

Classical Analysis and Nilpotent Lie Groups

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Classical Fourier analysis has an exact counterpart in group theory and in some areas of geometry. Here I'll describe how this goes for nilpotent Lie groups, for a class of Riemannian manifolds closely related to a nilpotent Lie group structure. There are also some infinite dimensional analogs but I won't go into that here. The analytic ideas are not so different from the classical Fourier transform and Fourier inversion theories in one real variable.

Here I'll give a few brief indications of this beautiful topic. References, proofs and related topics for the finite dimensional theory can be found a recent AMS Monograph/Survey volume [1]. If you are interested in the infinite dimensional theory you may also wish to look at the article [2] appear soon in *Mathematische Annalen*.

In Section 1 I'll recall a few basic facts on classical Fourier theory and note the connection with the theory of locally compact abelian groups and their unitary representations. In Section 2 we look at the first noncommutative locally compact groups, the Heisenberg groups H_n . We describe their unitary representations, Fourier transform theory and Fourier inversion formula.

The coadjoint orbit picture is the best way to understand representations of nilpotent Lie groups. It is guided by the example of the Heisenberg group. We indicate that theory in Section 3. Then in Section 4 we come to a class of connected, simply connected, nilpotent Lie groups with many of the good analytic properties of vector groups and Heisenberg groups. Those are the simply connected, nilpotent Lie groups with square integrable representations.

In Section 5 we push some of the analysis to a class of homogeneous spaces where the techniques and results are analogous to those of locally compact abelian groups. Those are the commutative spaces G/K , i.e. the Gelfand pairs (G, K) . We have already had a glimpse of this in Section 2 for the semidirect products $H_n \rtimes K$ and the Riemannian homogeneous spaces $(H_n \rtimes K)/K$ where K is a compact group of automorphisms of H_n . In Section 6 we look more generally at Fourier transform theory and the Fourier inversion formula for commutative nilmanifolds $(N \rtimes K)/K$ where N is a simply connected, nilpotent Lie groups with square integrable representations and K is a compact group of automorphisms of N .

As indicated earlier, Fourier analysis for nilpotent Lie groups N and commutative nilmanifolds $(N \rtimes K)/K$, where N has square integrable representations, has recently been extended to some classes direct limit groups and spaces.

1 Classical Fourier Series

Let's recall the Fourier series development for a function f of one variable that is periodic of period 2π . One views f as a function on the circle $S = \{e^{ix}\}$. The circle S is a multiplicative group and we expand f in terms of the unitary characters

$$\chi_n : S \rightarrow S \text{ by } \chi_n(x) = e^{inx}, \text{ continuous group homomorphism.}$$

Then the Fourier inversion formula is

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\chi_n$$

where the Fourier transform

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx} dx = \langle f, \chi_n \rangle_{L^2(S)}.$$

The point is that f is a linear combination of the χ_n with coefficients given by the Fourier transform \widehat{f} . This uses the topological group structure and the rotation-invariant measure on S .

One has a similar situation when the compact group S is replaced by a finite dimensional real vector space V . Let V^* denote its linear dual space. If $f \in L^1(V) \cap L^2(V)$ the Fourier inversion formula is

$$f(x) = \left(\frac{1}{2\pi}\right)^{m/2} \int_{V^*} \widehat{f}(\xi)e^{ix \cdot \xi} d\xi$$

where the Fourier transform is

$$\widehat{f}(\xi) = \left(\frac{1}{2\pi}\right)^{m/2} \int_V f(x)e^{-ix \cdot \xi} dx = \langle f, \chi_\xi \rangle_{L^2(V)}.$$

Again, f is a linear combination (this time it is a continuous linear combination) of the unitary characters $\chi_\xi(x) = e^{x \cdot \xi}$ on V , and the coefficients of the linear combination are given by the Fourier transform \widehat{f} .

It is the same story for locally compact abelian groups G . The unitary characters form a group

$$\widehat{G} = \{\chi : G \rightarrow S \text{ continuous homomorphisms}\}$$

with composition $(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x)$. It is locally compact with the weak topology for the evaluation maps $ev_x : \chi \mapsto \chi(x)$. If $f \in L^1(G) \cap L^2(G)$ the Fourier inversion formula is

$$f(x) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(x)d\chi$$

where the Fourier transform

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx = \langle f, \chi \rangle_{L^2(G)}.$$

As in the Euclidean cases, Fourier inversion expresses the function f as a (possibly continuous) linear combination of unitary characters on G , where the coefficients of the linear combination are given by the Fourier transform.

In this context, $f \mapsto \widehat{f}$ preserves L^2 norm and extends by continuity to an isometry of $L^2(G)$ onto $L^2(\widehat{G})$. In effect this expresses $L^2(G)$ as a (possibly continuous) sum of G -modules,

$$L^2(G) = \int_{\widehat{G}} \mathbb{C}_\chi d\chi \text{ where } \mathbb{C}_\chi \text{ is spanned by } \chi.$$

In this direct integral decomposition $d\chi$ could be replaced by any equivalent measure, so that decomposition is not as precise as the Fourier inversion formula.

2 The Heisenberg group

Next, we see what happens when we weaken the commutativity condition. The first case of that is the case of the Heisenberg group. There the Fourier transform and Fourier inversion are in some sense the same as in the classical case of a vector group, except that some of the integration occurs in the character formula and the rest in integration over the unitary dual.

