$((a \otimes b) \circ x) \circ (c \otimes d) = (a \otimes b) \circ (x \circ (c \otimes d)).$$

So $X$ is an $A \widehat{\otimes} A$-bimodule for the action $\circ$. Also since the net $(e_i)$ is bounded, it can be easily seen that $X$ is indeed a Banach $A \widehat{\otimes} A$-bimodule for $\circ$. For a Banach $A \widehat{\otimes} A^{op}$-bimodule $X$, $X_\dagger$ denotes $X$ as an $A \widehat{\otimes} A$-bimodule (via the action $\circ$).

Now if $Y$ is a closed submodule of $X$ and $f \in Z_{A\widehat{\otimes}A^{op}}(Y^*)$, we show that $f \in Z_{A\widehat{\otimes}A}(Y^*_\dagger)$.
To prove the above statement we have:

$$\begin{aligned}(a \otimes b) \circ f &= \mathrm{wk}^* - \lim_i (\mathrm{a} \otimes \mathrm{e_i}) \bullet \mathrm{f} \bullet (\mathrm{e_i} \otimes \mathrm{b}) \\ &= \mathrm{wk}^* - \lim_i \mathrm{f} \bullet (\mathrm{a} \otimes \mathrm{e_i}) \bullet (\mathrm{e_i} \otimes \mathrm{b}) \\ &= \mathrm{wk}^* - \lim_i \mathrm{f} \bullet (\mathrm{ae_i} \otimes \mathrm{be_i}) \\ &= f \bullet (a \otimes b).\end{aligned}$$

Similarly

$$f \circ (a \otimes b) = (a \otimes b) \bullet f.$$

Thus

$$f \in Z_{A\widehat{\otimes}A}(Y^*_\dagger).$$

Now from [7, Theorem 1] , $f$ has an extension to an $\tilde{f} \in Z(A\widehat{\otimes}A, X^*_\dagger)$ .
We show that $\tilde{f} \in Z(A\widehat{\otimes}A^{op}, X^*)$ For this purpose we have

$$\begin{aligned}(a \otimes b) \bullet \tilde{f} &= \mathrm{wk}^* - \lim_i - \mathrm{wk}^* - \lim_j ((\mathrm{a} \otimes \mathrm{e_i})(\mathrm{e_j} \otimes \mathrm{b})) \bullet \tilde{\mathrm{f}} \bullet (\mathrm{e_i} \otimes \mathrm{e_j}) \\ &= \mathrm{wk}^* - \lim_i (\mathrm{a} \otimes \mathrm{e_i})(\mathrm{wk}^* - \lim_j (\mathrm{e_j} \otimes \mathrm{b}) \bullet \tilde{\mathrm{f}} \bullet (\mathrm{e_i} \otimes \mathrm{e_j})) \\ &= \mathrm{wk}^* - \lim_i (\mathrm{a} \otimes \mathrm{e_i}) \bullet (\tilde{\mathrm{f}} \circ (\mathrm{e_i} \otimes \mathrm{b})) \\ &= \mathrm{wk}^* - \lim_i (\mathrm{a} \otimes \mathrm{e_i}) \bullet ((\mathrm{e_i} \otimes \mathrm{b}) \circ \tilde{\mathrm{f}}) \\ &= \mathrm{wk}^* - \lim_i (\mathrm{a} \otimes \mathrm{e_i}) \bullet (\mathrm{wk}^* - \lim_j (\mathrm{e_i} \otimes \mathrm{e_j}) \bullet \tilde{\mathrm{f}} \bullet (\mathrm{e_j} \otimes \mathrm{b})) \\ &= \mathrm{wk}^* - \lim_i \mathrm{wk}^* - \lim_j (\mathrm{ae_i} \otimes \mathrm{e_j e_i}) \bullet \tilde{\mathrm{f}} \bullet (\mathrm{e_j} \otimes \mathrm{b}) \\ &= \mathrm{wk}^* - \lim_i (\mathrm{a} \otimes \mathrm{e_i}) \bullet \tilde{\mathrm{f}} \bullet (\mathrm{e_i} \otimes \mathrm{b}) \\ &= (a \otimes b) \circ \tilde{f}.\end{aligned}$$

similarly we have $\tilde{f} \bullet (a \otimes b) = \tilde{f} \circ (a \otimes b)$ and since $\tilde{f} \in Z_{A\widehat{\otimes}A}(X^*_\dagger)$, then $(a \otimes b) \bullet \tilde{f} = \tilde{f} \bullet (a \otimes b)$. Hence

$$\tilde{f} \in Z_{A\widehat{\otimes}A^{op}}(X^*).$$

Since $Y$ was an arbitrary closed submodule of $X$ and $f$ was arbitrary in $Z_{A\widehat{\otimes}A^{op}}(Y^*)$, again by exploiting [7,Theorem 1], we have that $A\widehat{\otimes}A^{op}$ is amenable. □

**Theorem 2.8.** *Suppose that $A\widehat{\otimes}A$ is amenable and $A$ has a bounded approximate identity. Then $A$ is amenable.*

**Proof:** By the preceding Theorem we have that $A\widehat{\otimes}A^{op}$ is amenable. Since $A$ has a bounded approximate identity, from Theorem 2.3 , $A$ is amenable. $\square$

Since having a bounded approximate identity is a necessary condition for an algebra to be amenable, the Theorem 2.8 has the minimum conditions. If we can prove that amenability of $A\widehat{\otimes}A$ implies that $A$ has a bounded approximate identity, then we can even drop the condition in Theorem 2.8 that $A$ has a bounded approximate identity.

