ON THE SET WHERE THE ITERATES OF AN ENTIRE FUNCTION ARE BOUNDED

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ABSTRACT. We show that for a transcendental entire function the set of points whose orbit under iteration is bounded can have arbitrarily small positive Hausdorff dimension.

1. INTRODUCTION

The main objects studied in complex dynamics are the *Fatou set* $F(f)$ of a rational or entire function f, defined as the set of all points where the iterates f^n of f form a normal family, and the *Julia set* $J(f)$, which is the complement of $F(f)$. In the dynamics of transcendental entire functions – and this is the case we shall be concerned with – a fundamental role is also played by the escaping set

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

The first systematic study of this set was undertaken by Eremenko [9] who, among other results, showed that $I(f) \neq \emptyset$ and in fact $I(f) \cap J(f) \neq \emptyset$ for every transcendental entire function f. Moreover, $J(f) = \partial I(f)$. In this paper we will consider the set

$$K(f) = \{z \in \mathbb{C} : (f^n(z)) \text{ is bounded}\}.$$

As repelling periodic points are dense in the Julia set [1], the properties of $I(f)$ mentioned above also hold for $K(f)$; that is, $K(f) \cap J(f) \neq \emptyset$ and $J(f) = \partial K(f)$.

For a polynomial f the set $K(f)$ is called the *filled Julia set* of f and we have $K(f) = \mathbb{C} \setminus I(f)$, but for a transcendental entire function f there are points which are neither in $K(f)$ nor in $I(f)$, for example there are points in $J(f)$ whose orbit is dense in $J(f)$. However, there may also be points in $F(f)$ which are neither in $K(f)$ nor in $I(f)$; see [10, Example 1].

We denote the Hausdorff dimension and the packing dimension of a subset A of $\mathbb{C}$ by $\dim_{\mathrm{H}} A$ and $\dim_{\mathrm{P}} A$, respectively. We refer to Falconer's book [11] for the definition of these dimensions and further information. Here we only note that we always have $\dim_{\mathrm{H}} A \leq \dim_{\mathrm{P}} A$; see [11, p. 48]. By a result of Baker [2], the Julia set of a transcendental entire function f contains continua. In fact, even $I(f) \cap J(f)$ contains continua and thus $\dim_H(I(f) \cap J(f)) \geq 1$; cf. [16, Theorem 5] and [18, Theorem 1.3].

A major open question in transcendental dynamics is whether $\dim_H J(f) > 1$ for every transcendental entire function f. It was proved by Stallard ([20], see

1991 *Mathematics Subject Classification.* 37F10; 30D05; 37F35.
Supported by a Chinese Academy of Sciences Visiting Professorship for Senior International Scientists, Grant No. 2010 TIJ10. Also supported by the Deutsche Forschungsgemeinschaft, Be 1508/7-1, the EU Research Training Network CODY and the ESF Networking Programme HCAA..

also [7, 17]) that this is the case for functions in the Eremenko-Lyubich class B which consists of all transcendental entire functions for which the set of critical values and finite asymptotic values is bounded. Barański, Karpińska and Zdunik [3] showed that for $f \in B$ there exists a compact, invariant Cantor subset C of $J(f)$ with $\dim_{\mathrm{H}} C > 1$. In particular, $\dim_{\mathrm{H}}(K(f) \cap J(f)) > 1$ for $f \in B$. On the other hand, Rempe and Stallard [15] showed that there are functions $f \in B$ for which $\dim_{\mathrm{H}} I(f) = 1$.

We consider the dimensions of $K(f)$ for entire functions which need not be in Eremenko-Lyubich class. The following result is a special case of a result of Rempe [14, Corollary 2.11] who proved that the hyperbolic dimension of an Ahlfors islands map is positive.

Theorem 1. *If f is a transcendental entire function, then* $\dim_{\mathrm{H}}(K(f) \cap J(f)) > 0$.