The Heisenberg group of real dimension $2m + 1$ is

$$H_m = \text{Im } \mathbb{C} + \mathbb{C}^m \text{ with group law } (z, w)(z', w') = (z + z' + \text{Im } \langle w, w' \rangle, w + w')$$

where Im denotes imaginary component (as opposed to the coefficient of $\sqrt{-1}$), $z, z' \in Z := \text{Im } \mathbb{C}$ and $w, w' \in W := \mathbb{C}^m$. Its Lie algebra, the Heisenberg algebra, is

$$\mathfrak{h}_m = \mathfrak{z} + \mathfrak{w} = \text{Im } \mathbb{C} + \mathbb{C}^m \text{ with } [z + w, z' + w'] = (z + z' + 2 \text{Im } \langle w, w' \rangle) + (w + w').$$

Here $Z = \exp(\mathfrak{z})$ is both the center and the derived group of H_m , and its complement $W = \exp(\mathfrak{w}) \cong \mathbb{R}^{2m}$.

Unitary characters have to annihilate the derived group of H_m , in other words factor through H_m/Z , so the only functions that can be expanded in unitary characters are the functions that are lifted from H_m/Z . Thus we have to consider something more general.

The space \widehat{H}_m of (equivalence classes of) irreducible unitary representations of H_m breaks into two parts, one consisting of the 1-dimensional representations and the other of the infinite dimensional representation. This goes as follows.

- One-dimensional representations. These are the ones that annihilate the center Z , and are given by the unitary characters χ_ξ , $\xi \in W^*$, on $W \cong \mathbb{R}^{2m}$.
- Infinite dimensional representations. These are the $\pi_\zeta = \text{Ind}_N^{H_m}(\chi_\zeta)$ where

$$N = \text{Im } \mathbb{C} + \mathbb{R}^m \subset H_m \text{ and } \zeta \in \mathfrak{z}^* \setminus \{0\}.$$

Recall the definition of the induced representation $\pi_\zeta = \text{Ind}_N^{H_m}(\chi_\zeta)$. Here χ_ζ extends from Z to N by $\chi_\zeta(z, w) = \chi_\zeta(z)$. Thus we have a unitary line bundle over H_m/N associated to the principal N -bundle $H_m \rightarrow H_m/N$ by the action $w : t \mapsto \chi_\zeta(w)t$ of N on \mathbb{C} . Now π_ζ is the natural action of H_m on the space of L^2 sections of that line bundle.

The classical ‘‘Uniqueness of the Heisenberg commutation relations’’ says that ζ determines the class $[\pi_\zeta] \in \widehat{H_m}$. And restriction to Z shows that $[\pi_\zeta] = [\pi_{\zeta'}]$ just when $\zeta = \zeta'$.

Using the fact that ζ determines $[\pi_\zeta]$, one realizes $[\pi_\zeta]$ by an action of H_m on the Hilbert space \mathcal{H}_m of Hermite polynomials on \mathbb{C}^m . The maximal compact subgroup of $\text{Aut}(H_m)$ is the unitary group $U(m)$. Its action is

$$g : (z, w) \mapsto (z, g(w)).$$

Result: π_ζ extends to an irreducible unitary representation $\tilde{\pi}_\zeta$ of the semidirect product $H_m \rtimes U(m)$ on \mathcal{H}_m . So if K is any closed subgroup of $U(m)$ then $\tilde{\pi}_\zeta|_{H_m \rtimes K}$ is an irreducible unitary representation of $H_m \rtimes K$ on \mathcal{H}_m . The Mackey Little Group theory says that $\widehat{H_m \rtimes K} = \{[\tilde{\pi}_\zeta \otimes \kappa] \mid [\kappa] \in \widehat{K} \text{ and } \zeta \in \mathfrak{z}^* \setminus \{0\}\}$.

3 Representations and Coadjoint Orbits

Kirillov theory for connected simply connected nilpotent Lie groups N realizes their unitary representations in terms of the the coadjoint representation of N , that is, the representation ad^* of N on the linear dual space \mathfrak{n}^* of its Lie algebra \mathfrak{n} .

On the group level the coadjoint representation is given by $(\text{Ad}^*(n)f)(\xi) = f(\text{Ad}(n)^{-1}\xi)$. Write \mathcal{O}_f for the (coadjoint) orbit $\text{Ad}^*(N)f$ of the linear functional f . Consider the anti-symmetric bilinear form b_f on \mathfrak{n}^* given by $b_f(\xi, \eta) = f([\xi, \eta])$. The kernel of b_f is the Lie algebra of the isotropy subgroup of N at f . Thus b_f defines an $\text{Ad}^*(N)$ -invariant symplectic form ω_f on the coadjoint orbit \mathcal{O}_f . The symplectic homogeneous space $(\mathcal{O}_f, \omega_f)$ leads to a unitary representation class $[\pi_f] \in \widehat{N}$, as follows.

Let N_f denote the $\text{Ad}^*(N)$ -stabilizer of f . Its Lie algebra \mathfrak{n}_f is the annihilator of f , in other words $\mathfrak{n}_f = \{\nu \in \mathfrak{n} \mid f(\nu, \mathfrak{n}) = 0\}$. A (real) **polarization** for f is a subalgebra $\mathfrak{p} \subset \mathfrak{n}$ that contains \mathfrak{n}_f , has dimension given by $\dim(\mathfrak{p}/\mathfrak{n}_f) = \frac{1}{2} \dim(\mathfrak{n}/\mathfrak{n}_f)$, and satisfies $f([\mathfrak{p}, \mathfrak{p}]) = 0$. Under the differential $\mathfrak{n} \rightarrow T_f(\mathcal{O}_f)$ of $N \rightarrow \mathcal{O}_f$, real polarizations for f are in one to one correspondence with N -invariant integrable Lagrangian distributions on $(\mathcal{O}_f, \omega_f)$.

