Integrability of weak distributions on Banach manifolds

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Abstract

This paper concerns the problem of integrability of non closed distributions on Banach manifolds. We introduce the notion of weak distribution and we look for conditions under which these distributions admit weak integral submanifolds. We give some applications to Banach Lie algebroid and Banach Lie-Poisson manifold. The main results of this paper generalize the works presented in [ChSt], [Nu] and [Gl].

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1 Introduction

In differential geometry, a *distribution* on a smooth manifold M is an assignment $\mathcal{D} : x \mapsto \mathcal{D}_x \subset T_xM$ on M, where $\mathcal{D}_x$ is a subspace of T_xM. The distribution is *integrable* if, for any $x \in M$ there exists an immersed submanifold $f : L \to M$ such that $x \in f(L)$ and for any $z \in L$ we have $f_*T_zL = \mathcal{D}_{f(z)}$. On the other hand, $\mathcal{D}$ is called *involutive* if, for any vector fields X and Y on M which are tangent to $\mathcal{D}$, the Lie Bracket $[X, Y]$ is also tangent to $\mathcal{D}$. The distribution is *invariant* if for any vector field X tangent to $\mathcal{D}$, the flow ϕ_t^X leaves $\mathcal{D}$ invariant (see 2.1).

On a finite dimensional manifold, when $\mathcal{D}$ is a subbundle of TM, the classical Frobenius Theorem gives an equivalence between integrability and involutivity. In the other case, the distribution is "singular" and even under assumption of smoothness on $\mathcal{D}$, in general, the involutivity is not a sufficient condition for integrability (we need some more additional local conditions). These problems were clarified and resolved essentially in [Su], [St] and [Ba].

In the context of Banach manifold, the Frobenius Theorem is again true for distributions which are complemented subbundles of the tangent bundle. For singular distributions, some papers ([ChSt],[Nu]) show that, when the distribution is closed and complemented (i.e. $\mathcal{D}_x$ is a complemented Banach subspace of T_xM), we have equivalence between integrability and invariance. Some results of sufficient condition about local involutivity give also a result of integrability. A more recent result ([Gl]) proves such results without the assumption that the distribution is complemented.

In this paper, in reference to "weak submanifolds" in a Banach manifold, (([El],[Pe]), we consider "weak distributions": $\mathcal{D}_x$ can be not closed in T_xM but endowed with its own Banach structure so that the inclusion $\mathcal{D}_x \to T_xM$ is continuous. Such a category of distributions takes naturally place in the framework of Banach Lie algebroids (morphisms from a Banach bundle over a Banach manifold into the tangent bundle of this manifold). Under conditions of "local triviality", our results can be seen at the same time as a generalization of results of [Su],[St] and results of

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[ChSt], [Nu] and [Gl] too.

The first section contains the essential definition and property about weak distributions and also the first result of equivalence between involutivity and invariance under local lower triviality assumption (Theorem 1). In the second section, we adapt the context of [ChSt] to a generalization of their results of involutivity under condition of "Lie invariance"(Theorem 2). In the second part, under the assumption of "strong upper triviality", we give condition of "local involutivity" under which we have an integrability property (Theorem 4). In the last section, we give applications of these results in the context of Banach Lie Algebroid (see [An]) and Banach Lie-Poisson manifold as it is exposed in [OdRa1] and [OdRa2]

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2 Integrability and invariance

2.1 Preliminaries and context

Let M be a smooth connected Banach manifold modeled on a Banach space E. We denote by $\mathcal{A}(M)$ the ring of smooth functions on M and by $\mathcal{X}(M)$ the Lie algebra of smooth vector fields on M. A **local vector field** X on M is a smooth section of the tangent bundle TM defined on an open set of M (denoted by $\mathrm{Dom}(X)$). Denote by $\mathcal{X}_L(M)$ the set of all local vector fields on M. Such a vector field $X \in \mathcal{X}_L(M)$ has a flow ϕ_t^X which is defined on a maximal open set Ω_X of $M \times \mathbb{R}$.