Theorem 1 is also implicit in Stallard's [19] proof that $\dim_{\mathrm{H}} J(f) > 0$ for transcendental meromorphic functions f. The proofs in [14, 19] are based on suitable versions of the Ahlfors islands theorem; see [12, Theorem 6.2] or, for an alternative proof, [5]. This is used to to obtain an iterated function scheme (see [11]), whose invariant set is a (hyperbolic) Cantor subset of $K(f) \cap J(f)$ which can be shown to have positive Hausdorff dimension. We note, however, that for entire and meromorphic functions different versions of the Ahlfors islands theorem have to be used; see the discussion in [6, Section 6.4]. For entire functions such a hyperbolic, invariant Cantor subset of $K(f) \cap J(f)$ is also constructed in [8].

It is the purpose of this note to show that Theorem 1 is best possible even for entire functions.

Theorem 2. *For every $\varepsilon > 0$ there exists a transcendental entire function f such that* $\dim_{\mathrm{H}} K(f) \leq \dim_P K(f) < \varepsilon$.

For an introduction to the dynamics of transcendental entire (and meromorphic) functions we refer to [4]. Results on dimensions of Julia sets of transcendental functions are surveyed in [21].

Acknowledgment. I thank Lasse Rempe, Phil Rippon and Gwyneth Stallard for helpful comments.

2. Proof of Theorem 2

Let C be a large positive constant and define $(a_k)_{k \geq 1}$ recursively by $a_1 = 1$ and

$$a_{k+1} = 8C\, a_k \prod_{j=1}^{k-1} \frac{a_k}{a_j} \tag{1}$$

for $k \geq 1$. (Here $\prod_{j=1}^{0} a_1/a_j = 1$ so that $a_2 = 8Ca_1 = 8C$.) Induction shows that (a_k) increases and that

$$\frac{a_{k+1}}{a_k} \geq 8C \prod_{j=1}^{k-1} \frac{a_k}{a_{k-1}} \geq (8C)^k \tag{2}$$

for all k. Thus

$$f(z) = C\, z \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right)$$

defines an entire function f. For $k \geq 1$ we put

$$r_k = \frac{2k+1}{2k+2} a_k \quad \text{and} \quad s_k = 10 a_k$$

and we set $r_0 = 0$ and $s_0 = 16/C$. For large C we have $r_k < s_k < r_{k+1}$ for $k \geq 0$. We define, for $k \geq 0$,

$$A_k = \{z \in \mathbb{C} : r_k \leq |z| \leq s_k\} \quad \text{and} \quad B_k = \{z \in \mathbb{C} : s_k < |z| < r_{k+1}\}.$$

We will show that

$$f(B_k) \subset B_{k+1} \tag{3}$$

for all $k \geq 1$. In order to do so we note first that by (2) we can achieve that

$$\frac{a_{k+1}}{a_k} > 320e(k+1) \geq 2k+4 \tag{4}$$

for all $k \geq 1$ by choosing C sufficiently large. We deduce that if $1 \leq j \leq k-1$, then $(2k+2)a_j \leq (2k+2)a_{k-1} \leq a_k$ and hence

$$1 + \frac{r_k}{a_j} \leq \frac{a_k}{(2k+2)a_j} + \frac{r_k}{a_j} = \frac{a_k}{a_j} \tag{5}$$

and

$$\frac{r_k}{a_j} - 1 \geq \frac{r_k}{a_j} - \frac{a_k}{(2k+2)a_j} = \frac{k}{k+1}\frac{a_k}{a_j}. \tag{6}$$

Moreover, it follows from (2) that we can achieve that

$$\prod_{j=k+1}^{\infty}\left(1 + \frac{10a_k}{a_j}\right) \leq 2 \quad \text{and} \quad \prod_{j=k+1}^{\infty}\left(1 - \frac{10a_k}{a_j}\right) \geq \frac{9}{10} \geq \frac{1}{2} \tag{7}$$

for all $k \geq 1$ by choosing C large.