Fix a real polarization \mathfrak{p} for f and let $P = \exp(\mathfrak{p})$. It is the analytic subgroup of N for \mathfrak{p} and it is a closed, connected, simply connected subgroup of N . In particular $e^{if} : P \rightarrow \mathbb{C}$ is a well defined unitary character. That defines a unitary representation

$$\pi_f = \pi_{f,\mathfrak{p}} = \text{Ind}_P^N(e^{if})$$

of N . The basic facts are given by

Theorem 3.1 *Let N be a connected simply connected nilpotent Lie group and $f \in \mathfrak{n}^*$.*

1. *There exist real polarizations \mathfrak{p} for f .*
2. *If \mathfrak{p} is a real polarization for f then the unitary representation $\pi_{f,\mathfrak{p}}$ is irreducible.*
3. *If \mathfrak{p} and \mathfrak{p}' are real polarizations for f then the unitary representations $\pi_{f,\mathfrak{p}}$ and $\pi_{f,\mathfrak{p}'}$ are equivalent, so the class $[\pi_f] \in \widehat{N}$ is well defined.*
4. *If $[\pi] \in \widehat{N}$ then there exists $h \in \mathfrak{n}^*$ such that $[\pi] = [\pi_h]$.*

In other words, $f \mapsto \pi_{f,\mathfrak{p}}$ induces a one to one map of $\mathfrak{n}^*/\text{Ad}^*(N)$ onto \widehat{N} .

To see just how this works, consider the case where N is the Heisenberg group H_m , and let $f \in \mathfrak{h}_m^*$. Here the center $\mathfrak{z} = \text{Im } \mathbb{C}$ and its complement $\mathfrak{v} = \mathbb{C}^n$. Decompose $\mathfrak{v} = \mathfrak{u} + \mathfrak{w}$ where $\mathfrak{u} = \mathbb{R}^n$ and $\mathfrak{w} = i\mathbb{R}^n$. Note $\text{Im } \langle \mathfrak{u}, \mathfrak{u} \rangle = 0 = \text{Im } \langle \mathfrak{w}, \mathfrak{w} \rangle$. If $f(\mathfrak{z}) = 0$ we have the real polarization $\mathfrak{p} = \mathfrak{h}_m$. If $f(\mathfrak{z}) \neq 0$ we have the real polarization $\mathfrak{p} = \mathfrak{z} + \mathfrak{u}$. That demonstrates Theorem 3.1(1).

If $f(\mathfrak{z}) = 0$ then $\pi_{f,\mathfrak{p}}$ is a unitary character on H_n , automatically irreducible. Now suppose $f(\mathfrak{z}) \neq 0$ and $\mathfrak{p} = \mathfrak{z} + \mathfrak{u}$. Then $\pi_{f,\mathfrak{p}}$ is a representation of H_n on $L^2(G/P) = L^2(W)$ where $W = \exp(\mathfrak{w}) \cong \mathbb{R}^n$. Then $\pi_{f,\mathfrak{p}}(H_n)$ acts by all translations on W and by scaling that distinguishes the integrands of $L^2(W) = \int_{\mathfrak{w}^*} e^{i\langle \xi, \cdot \rangle} \mathbb{C} d\xi$, so it is irreducible. That demonstrates Theorem 3.1(2).

If $f(\mathfrak{z}) = 0$ then \mathfrak{h}_n is the only real polarization for f . Now suppose $f(\mathfrak{z}) \neq 0$ and consider the case where $\mathfrak{p} = \mathfrak{z} + \mathfrak{u}$ and $\mathfrak{p}' = \mathfrak{z} + \mathfrak{w}$. Then $\omega = \text{Im } h(\cdot, \cdot)$ pairs \mathfrak{u} with \mathfrak{w} , and the Fourier transform $\mathcal{F} : L^2(U) \cong L^2(W)$ intertwines $\pi_{f,\mathfrak{p}}$ with $\pi_{f',\mathfrak{p}'}$. More generally, if \mathfrak{p} and \mathfrak{p}' are any two real polarizations for f , then we write $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{p}') + \mathfrak{u}'$, $\mathfrak{p}' = (\mathfrak{p} \cap \mathfrak{p}') + \mathfrak{w}'$ and $\mathfrak{p} \cap \mathfrak{p}' = \mathfrak{z} + \mathfrak{v}'$. That done, ω pairs \mathfrak{u}' with \mathfrak{w}' , and the corresponding Fourier transform $\mathcal{F} : L^2(U') \cong L^2(W')$ combines with the identity transformation of $L^2(V')$ to give a map $L^2(V') \widehat{\otimes} L^2(U') \cong L^2(V') \widehat{\otimes} L^2(W')$ that intertwines $\pi_{f,\mathfrak{p}}$ with $\pi_{f',\mathfrak{p}'}$. This demonstrates Theorem 3.1(3).

Now Theorem 3.1(4) follows from the considerations we outlined in Section 2. In the terminology there, the infinite dimensional irreducible unitary representation π_ζ of H_n is equivalent to π_f whenever $f \in \mathfrak{n}^*$ such that $f(z) = \langle \zeta, x \rangle$ for every $z \in \mathfrak{z}$. In particular, if $f(\mathfrak{z}) \neq 0$ where \mathfrak{z} is the center of the Heisenberg algebra, then the coadjoint orbit $\mathcal{O}_f = f + \mathfrak{z}^\perp$, where $\mathfrak{z}^\perp := \{h \in \mathfrak{n}^* \mid h(\mathfrak{z}) = 0\}$. Of course one can also verify this by direct computation.