A **weak submanifold** of M is a pair (N, f) of a Banach manifold N (modeled on a Banach space F) and a smooth map $f : N \to M$ such that : ([El],[Pe])

1. $F \subset E$ and the natural inclusion $i : F \to E$ between these two Banach spaces is continuous
2. f is injective and the tangent map $T_x f : T_x N \to T_{f(x)} M$ is injective for all $x \in N$.

Remark 2.1
Given a weak submanifold $f : N \to M$, on the subset $f(N)$ in M we have two topologies:

1. *the induced topology from M*
2. *the topology for which f is a homeomorphism from N to $f(N)$.*

With this last topology, via f, we get a structure of Banach manifold modeled on F. Moreover, the inclusion from $f(N)$ into M is continuous as map from the Banach manifold $f(N)$ to M. In particular, if U is an open of M, then, $f(N) \cap U$ is an open set for the topology of the Banach manifold on $f(N)$.

In this work, a **weak distribution** on a M is a map $\mathcal{D} : x \to \mathcal{D}_x$ which, for every $x \in M$, associates a vector subspace $\mathcal{D}_x$ in $T_x M$ (not necessarily closed) endowed with a norm $|| \, ||_x$ so that $(\mathcal{D}_x, || \, ||_x)$ is a Banach space (denoted by $\tilde{\mathcal{D}}_x$) and such that the natural inclusion $i_x : \tilde{\mathcal{D}}_x \to T_x M$ is continuous.

Remark 2.2
When $\mathcal{D}_x$ is closed, via any chart, we get a norm on $T_x M$ which induces a Banach structure on $\mathcal{D}_x$. So if $\mathcal{D}_x$ is closed for all $x \in M$, the previous assumption on the Banach structure $\tilde{\mathcal{D}}_x$ is always satisfied, and we get the usual definition of a distribution on M (compare with [Gl], [ChSt], [Nu]). In this last situation we always endow $\tilde{\mathcal{D}}_x$ with this induced Banach structure and we say that $\mathcal{D}$ is **closed**.

Examples 2.3

(1) *Let* l^p *(resp.* l^∞*) be the Banach space of real sequences* (x_k) *such that* $\sum_{k=1}^{\infty} |x_k|^p < \infty$ *(resp. absolutely bounded) and denote by* I_p *the natural inclusion of* l^1 *in* l^p *,* $p > 1$ *or* $p = \infty$*. On the Banach space* l^p*,* $x \mapsto \mathcal{D}_x = x + I_p(l^1)$ *is a weak distribution which is not closed.*

(2) *Let* E *and* F *be two Banach spaces and* $T : F \to E$ *a continuous operator. Denote by* $\hat{T} : F/\ker T \to E$ *the canonical quotient bijection associated to* T *that is*

$$\begin{array}{ccc} F & \xrightarrow{q} & F/\ker T \\ {\scriptstyle T}\downarrow & \swarrow {\scriptstyle \hat{T}} & \\ T(F) & & \end{array} \qquad (1)$$

We can endow $T(F)$ *with the structure of Banach space such that* $\hat{T}$ *is an isometry. On* E*, the assignment* $x \mapsto \mathcal{D}_x = x + T(F)$ *is a weak distribution. This distribution is closed if and only if* $T(F)$ *is closed in* E*.*

(3) *Let* $L(F, E)$ *be the set of continuous operators between the Banach spaces* F *and* E*. Given a smooth map* $\Psi : E \to L(F, E)$*, we denote by* Ψ_x *the continuous operator associated to* $x \in E$*. As in (1) denote by* $\hat{\Psi}_x$ *the canonical bijection associated* Ψ_x *and we endow* $\mathcal{D}_x = \Psi(F, x)$ *with the Banach structure such that* $\hat{\Psi}_x$ *is an isometry. Then,* $x \to \mathcal{D}_x$ *is a weak distribution on* E *which is closed if and only if* $\mathcal{D}_x$ *is closed for any* $x \in E$*.*

A vector field $Z \in \mathcal{X}_L(M)$ is **tangent** to $\mathcal{D}$, if for all $x \in \mathrm{Dom}(Z)$, $Z(x)$ belongs to $\mathcal{D}_x$. The set of local vector fields tangent to $\mathcal{D}$ will be denote by $\mathcal{X}_{\mathcal{D}}$.