## 3 Some results in commutative Banach algebras

Now we go to the case where our algebra $A$ is commutative. First we prove the following general result.
For the Banach algebra $A$, we define

$$A^2 = \mathrm{Lin}\{\mathrm{ab} : \mathrm{a}, \mathrm{b} \in \mathrm{A}\}.$$

**Theorem 3.1.** *Suppose that $B$ is a Banach algebra and $A$ is a closed subalgebra of $B$ such that $A\widehat{\otimes}B$ is weakly amenable. Then* $(A^2)^- = A$

**Proof:** *Suppose* that $(A\widehat{\otimes}B)$ is weakly amenable and $(A^2)^- \neq A$. Then from Hahn-Banach Theorem there exists a $\lambda \in A^*$ such that $\lambda|_{A^2} = 0$ and $\lambda \neq 0$. So there exists an $a_0 \in A$ such that $\lambda(a_0) = 1$. We denote a Hahn-Banach extension of $\lambda$ on $B$ by $\tilde{\lambda}$. So $\tilde{\lambda} \in B^*$ and we specify $D : (A\widehat{\otimes}B) \longrightarrow (A\widehat{\otimes}B)^*$ by

$$D(a \otimes b) = \tilde{\lambda}(a)\tilde{\lambda}(b)(\tilde{\lambda} \otimes \tilde{\lambda}) \qquad (a \in A, b \in B),$$

where $(\tilde{\lambda} \otimes \tilde{\lambda})(c \otimes d) = \tilde{\lambda}(c)\tilde{\lambda}(d)$.

Then we have

$$D((a \otimes b)(c \otimes d)) = D(ac \otimes bd) = \tilde{\lambda}(ac)\tilde{\lambda}(bd)(\tilde{\lambda} \otimes \tilde{\lambda}) = 0$$

On the other hand for $a, c, x \in A$ and $b, d, y \in B$ we have

$$\langle (a \otimes b)D(c \otimes d), x \otimes y\rangle = \langle D(c \otimes d), xa \otimes yb\rangle = \tilde{\lambda}(c)\tilde{\lambda}(d)\tilde{\lambda}(xa)\tilde{\lambda}(yb) = 0$$

and similarly

$$\langle D(a \otimes b).(c \otimes d), x \otimes y\rangle = \langle D(a \otimes b), cx \otimes dy\rangle = \tilde{\lambda}(a)\tilde{\lambda}(b)\tilde{\lambda}(cx)\tilde{\lambda}(dy) = 0.$$

So $D : A\widehat{\otimes}B \longrightarrow (A\widehat{\otimes}B)^*$ is a continuous derivation and hence from weak amenability of $(A\widehat{\otimes}B)$ it follows that $D = ad(\xi)$ for some $\xi \in (A\widehat{\otimes}B)^*$.

So

$$\begin{aligned}\langle D(a_0 \otimes a_0), (a_0 \otimes a_0)\rangle &= \langle (a_0 \otimes a_0).\xi - \xi.(a_0 \otimes a_0), a_0 \otimes a_0\rangle \\ &= \langle \xi, (a_0^2 \otimes a_0^2) - (a_0^2 \otimes a_0^2)\rangle \\ &= 0\end{aligned}$$

But we have:

$$\langle D(a_0 \otimes a_0), (a_0 \otimes a_0)\rangle = \tilde{\lambda}(a_0)\tilde{\lambda}(a_0)(\tilde{\lambda} \otimes \tilde{\lambda})(a_0 \otimes a_0) = (\tilde{\lambda}(a_0))^4 = 1.$$

So we have come up with a contradiction and hence $(A^2)^- = A$ □

**Theorem 3.2.** *Suppose that $B$ is a commutative Banach algebra and $A$ is an ideal in $B$ such that $A\widehat{\otimes}B$ is weakly amenable. Then $A$ is weakly amenable.*

**Proof** *Suppose* that $A\widehat{\otimes}B$ is weakly amenable. Then we define $\varphi : A\widehat{\otimes}B \longrightarrow A$ by $\varphi(a \otimes b) = ab$. It can be easily seen that $\varphi$ is continuous and is an algebra homomorphism. Also by Theorem 3.1 we have $\varphi(A)^- = A$. Hence from [3, Proposition 2.11], $A$ is weakly amenable. □

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