4 Square Integrable Representations

In this section N is a connected simply connected nilpotent Lie group and Z is its center. If $\zeta \in \widehat{Z}$ we denote $\widehat{N}_\zeta = \{[\pi] \in \widehat{N} \mid \pi|_Z \text{ is a multiple of } \zeta\}$. The corresponding L^2 space is

$$(4.2) \quad L^2(N/Z : \zeta) := \left\{ f : N \rightarrow \mathbb{C} \text{ measurable} \mid \begin{array}{l} f(nz) = \zeta(z)^{-1} f(n) \text{ and} \\ \int_{N/Z} |f(n)|^2 d\mu_{N/Z}(nZ) < \infty \end{array} \right\}.$$

The inner product $\langle f, h \rangle_\zeta = \int_{N/Z} f(n) \overline{h(n)} d\mu_{N/Z}(nZ)$ is well defined on the relative L^2 space $L^2(N/Z : \zeta)$. Each \widehat{N}_ζ is a measurable subset of \widehat{N} , and $\widehat{N} = \bigcup_{\zeta \in \widehat{Z}} \widehat{N}_\zeta$. Here $L^2(N) = \int_{\widehat{Z}} L^2(N/Z : \zeta)$. This decomposes the left regular representation of N as

$$\ell = \text{Ind}_{\{1\}}^N(1) = \text{Ind}_Z^N \text{Ind}_{\{1\}}^Z(1) = \text{Ind}_Z^N \int_{\widehat{Z}} \zeta d\zeta = \int_{\widehat{Z}} \text{Ind}_Z^N \zeta d\zeta = \int_{\widehat{Z}} \ell_\zeta d\zeta$$

where $\ell_\zeta = \text{Ind}_Z^N \zeta$ is the left regular representation of N on $L^2(N/Z : \zeta)$. The corresponding expansion for functions,

$$f(n) = \int_{\widehat{Z}} f_\zeta(n) d\zeta \text{ where } f_\zeta(n) = \int_Z f(nz) \zeta(z) d\mu_Z(z),$$

is just Fourier inversion on the commutative locally compact group Z .

Now we describe some results of Moore and myself on square integrable representations in this context. The first observation is

Theorem 4.3 *Let N be a connected simply connected nilpotent Lie group and $\zeta \in \widehat{Z}$. If $[\pi] \in \widehat{N}_\zeta$ then the following conditions are equivalent.*

1. *There exist nonzero $u, v \in H_\pi$ such that $|f_{u,v}| \in L^2(N/Z)$, i.e., $f_{u,v} \in L^2(N/Z : \zeta)$.*
2. *The coefficient $|f_{u,v}| \in L^2(N/Z)$, equivalently $f_{u,v} \in L^2(N/Z : \zeta)$, for all $u, v \in H_\pi$.*
3. *$[\pi]$ is a discrete summand of ℓ_ζ .*

A representation class $[\pi] \in \widehat{N}$ is L^2 or **square integrable** or **relative discrete series** if its coefficients $f_{u,v}(n) = f_{\pi \cdot u, v}(n) := \langle u, \pi(n)v \rangle$ satisfy $|f_{u,v}| \in L^2(N/Z)$, in other words if its coefficients are square integrable modulo Z . Theorem 4.3 says that it is sufficient to check this for just one nonzero coefficient, and Theorem 4.3(3) justifies the term ‘‘relative discrete series’’.

We say that N **has square integrable representations** if at least one class $[\pi] \in \widehat{N}$ is square integrable. These representations satisfy an analog of the Schur orthogonality relations:

Theorem 4.4 *Let N be a connected simply connected nilpotent Lie group. If $\zeta \in \widehat{Z}$ and $[\pi] \in \widehat{N}_\zeta$ is square integrable then there is a number $\deg(\pi) > 0$ such that the coefficients of π satisfy*

$$(4.5) \quad \int_{N/Z} f_{u_1, v_1}(n) \overline{f_{u_2, v_2}(n)} d\mu_{N/Z}(nZ) = \frac{1}{\deg(\pi)} \langle u_1, u_2 \rangle \overline{\langle v_1, v_2 \rangle}$$

for all $u_i, v_i \in H_\pi$. If $[\pi_1], [\pi_2] \in \widehat{N}_\zeta$ are inequivalent square integrable representations then their coefficients are orthogonal in $L^2(N/Z : \zeta)$,

$$(4.6) \quad \int_{N/Z} f_{\pi_1: u_1, v_1}(n) \overline{f_{\pi_2: u_2, v_2}(n)} d\mu_{N/Z}(nZ) = 0,$$

for all $u_1, v_1 \in H_{\pi_1}$ and $u_2, v_2 \in H_{\pi_2}$.

The number $\deg(\pi)$ is the **formal degree** of $[\pi]$. It plays the same role in Theorem 4.4 as that played by the degree in the Schur orthogonality relations for compact groups. In general, $\deg(\pi)$ depends on normalization of Haar measure: a rescaling of Haar measure $\mu_{N/Z}$ of N/Z to $c\mu_{N/Z}$ rescales formal degrees $\deg(\pi)$ to $\frac{1}{c}\deg(\pi)$. We don't see this for compact groups because there we always scale Haar measure to total mass 1.