We say that $\mathcal{D}$ is **is generated by a subset** $\mathcal{X} \subset \mathcal{X}_L(M)$ if, for every $x \in M$, the vector space $\mathcal{D}_x$ is the linear hull of the set $\{Y(x)\ ,\ Y \in \mathcal{X}\ ,\ x \in \mathrm{Dom}(Y)\}$.

For a weak distribution $\mathcal{D}$, on M we have the following definitions (compare with the definition of "smooth distribution" in [Gl]):

• $\mathcal{D}$ is **lower (locally) trivial** if for each $x \in M$, there exists an open neighbourhood V of x, a smooth map $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$ (called **lower trivialization**) such that :

(i) $\Theta(\tilde{\mathcal{D}}_x \times \{y\}) \subset \mathcal{D}_y$ for each $y \in V$

(ii) for each $y \in V$, $\Theta_y \equiv \Theta(\ ,y) : \tilde{\mathcal{D}}_x \to T_yM$ is a continuous operator and $\Theta_x : \tilde{\mathcal{D}}_x \to T_xM$ is the natural inclusion i_x

(iii) there exists a continuous operator $\tilde{\Theta}_y : \tilde{\mathcal{D}}_x \to \tilde{\mathcal{D}}_y$ such that $i_y \circ \tilde{\Theta}_y = \Theta_y$, $\tilde{\Theta}_y$ is an isomorphism from $\tilde{\mathcal{D}}_x$ onto $\Theta_y(\tilde{\mathcal{D}}_x)$ and $\tilde{\Theta}_x$ is the identity of $\tilde{\mathcal{D}}_x$

• an **integral manifold** of $\mathcal{D}$ through x is a weak submanifold $f : N \to M$ such that there exists $u_0 \in N$ such that $f(u_0) = x$ and $T_uf(T_uN) = \mathcal{D}_{f(u)}$ for all $u \in N$.

• $\mathcal{D}$ is called **integrable** if for any $x \in M$ there exists an integral manifold N of $\mathcal{D}$ through x.

• if $\mathcal{D}$ is generated by $\mathcal{X} \subset \mathcal{X}_{\mathcal{L}}(\mathcal{M})$, then $\mathcal{D}$ is called **$\mathcal{X}$- invariant** if for any $X \in \mathcal{X}$, the tangent map $T_x\phi_t^X$ send $\mathcal{D}_x$ onto $\mathcal{D}_{\phi_t^X(x)}$ for all $(x,t) \in \Omega_X$. $\mathcal{D}$ is **invariant** if $\mathcal{D}$ is $\mathcal{X}_{\mathcal{D}}-$ invariant.

For an illustration of the property of lower local triviality in the previous context, we give the following result, which is a generalization of Example 2.3 (3)

Proposition 2.4
Let $\mathcal{D} : x \to \mathcal{D}_x \subset T_xM$ *be a field on* M *of normed subspaces. Suppose that for each* $x \in M$*, there exists an open neighbourhood* V *of* x*, a Banach space* F *and a smooth map* $\Psi : F \times V \to TM$ *such that :*

(i) $\Psi(F \times \{y\}) = \mathcal{D}_y$ *for each* $y \in V$

(ii) *for each* $y \in V$, $\Psi_y \equiv \Psi(\ ,y) : F \to T_yM$ *is a continuous operator such that* $\Psi_y(F) = \mathcal{D}_y$

We have the following properties:

1. $\mathcal{D}_x$ *has a natural structure of Banach space (again denoted by* $\tilde{\mathcal{D}}_x$*) such that the canonical continuous operator* $\hat{\Psi}_x : F/\ker\Psi_x \to \tilde{\mathcal{D}}_x$ *is an isometry, in particular,* $\mathcal{D}$ *is a weak distribution.*

2. *There exists a neighbourhood* W *of* x *and, for any* $y \in W$*, a continuous surjective operator* $\tilde{\Psi}_y : F \to \tilde{\mathcal{D}}_y$ *such that* $i_y \circ \tilde{\Psi}_y = \Psi_y$*, where* $i_y : \tilde{\mathcal{D}}_y \to T_yM$ *is the natural inclusion.*