For $k \geq 1$ we deduce from (1), (5) and (7) that if $|z| = r_k$, then

$$\begin{aligned} |f(z)| &\leq C\, r_k \prod_{j=1}^{k-1}\left(1 + \frac{r_k}{a_j}\right) \cdot \left(1 + \frac{r_k}{a_k}\right) \cdot \prod_{j=k+1}^{\infty}\left(1 + \frac{r_k}{a_j}\right) \\ &\leq 4C\, a_k \prod_{j=1}^{k-1} \frac{a_k}{a_j} = \frac{1}{2} a_{k+1} < r_{k+1}. \end{aligned}$$

Similarly, (1), (4), (6) and (7) yield that if $|z| = r_k$, then

$$\begin{aligned} |f(z)| &\geq C\, r_k \prod_{j=1}^{k-1}\left(\frac{r_k}{a_j} - 1\right) \cdot \left(1 - \frac{r_k}{a_k}\right) \cdot \prod_{j=k+1}^{\infty}\left(1 - \frac{r_k}{a_j}\right) \\ &\geq C\left(\frac{k}{k+1}\right)^k a_k \prod_{j=1}^{k-1} \frac{a_k}{a_j} \cdot \frac{1}{2k+2} \cdot \frac{1}{2} \qquad (8) \\ &\geq \frac{C}{2e(2k+2)} a_k \prod_{j=1}^{k-1} \frac{a_k}{a_j} = \frac{a_{k+1}}{32e(k+1)} > 10a_k = s_k. \end{aligned}$$

The last two inequalities imply that

$$f(z) \in B_k \quad \text{for } |z| = r_k \tag{9}$$

if $k \geq 1$. Next we note that if $k \geq 1$ and and $|z| = s_k$, then

$$
\begin{aligned}
|f(z)| &\geq C\, s_k \prod_{j=1}^{k-1} \left(\frac{s_k}{a_j} - 1 \right) \cdot \left(\frac{s_k}{a_k} - 1 \right) \cdot \prod_{j=k+1}^{\infty} \left(1 - \frac{s_k}{a_j} \right) \\
&\geq 10C\, a_k \prod_{j=1}^{k-1} \frac{9a_k}{a_j} \cdot 9 \cdot \frac{9}{10} = \frac{9^{k+1}}{8} a_{k+1} > s_{k+1}.
\end{aligned} \tag{10}
$$

Similarly as in (7) we also see that if $|z| = s_0 = 16/C$, then

$$
|f(z)| \geq C\, s_0 \prod_{j=1}^{\infty} \left(1 - \frac{s_0}{a_j} \right) \geq C\, s_0 \frac{9}{10} = \frac{16 \cdot 9}{10} > 10 = s_1, \tag{11}
$$

provided C is chosen large enough. Also, since $s_k < r_{k+1}$ for all $k \geq 0$, we deduce from (9), with k replaced by $k+1$, that $|f(z)| < r_{k+2}$ for $|z| = s_k$. Together with (10) and (11) this yields that

$$
f(z) \in B_{k+1} \quad \text{for } |z| = s_k \tag{12}
$$

if $k \geq 0$. Combining this with (9) we obtain (3).

Next we show that with $L = C/(4e)$ we have

$$
|f'(z)| \geq 2^k L \quad \text{for } z \in A_k. \tag{13}
$$

In order to do so we note first that if p is a real polynomial with real zeros, then each interval bounded by two adjacent zeros of p contains exactly one zero of p', and besides multiple zeros of p there are no further zeros of p'. In particular, p' has only real zeros. Moreover, we see that p has no positive local minima and no negative local maxima.