Theorems 4.3 and 4.4 only require that N be a locally compact group of Type I and that Z be a closed subgroup of the center of N . They can be understood as special cases of Hilbert algebra theory. Here we related them to the Kirillov theory.

Given $f \in \mathfrak{n}^*$ we have the bilinear form $b_f(x, y) = f([x, y])$, the coadjoint orbit $\mathcal{O}_f = \text{Ad}^*(N)f$, the associated representation $[\pi_f]$, and the character $\zeta \in \widehat{Z}$ such that $[\pi_f] \in \widehat{N}_\zeta$. Note that $f|_{\mathfrak{z}}$ determines the affine subspace $f + \mathfrak{z}^\perp$ in \mathfrak{n}^* .

Theorem 4.7 *Let N be a connected simply connected nilpotent Lie group and $f \in \mathfrak{n}^*$. Then the following conditions are equivalent.*

1. $[\pi_f]$ is square integrable.
2. The left regular representation ℓ_ζ of N on $L^2(N/Z : \zeta)$ is primary.
3. $\mathcal{O}_f = f + \mathfrak{z}^\perp$, determined by the restriction $f|_{\mathfrak{z}}$.
4. b_f is nondegenerate on $\mathfrak{n}/\mathfrak{z}$.

Recall the notion of the Pfaffian $\text{Pf}(\omega)$ of an antisymmetric bilinear form ω on a finite dimensional real vector space V relative to a volume form ν on V . If $\dim V$ is odd then by definition $\text{Pf}(\omega) = 0$. If $\dim V = 2m$ even, then ω^m is a multiple of ν , and by definition that multiple is $\text{Pf}(\omega)$; in other words $\omega^m = \text{Pf}(\omega)\nu$. The Pfaffian is the square root of the determinant on antisymmetric bilinear forms.

Fix a volume element ν on $\mathfrak{v} := \mathfrak{n}/\mathfrak{z}$. If $f \in \mathfrak{n}^*$ we view $\omega_f(x, y) = f([x, y])$ as an antisymmetric bilinear form on \mathfrak{v} . Define $P(f) := \text{Pf}(\omega_f)$. Then P is a homogeneous polynomial function on \mathfrak{n}^* , and $P(f)$ depends only on $f|_{\mathfrak{z}}$. So there is a homogeneous polynomial function (which we also denote P) on \mathfrak{z}^* such that $P(f) = P(f|_{\mathfrak{z}})$.

In the case of the Heisenberg group H_m , P is the homogeneous polynomial $P(\zeta) = \zeta(z_0)^m$ of degree m on \mathfrak{z}^* . Here the choice of nonzero $z_0 \in \mathfrak{z}$ is a normalization, in effect a choice of unit vector. As described in Theorem 4.10 below, this also gives the formal degree of $[\pi_f]$ where $\zeta = f|_{\mathfrak{z}}$. Further, as described in Theorem 4.12, it gives the Plancherel measure on $\widehat{H_m}$.

In view of Theorem 4.7 we now have

Theorem 4.8 *The representation π_f is square integrable if and only if the Pfaffian polynomial $P(f|_{\mathfrak{z}}) \neq 0$. In particular $\phi : f|_{\mathfrak{z}} \mapsto [\pi_f]$ defines a bijection from $\{\lambda \in \mathfrak{z}^* \mid P(\lambda) \neq 0\}$ onto $\{[\pi] \in \widehat{N} \mid [\pi] \text{ is square integrable}\}$.*

One can view the polynomial P as an element of the symmetric algebra $S(\mathfrak{z})$, and since \mathfrak{z} is commutative that symmetric algebra is the same as the universal enveloping algebra \mathfrak{Z} . From this one can prove

Corollary 4.9 *The group N has square integrable representations if and only if the inclusion $\mathfrak{Z} \hookrightarrow \mathfrak{N}$ of universal enveloping algebras, induced by $\mathfrak{z} \hookrightarrow \mathfrak{n}$, maps \mathfrak{Z} onto the center of \mathfrak{N} .*

Both formal degree and the polynomial P are scaled by $1/c$ when Haar measure on N/Z is scaled by c , so the following is independent of normalization of Haar measure on N/Z .

Theorem 4.10 *The formal degree of a square integrable representation $[\pi_f] = \phi(f|_{\mathfrak{z}})$ is given by $\text{deg}(\pi_f) = |P(f|_{\mathfrak{z}})|$.*

As in the semisimple case, the **infinitesimal character** of a representation class $[\pi] \in \widehat{N}$ is the associative algebra homomorphism $\xi_\pi : \text{Cent}(\mathfrak{N}) \rightarrow \mathbb{C}$ from the center of the enveloping algebra, such that $d\pi(\zeta)$ is scalar multiplication by $\chi_\pi(\zeta)$. (Initially this holds only on C^∞ vectors, but they are dense in H_π , so by continuity it holds on all vectors.) If $\zeta \in \mathfrak{z}$ then $\chi_{\pi_f}(\zeta) = if(\zeta)$. Now, from Theorem 4.10,

Corollary 4.11 *If $[\pi] \in \widehat{N}$ then the formal degree $\text{deg}(\pi) = |\chi_\pi(P)|$ where we understand the formal degree of a non square integrable representation to be zero.*

For the Plancherel formula and Fourier inversion we must normalize Haar measures. Choose Haar measures μ_Z and $\mu_{N/Z}$; they define a Haar measure μ_N by $d\mu_N = d\mu_{N/Z} d\mu_Z$, i.e.

$$\int_N f(n) d\mu_N(n) = \int_{N/Z} \left(\int_Z f(nz) d\mu_Z(z) \right) d\mu_{N/Z}(nZ).$$

Now we have Lebesgue measures ν_Z , $\nu_{N/Z}$ and ν_N on \mathfrak{z} , $\mathfrak{n}/\mathfrak{z}$ and \mathfrak{n} specified by the condition that the exponential map have Jacobian 1 at 0, and they satisfy $d\nu_N = d\nu_{N/Z} d\nu_Z$. Normalize Lebesgue measures on the dual spaces by the condition that Fourier transform is an isometry; that gives Lebesgue measures ν_Z^* , $\nu_{N/Z}^*$ and ν_N^* such that $d\nu_N^* = d\nu_{N/Z}^* d\nu_Z^*$.