3. *Assume that* $\ker\Psi_x$ *is complemented (i.e.* $F = \ker\Psi_x \oplus S$*). Then there exists an open neighbourhood* W *of* x *such that the restriction* θ_y *of* Ψ_y *to* S *is injective for any* $y \in W$*, and then* $\Theta(u, y) = (\theta_y \circ [\theta_x]^{-1}(u), y)$ *is a lower trivialization of* $\mathcal{D}$*.*

Definition 2.5

1. *A weak distribution* $\mathcal{D}$ *which has the properties (i),and (ii) of Proposition 2.4 is called* **upper trivial***, each map* $\Psi : F \times V \to TM$ *is called an* **upper trivialization***.*

2. *A weak distribution* $\mathcal{D}$ *is called* **strong upper trivial** *if, for any* $x \in M$*, there exists an upper trivialization* $\Psi : F \times V \to TM$ *such* $\ker\Psi_x$ *is complemented and the property (3) of Proposition 2.4 is true on* V*.*
   In this case Ψ *is called a* **strong upper trivialization**

3. *For strong upper trivialization* $\Psi : F \times V \to TM$*, the lower trivialization* Θ*, given in Proposition 2.4, is called the* **associated lower trivialization***.*

The context of Proposition 2.4 can be found in the framework of Banach Lie-Poisson manifold $(M, \{\ ,\ \})$ where $\Psi : T^*M \to TM \subset T^{**}M$ is the canonical morphism associated to the Poisson structure (see for instance [OdRa1] and [OdRa2]).

Corollary 2.6
Let $\pi : \mathcal{F} \to M$ *be a Banach fiber bundle over* M *with typical fiber* F *and* $\Psi : \mathcal{F} \to TM$ *a morphism of bundle. Then* $\mathcal{D} = Im\ \Psi$ *is an upper trivial weak distribution. If, moreover the kernel of* Ψ *is complemented in each fiber, then* $\mathcal{D}$ *is a strong upper trivial weak distribution and also a lower trivial weak distribution.*

We end this section with the proof of Proposition 2.4 and its corollary. For this, we need the following Lemma which will be also used later:

Lemma 2.7

1. *Consider two Banach spaces* E_1 *and* E_2 *and* $i : E_1 \to E_2$ *an injective continuous operator. Let* Θ_y *be a smooth field of continuous operators of* $L(E_1, E_2)$ *on an open neighbourhood* V *of* $x \in E_1$ *such that* $\Theta_x = i$*. Then there exists a neighbourhood* W *in* V *such that* Θ_y *is an injective operator on* W*.*

2. *Let* $f : U \to V$ *be a* C^1 *map from two open sets* U *and* V *in Banach spaces* E_1 *and* E_2 *respectively such that* T_uf *is injective at* $u \in U$*. Then there exists an open neighbourhood* W *of* u *in* U *such that the restriction of* f *to* W *is injective.*

Proof of Lemma 2.7

There exists an open ball $B(x,r)$ included in V such that $||\Theta_y - \Theta_x|| \leq K||y-x]]$ for any $y \in B(x,r)$. We can suppose that $r<1$. Assume that the conclusion of Lemma 2.7 (1) is not true. So, for each $n \in \mathbb{N}^*$, there exists $x_n \in B(x, r/n)$ and $h_n \in E_1$ with $||h_n|| = 1$ such that $\Theta_{x_n}(h_n) = 0$. We have of course:

$$< \alpha, \Theta_{x_n}(h_n) >= 0 \tag{2}$$

for all $\alpha \in E_2^*$.
It follows that we have:

$$| < \alpha, \Theta_x(h_n) > | = | < \alpha, (\Theta_x - \Theta_{x_n})(h_n) > | \leq \frac{K||\alpha||}{n} \tag{3}$$

On the other hand, $\Theta_x = i$ is a continuous bijective operator from the Banach space E_1 onto the normed subspace $F = i(E_1)$ in E_2. So, the transpose operator $i^* \in L(F^*, E_1^*)$ is a monomorphism[1] with a dense range (see [Ha], [HaMb]). From Hahn-Banach Theorem, there exists $\beta_n \in E_1^*$ such that $< \beta_n, h_n >= 1$ with $||\beta_n|| = 1$. From the density of $i^*(F^*)$, there exists $\alpha_n \in F^*$ such that $||\beta_n - i^*(\alpha_n)|| < \frac{1}{4}$, i.e. such that $\frac{3}{4} \leq ||i^*(\alpha_n)|| \leq \frac{5}{4}$. From these inequalities we get:

- $| < i^*(\alpha_n), h_n > -1| = | < i^*(\alpha_n) - \beta_n, h_n > | \leq \frac{1}{4}$

so we have $| < i^*(\alpha_n), h_n > | \geq \frac{3}{4}$

- as $||i^*(\alpha_n)|| \leq \frac{5}{4}$ and as i^* is a monomorphism, we have $||i^*(\alpha_n)|| \geq k||\alpha_n||$ for some $k>0$

and finally we get $||\alpha_n|| \leq \frac{5}{4k}$.

On the other hand we can write:

$$| < i^*(\alpha_n), h_n > | = | < \alpha_n, i(h_n) > | \geq \frac{3}{4} \tag{4}$$

for any n.
From Hahn-Banach Theorem, we obtain the same relation (4) with $\alpha_n \in E_2^*$. But from (3) we get:
$| < \alpha_n, \Theta_x(h_n) > | \leq \frac{K||\alpha||}{n}$ and so $| < \alpha_n, i(h_n) > | \leq \frac{K}{n}\frac{5}{4k}$
for any n.
So we get a contradiction with (4) for n large enough. So we have completed the proof of the part (1).

Let be $f : U \to V$ a map of class C^1. As the problem is local, without loss of generality, we can suppose that U is an open ball of center $0 \in E_1$. As f is C^1, there exists an open ball $B(0,r)$ such that

$$||T_u f - T_v f|| \leq K||u-v|| \text{ for } u, v \in B(0,r) \tag{5}$$

Moreover, we can choose r so that $r<1$.

Suppose that f is not locally injective around 0. Given any pair $(u,v) \in [B(0,r)]^2$ such that $u \neq v$ but $f(u) = f(v)$, we set $h = v-u$. For any $\alpha \in E_2^*$ we consider the smooth curve $c_\alpha : [0,1] \to \mathbb{R}$ defined by:

$$c_\alpha(t) =< \alpha, f(u+th) - f(u) >$$

[1] an operator T between two Banach space E and F is a monomorphism if we have $\inf\{||T(u)||_F;\ ||u||_E = 1\} \geq k > 0$

Of course we have $\dot{c}_\alpha(t) =< \alpha, T_{u+th}f(h) >$.

Denote by $]u, v[$ the set of points $\{w = u + th, t \in]0, 1[\}$. As we have $c_\alpha(0) = c_\alpha(1) = 0$, from Rolle's Theorem, there exists $u_\alpha \in]u, v[$ such that

$$< \alpha, T_{u_\alpha} f(h) >= 0 \tag{6}$$

Replacing h by $\dfrac{h}{||h||}$, we can suppose in (6) that $||h|| = 1$.

From our assumption it follows that, for each $n \in \mathbb{N}^*$, there exists u_n and v_n in $B(x, r/n)$ so that $u_n \neq v_n$ but with $f(u_n) = f(v_n)$. So from the previous argument, for any $\alpha \in E_2^*$, we have

$$< \alpha, T_{u_{\alpha,n}} f(h_n) >= 0 \tag{7}$$

for some $u_{\alpha,n} \in]u_n, v_n[$ and with $h_n = \dfrac{v_n - u_n}{||v_n - u_n||}$

From (5) and (7), we get

$$| < \alpha, T_0 f(h_n) > | = | < \alpha, [T_0 f - T_{u_{\alpha,n}} f](h_n) > | \leq ||\alpha||.\frac{Kr}{n} < ||\alpha|| \frac{K}{n}. \tag{8}$$

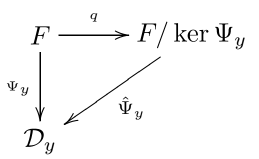
for any $\alpha \in E_2^*$.

Now, we can use the same argument as in part (1) which again leads to a contradiction.