Since our function f is a limit of real polynomials with real, non-negative zeros, f' is also a limit of such polynomials. It follows that f' has no positive local minima and no negative local maxima. This implies that if a compact interval contains no zero of f', then $|f'|$ assumes its minimum in the interval at one of the endpoints of the interval. The fact that f' is a limit of real polynomials with real, non-negative zeros also implies that $|f'|$ takes its minimum on a circle around the origin at the intersection of this circle with the positive real axis. We will see that f' has no zeros in the intervals $[r_k, s_k]$. The above arguments then imply that

$$
\min_{z \in A_k} |f'(z)| = \min\{|f'(r_k)|, |f'(s_k)|\}. \tag{14}
$$

In order to prove that f' has no zeros in the intervals $[r_k, s_k]$, we note that if $r_k \leq x < a_k$ and $1 \leq j \leq k-1$, then $x > 2a_j$ by (2) and hence $x/(x - a_j) < 2$. Thus

$$
\begin{aligned}
\frac{xf'(x)}{f(x)} &= 1 + \sum_{j=1}^{\infty} \frac{x}{x - a_j} \leq 1 + \sum_{j=1}^{k-1} \frac{x}{x - a_j} + \frac{r_k}{r_k - a_k} \\
&\leq 1 + 2(k-1) - (2k+1) = -2 < 0 \quad \text{for } r_k \leq x < a_k.
\end{aligned} \tag{15}
$$

On the other hand, using (2) it is not difficult to see that by choosing C large we can achieve that if $k \geq 1$, then

$$
\frac{xf'(x)}{f(x)} \geq 1 - \sum_{j=k+1}^{\infty} \frac{s_k}{a_j - s_k} \geq \frac{1}{2} \quad \text{for } a_k < x \leq s_k. \tag{16}
$$

With $a_0 = 0$ this also holds for $k = 0$ if C is large. It follows from (15) and (16) that f' has no zeros in the intervals $[r_k, s_k]$. Thus (14) holds. Moreover, (2), (8) and (15) yield that

$$|f'(r_k)| \geq 2\frac{|f(r_k)|}{r_k} \geq 2\frac{a_{k+1}}{32e(k+1)a_k} \geq \frac{(8C)^k}{16e(k+1)} \geq \frac{C}{4e}2^k = 2^k L \tag{17}$$

for $k \geq 1$ while (2), (10) and (16) give

$$|f'(s_k)| \geq \frac{1}{2}\frac{|f(s_k)|}{s_k} \geq \frac{1}{2}\frac{9^{k+1}a_{k+1}}{80a_k} \geq \frac{1}{2}\frac{9^{k+1}(8C)^k}{80} \geq 4C\,2^k \geq 2^k L \tag{18}$$

for $k \geq 1$. Finally, $f'(0) = C \geq L$ and (11) implies that

$$|f'(s_0)| \geq \frac{1}{2}\frac{s_1}{s_0} = \frac{10}{32}C \geq L. \tag{19}$$

Now (13) follows from (14), (17), (18) and (19).

To estimate the dimension of $K(f)$, we fix $N \in \mathbb{N}$ and put

$$K_N(f) = \{z \in \mathbb{C} : |f^n(z)| \leq s_N \text{ for all } n \in \mathbb{N}\}$$

It follows from (3) that $K_N(f)$ consists of all points z for which $f^n(z) \in \bigcup_{k=0}^N A_k$ for all $n \in \mathbb{N}$. Thus, assuming that C is chosen such that $L = C/(4e) > 1$, we deduce from (13) that $K_N(f)$ is a conformal repeller; see [13, Chapter 8] and [22, Chapter 5] for the definition and properties of conformal repellers. It follows (see [13, Corollary 8.1.7] or [22, Theorem 5.12]) that the Minkowski dimension, packing dimension and Hausdorff dimension of $K_N(f)$ all coincide and are given by Bowen's formula. This formula says that with $F = f|_{K_N(f)}$ these dimensions are equal to the unique zero of the pressure function $t \to P(F,t)$ defined by

$$P(F,t) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{z\in F^{-n}(a)}|(F^n)'(z)|^{-t}\right),$$

for some $a \in K_N(f)$.