Theorem 4.12 *Let N have square integrable representations. Let $c = m!2^m$ where $2m$ is the maximum dimension of the $\text{Ad}^*(N)$ -orbits in \mathfrak{n}^* . Then Plancherel measure for N is concentrated on the square integrable classes and its image under the map*

$$\phi^{-1} : \{[\pi] \in \widehat{N} \mid [\pi] \text{ is square integrable}\} \rightarrow \{\lambda \in \mathfrak{z}^* \mid P(\lambda) \neq 0\}$$

of Theorem 4.8 is $c|P(x)|d\nu_Z^*(x)$.

5 Commutative Spaces – Generalities

We just described the Fourier transform and Fourier inversion formulae for H_m — and a somewhat larger class of connected simply connected nilpotent Lie groups. Now we edge toward a more geometric setting, that of commutative spaces, which is a common generalization of Riemannian symmetric spaces, locally compact abelian groups and homogeneous graphs. It is interesting and precise for the cases that involve connected simply connected nilpotent Lie groups with square integrable representations.

A *commutative space* G/K , equivalently a *Gelfand pair* (G, K) , consists of a locally compact group G and a compact subgroup K such that the convolution algebra $L^1(K \backslash G/K)$ is commutative. There are several other formulations. Specifically, the following are equivalent.

1. (G, K) is a Gelfand pair, i.e. $L^1(K \backslash G/K)$ is commutative under convolution.
2. If $g, g' \in G$ then $\mu_{KgK} * \mu_{Kg'K} = \mu_{Kg'K} * \mu_{KgK}$ (convolution of measures on $K \backslash G/K$).
3. $C_c(K \backslash G/K)$ is commutative under convolution.
4. The measure algebra $\mathcal{M}(K \backslash G/K)$ is commutative.
5. The (left regular) representation of G on $L^2(G/K)$ is multiplicity free.

If G is a connected Lie group one can add

6. The algebra $\mathcal{D}(G, K)$ of G -invariant differential operators on G/K is commutative.

Commutative spaces G/K are important for a number of reasons. First, they are manageable because their basic harmonic analysis is very similar to that of locally compact abelian groups. We will describe that in a moment. Second, in the Lie group cases, most of the G/K carry invariant weakly symmetric Riemannian metrics, which have properties very similar to those of Riemannian symmetric spaces. Third, the invariant differential operators and corresponding spherical functions play a definite role in special function theory. And fourth,

in the nilpotent Lie groups setting, there is some interesting interplay between geometry and hypoellipticity.

We only consider the basic harmonic analysis, here in general and in Section 6 for the case of commutative nilmanifolds.

Analysis on locally compact abelian groups is based on decomposition of functions in terms of unitary characters. In the classical euclidean case these are just the complex exponentials $\chi_\xi : V \rightarrow \mathbb{C}$, $\xi \in V^*$, given by $\chi_\xi = e^{ix \cdot \xi}$. For a commutative space G/K the appropriate replacements are the positive definite spherical functions, defined as follows.

A continuous K -bi-invariant function $\varphi : G \rightarrow \mathbb{C}$ is K -spherical if $\varphi(1) = 1$ and $f \mapsto (f * \varphi)(1)$ is a homomorphism $C_c(K \backslash G / K) \rightarrow \mathbb{C}$. Equivalent: φ is not identically zero, and if $g_1, g_2 \in G$ then

$$\varphi(g_1)\varphi(g_2) = \int_K \varphi(g_1 k g_2) dk.$$

A function $\varphi : G \rightarrow \mathbb{C}$ is *positive definite* if $\sum \varphi(g_j^{-1} g_i) \overline{c_j} c_i \geq 0$ whenever $\{c_1, \dots, c_n\} \subset \mathbb{C}$ and $\{g_1, \dots, g_n\} \subset G$.

Denote $\mathcal{P} = \mathcal{P}(G, K)$: positive definite K -spherical functions on G . There is a one-one correspondence $\varphi \leftrightarrow \pi_\varphi$ between \mathcal{P} and the irreducible unitary representations π of G that have a K -fixed unit vector v . It is given by $\varphi(g) = \langle v, \pi(g)v \rangle_{\mathcal{H}_\pi}$. We have the *spherical transform*

$$\mathcal{S} : f \mapsto \widehat{f} \text{ from } L^1(K \backslash G / K) \text{ to functions on } \mathcal{P}$$

defined by

$$\mathcal{S}(f)(\varphi) = \widehat{f}(\varphi) = (f * \varphi)(1) = \int_G f(g)\varphi(g^{-1})dg.$$

The corresponding spherical inversion formula is

$$f(g) = \int_{\mathcal{P}} \widehat{f}(\varphi)\varphi(g)d\mu(\varphi).$$

Here \mathcal{P} has natural structure of locally compact space and μ is called Plancherel measure. The spherical transform

$$\mathcal{S} : L^1(K \backslash G / K) \cap L^2(K \backslash G / K) \rightarrow L^2(\mathcal{P}, \mu)$$

preserves L^2 norm and extends by continuity to an isometry

$$\mathcal{S} : L^2(K \backslash G / K) \cong L^2(\mathcal{P}, \mu).$$

Note that \mathcal{S} can only be given by its defining integral expression when that integral converges. This is why it has to be extended by L^2 continuity. Of course this problem is already present with the classical Fourier transform on \mathbb{R} .

The Plancherel Formula $\mathcal{S} : L^2(K \backslash G / K) \cong L^2(\mathcal{P}, \mu)$ gives a continuous direct sum (direct integral) decomposition

$$L^2(K \backslash G / K) \cong \int_{\mathcal{P}} \mathbb{C}_{\varphi} d\mu(\varphi).$$

This extends to a continuous direct sum decomposition

$$L^2(G / K) \cong \int_{\mathcal{P}} \mathcal{H}_{\pi_{\varphi}} d\mu(\varphi).$$

Of course all this depends on knowledge of the Plancherel measure μ .

6 Commutative Nilmanifolds

Theorem of Carcano (special case): Let $K \subset U(m)$ acting on \mathbb{C}^m , where $\mathfrak{h}_m = \text{Im } \mathbb{C} + \mathbb{C}^m$ with center $\text{Im } \mathbb{C}$. Then $(H_m \rtimes K) / K$ is commutative if and only if the representation of $K_{\mathbb{C}}$, on the ring of all polynomials on \mathbb{C}^m , is multiplicity free.

Kač classified the connected K_m that are irreducible on \mathbb{C}^m :

	Group K_m	Acting on		Group K_m	Acting on
1	$SU(m)$	$\mathbb{C}^m, m \geq 2$	8	$U(n)$	$\mathbb{C}^m = \Lambda^2(\mathbb{C}^n)$
2	$U(m)$	$\mathbb{C}^m, m \geq 1$	9	$SU(\ell) \times SU(n)$	$\mathbb{C}^m = \mathbb{C}^{\ell} \otimes \mathbb{C}^n, \ell \neq n$
3	$Sp(n)$	$\mathbb{C}^m = \mathbb{C}^{2n}$	10	$S(U(\ell) \times U(n))$	$\mathbb{C}^m = \mathbb{C}^{\ell} \otimes \mathbb{C}^n$
4	$U(1) \times Sp(n)$	$\mathbb{C}^m = \mathbb{C}^{2n}$	11	$U(2) \times Sp(n)$	$\mathbb{C}^m = \mathbb{C}^2 \otimes \mathbb{C}^{2n}$
5a	$U(1) \times SO(2n)$	$\mathbb{C}^m = \mathbb{C}^{2n}$	12	$SU(3) \times Sp(n)$	$\mathbb{C}^m = \mathbb{C}^3 \otimes \mathbb{C}^{2n}$
5b	$U(1) \times SO(2n+1)$	$\mathbb{C}^m = \mathbb{C}^{2n+1}$	13	$U(3) \times Sp(n)$	$\mathbb{C}^m = \mathbb{C}^3 \otimes \mathbb{C}^{2n}$
6	$U(n), n \geq 2$	$\mathbb{C}^m = S^2(\mathbb{C}^n)$	14	$SU(n) \times Sp(4)$	$\mathbb{C}^m = \mathbb{C}^n \otimes \mathbb{C}^8$
7	$SU(n), n \text{ odd}$	$\mathbb{C}^m = \Lambda^2(\mathbb{C}^n)$	15	$U(n) \times Sp(4)$	$\mathbb{C}^m = \mathbb{C}^n \otimes \mathbb{C}^8$

A *commutative nilmanifold* is a commutative space G/K where some connected closed nilpotent subgroup of G is transitive on G/K .

Example: G/K where $G = H_m \rtimes K_m$ and $K = K_m$, where K_m occurs in the table above.

Fact: Let G/K be commutative. If a conn closed nilpotent subgroup N of G is transitive then N is the nilradical of G , N is abelian or 2-step nilpotent, and $G = N \rtimes K$.

In particular: Commutative nilmanifolds have form G/K where $G/K = (N \rtimes K) / K$, N is not so different from the Heisenberg group and $K \subset \text{Aut}(N)$.

More examples: a commutative nilmanifold $(G = N \rtimes K) / K$ is *irreducible* if K acts irreducibly on $\mathfrak{n} / [\mathfrak{n}, \mathfrak{n}]$, *maximal* if it is not of the form $(\tilde{G} / \tilde{Z}, \tilde{K} / \tilde{Z})$ with $\{1\} \neq \tilde{Z} \subset \tilde{K}$ central in \tilde{G} . They have been classified by Vinberg. Let $\mathfrak{n} = \mathfrak{z} + \mathfrak{w}$ where \mathfrak{z} is the center and $\text{Ad}(K)\mathfrak{w} = \mathfrak{w}$. Then \mathfrak{z} is the center of \mathfrak{n} , $\mathfrak{w} \cong \mathfrak{n} / [\mathfrak{n}, \mathfrak{n}]$ as K -module, and the classification is