In order to apply Bowen's formula we note that every point in $K_N(f)$ has $N+1$ preimages under F. Let $a \in A_k$. It follows from (9) and the maximum principle that F has no a-points in A_j for $0 \leq j \leq k-2$. Moreover, it follows from (9) and (12) that F and $F - a$ have the same number of zeros in A_j for $k \leq j \leq N$. Thus F has exactly one a-point in A_j for $k \leq j \leq N$. We conclude that a has $k-1$ preimages under F in A_{k-1}. It follows from the above discussion, together with (13), that for $a \in K_N(f)$ and $t > 0$ we have

$$\sum_{b\in F^{-1}(a)}|F'(b)|^{-t} \leq \sum_{k=0}^{N}\left(2^k L\right)^{-t} \leq L^{-t}\sum_{k=0}^{\infty}2^{-tk} = \frac{L^{-t}}{1-2^{-t}}.$$

Now

$$\begin{aligned}\sum_{z\in F^{-(n+1)}(a)}|(F^{n+1})'(z)|^{-t} &= \sum_{b\in F^{-1}(a)}\sum_{z\in F^{-n}(b)}|(F^{n+1})'(z)|^{-t}\\ &= \sum_{b\in F^{-1}(a)}|F'(b)|^{-t}\sum_{z\in F^{-n}(b)}|(F^n)'(z)|^{-t}.\end{aligned}$$

With

$$S_n(t) = \sup_{c\in K_N(f)} \sum_{z\in F^{-n}(c)} |(F^n)'(z)|^{-t}$$

we thus have

$$S_{n+1}(t) \leq \frac{L^{-t}}{1-2^{-t}} S_n(t).$$

Induction shows that

$$\sum_{z\in F^{-n}(a)} |(F^n)'(z)|^{-t} \leq S_n(t) \leq \left(\frac{L^{-t}}{1-2^{-t}}\right)^n \tag{20}$$

for all $a \in K_N(f)$. Thus

$$P(F,t) \leq \log \frac{L^{-t}}{1-2^{-t}}. \tag{21}$$

Given $t > 0$, we can achieve that the right hand side of (21) is negative by choosing C and hence L large. Then the Minkowski, packing and Hausdorff dimension of $K_N(f)$ are less than t for all N. Since $K(f) = \bigcup_{N=1}^{\infty} K_N(f)$, we deduce that $\dim_P K(f) \leq t$. As $t > 0$ can be chosen arbitrarily small, the conclusion follows.

Remark. The thermodynamic formalism of [13, 22] is not actually needed to obtain an *upper* bound for $\dim_{\mathrm{H}} K_N(f)$. As $K_N(f)$ does not intersect the postcritical set of F, there exists $\delta > 0$ such that Koebe's distortion theorem may be applied to all inverse branches of the iterates of F on the disk $D(a,\delta) = \{z \in \mathbb{C} : |z-a| < \delta\}$. We obtain

$$F^{-n}(D(a,\delta)) \subset \bigcup_{z\in F^{-n}(a)} D\left(z, \frac{C}{|(F^n)'(z)|}\right)$$

for some constant C. Now (20) shows that $F^{-n}(D(a,\delta))$ can be covered by $(N+1)^n$ sets V_j whose diameters satisfy

$$\sum_j (\operatorname{diam} V_j)^t \leq (2C)^t \left(\frac{L^{-t}}{1-2^{-t}}\right)^n.$$

The compact set $K_N(f)$ can be covered by finitely many, say M, disks $D(a,\delta)$. Hence we obtain a covering of $K_N(f) = F^{-n}(K_N(f))$ by $M(N+1)^n$ sets W_j satisfying

$$\sum_j (\operatorname{diam} W_j)^t \leq M(2C)^t \left(\frac{L^{-t}}{1-2^{-t}}\right)^n.$$

This implies that the t-dimensional Hausdorff measure of $K_N(f)$ is 0, provided L is again chosen such that $L^{-t} < 1 - 2^{-t}$.

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