	Group K	\mathfrak{w}	\mathfrak{z}
1	$SO(n)$	\mathbb{R}^n	Skew $\mathbb{R}^{n \times n} = \mathfrak{so}(n)$
2	$Spin(7)$	$\mathbb{R}^8 = \emptyset$	$\mathbb{R}^7 = \text{Im } \emptyset$
3	G_2	$\mathbb{R}^7 = \text{Im } \emptyset$	$\mathbb{R}^7 = \text{Im } \emptyset$
4	$U(1) \cdot SO(n), n \neq 4$	\mathbb{C}^n	$\text{Im } \mathbb{C}$
5	$(U(1) \cdot)SU(n)$	\mathbb{C}^n	$\Lambda^2 \mathbb{C}^n \oplus \text{Im } \mathbb{C}$
6	$SU(n), n$ odd	\mathbb{C}^n	$\Lambda^2 \mathbb{C}^n$
7	$SU(n), n$ odd	\mathbb{C}^n	$\text{Im } \mathbb{C}$
8	$U(n)$	\mathbb{C}^n	$\text{Im } \mathbb{C}^{n \times n} = \mathfrak{u}(n)$
9	$(U(1) \cdot)Sp(n)$	\mathbb{H}^n	$\text{Re } \mathbb{H}_0^{n \times n} \oplus \text{Im } \mathbb{H}$
10	$U(n)$	$S^2 \mathbb{C}^n$	\mathbb{R}
11	$(U(1) \cdot)SU(n), n \geq 3$	$\Lambda^2 \mathbb{C}^n$	\mathbb{R}
12	$U(1) \cdot Spin(7)$	\mathbb{C}^8	$\mathbb{R}^7 \oplus \mathbb{R}$
13	$U(1) \cdot Spin(9)$	\mathbb{C}^{16}	\mathbb{R}
14	$(U(1) \cdot)Spin(10)$	\mathbb{C}^{16}	\mathbb{R}
15	$U(1) \cdot G_2$	\mathbb{C}^7	\mathbb{R}
16	$U(1) \cdot E_6$	\mathbb{C}^{27}	\mathbb{R}
17	$Sp(1) \times Sp(n), n \geq 2$	\mathbb{H}^n	$\text{Im } \mathbb{H} = \mathfrak{sp}(1)$
18	$Sp(2) \times Sp(n)$	$\mathbb{H}^{2 \times n}$	$\text{Im } \mathbb{H}^{2 \times 2} = \mathfrak{sp}(2)$
19	$(U(1) \cdot)SU(m) \times SU(n)$ $m, n \geq 3$	$\mathbb{C}^m \otimes \mathbb{C}^n$	\mathbb{R}
20	$(U(1) \cdot)SU(2) \times SU(n)$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\text{Im } \mathbb{C}^{2 \times 2} = \mathfrak{u}(2)$
21	$(U(1) \cdot)Sp(2) \times SU(n), n \geq 3$	$\mathbb{H}^2 \otimes \mathbb{C}^n$	\mathbb{R}
22	$U(2) \times Sp(n)$	$\mathbb{C}^2 \otimes \mathbb{H}^n$	$\text{Im } \mathbb{C}^{2 \times 2} = \mathfrak{u}(2)$
23	$U(3) \times Sp(n), n \geq 2$	$\mathbb{C}^3 \otimes \mathbb{H}^n$	\mathbb{R}

where the optional $U(1)$ is required in (5) when n is odd, in (11) when n is even, in (19) when $m = n$, in (20) when $n = 2$, and in (21) when $n \leq 4$. Here (9) was the first known case where G/K is not weakly symmetric (Lauret).

To make this explicit one needs to know the positive definite spherical functions and the Plancherel measure.

In the connected Lie group cases, the K -spherical functions on G are just the joint eigenfunctions for $\mathcal{D}(G, K)$, and in many cases this is how one finds them. We look at a few of those cases.

Case $(G, K) = (\mathbb{R}^n \rtimes K, K)$ where K is transitive on the unit sphere in \mathbb{R}^n . Then the invariant differential operators are the polynomials in the Laplacian $\Delta = -\sum \partial^2 / \partial x_i^2$, and the K -spherical functions are the radial eigenfunctions of Δ for real non-negative eigenvalue. They are the

$$\varphi_\xi(x) = (\|\xi\|r)^{-(n-2)/2} J_{(n-2)/2}(\|\xi\|r)$$

where $r = \|x\|$ and J_ν is the Bessel function of first kind and order ν .

Case $(G, K) = (H_n \rtimes U(n), U(n))$. In the coordinate $(z, w) \in \text{Im } \mathbb{C} + \mathbb{C}^n = H_n$ the invariant differential operators are the polynomials in $\partial / \partial z$, $\square = -\sum \partial^2 / \partial w_i^2$ and $\bar{\square}$. The

positive definite spherical functions corresponding to 1–dimensional representations are the

$$\varphi_\xi(z, w, k) = \frac{2^{n-1}(n-1)!}{(\|\xi\|\|w\|)^{n-1}} J_{n-1}(\|\xi\|\|w\|) \text{ for } 0 \neq \xi \in \mathfrak{w}^*$$

and those for infinite dimensional representations are the

$$\varphi_{\zeta, m} : (z, w, k) \mapsto \begin{cases} e^{i\zeta(z)} L_m^{(n-1)}(\zeta(z)\|w\|^2) e^{-\zeta(z)\|w\|^2/4} & \text{if } \zeta(i) > 0, \\ \overline{\varphi_{-\zeta, m}(z, w, k)} & \text{if } \zeta(i) < 0, \end{cases}$$

where $\zeta \in (\text{Im } \mathbb{C})^*$ and $L_m^{(n-1)}$ is the generalized Laguerre polynomial of order $n-1$ normalized to $L_m^{(n-1)}(0) = 1$.

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