$\triangle$

Proof of Proposition 2.4

At first, for any $x \in M$, we have a natural Banach structure on $\mathcal{D}_x$ (again denoted by $\tilde{\mathcal{D}}_x$) such that the natural morphism $\tilde{\Psi}_x : F/\ker \Psi_x \to \tilde{\mathcal{D}}_x$ is an isometry. On the other hand, take a local trivialization of TM on a neighbourhood W of x; so we have $TM \equiv E \times W$. In this context, on W, Ψ can be identified with a smooth field of continuous operators $\Psi_y : F \to E$ such that $\mathcal{D}_y = \Psi_y(F) \times \{y\} \subset E \times \{y\} \equiv T_yM$. Let us consider the following commutative diagram:



where q is the natural projection and $\hat{\Psi}_y$ is the natural bijection induced by Ψ_y. So, if we consider the Banach structure $\tilde{\mathcal{D}}_y$, we get a continuous operator $\tilde{\Psi}_y = \hat{\Psi}_y \circ q : F \to \tilde{\mathcal{D}}_y$ so that $\Psi_y = i_y \circ \tilde{\Psi}$.

Assume that $F = \ker \Psi_x \oplus S$, for some Banach space $S \subset F$. Let θ_y be the restriction to S of Ψ_y for any $y \in W$. Clearly, $\theta(u, y) = (\theta_y(u), y)$ defines a smooth map from $S \times W$ into $E \times V \equiv TM$ and $\theta_y : S \times \{x\} \to E \times \{x\} \equiv T_xM$ is a continuous operator whose image is contained in $\mathcal{D}_y$.

On the other hand, let $\tilde{\theta}_y$ be the restriction of $\tilde{\Psi}_y$ to S, then, $\tilde{\theta}_y$ is a continuous operator from S to $\tilde{\mathcal{D}}_y$ so that $\theta_y = i_y \circ \tilde{\theta}_y$ for any $y \in W$. Of course, $\tilde{\theta}_x : S \to \tilde{\mathcal{D}}_x$ is an isometry and, in particular, it is an isomorphism. As, θ_x is injective, from Lemma 2.7, without loss of generality, we can suppose that θ_y is injective for any $y \in W$. It follows that $\tilde{\theta}_y$ is a continuous injective operator from S into $\tilde{\mathcal{D}}_y$. As θ_y is injective, we have $\ker \Psi_y \cap S = \{0\}$. It follows that $q_1 = q_{|S}$ is an isomorphism onto $q(S) \subset F/\ker \Psi_y$. Of course the restriction q_2 of the isomorphism $\hat{\Psi}_y : F/\ker \Psi_y \to \tilde{\mathcal{D}}_y$ to $q(S)$ is an isomorphism onto $\tilde{\theta}_y(S)$ such that $\tilde{\theta}_y = q_2 \circ q_1$. So $\tilde{\theta}_y$ is an isomorphism of S onto $\tilde{\theta}_y(S)$.

Finally, the map

$$\Theta : \tilde{\mathcal{D}}_x \times W \to E \times W \equiv TM$$

defined by $\Theta(u,y) = (\theta_y \circ [\theta_x]^{-1}(u), y)$ is clearly a lower trivialization of $\mathcal{D}$

△

Proof of Corollary 2.6

Given $x \in M$ there exists a local trivialization of $\mathcal{F}$ on an open set V around x. So we can identify $\mathcal{F}$ with $F \times V$ on V. In this context, in restriction to V, the morphism Ψ can be identified, as a map $\Psi : F \times V \to TM$ which satisfies assumption (i) and (ii) of Proposition 2.4

△

2.2 Results

Let be $\mathcal{D}$ a lower locally trivial trivial distribution on M. For any lower trivialization $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$ and any $u \in \tilde{\mathcal{D}}_x$,we consider

$$X(z) = \Theta(u,z) \tag{9}$$

Of course we also have $X(z) = i_z \circ \tilde{\Theta}(u,z)$ where $\tilde{\Theta}(u,z) = \tilde{\Theta}_z(u)$ and X is a local vector field on M tangent to $\mathcal{D}$ whose domain is V. Moreover, the set of all such local vector fields spans $\mathcal{D}$.

A **lower (local) section** of a lower locally trivia weak l distribution $\mathcal{D}$ is a map of type (9) for any lower trivialization Θ any $u \in \tilde{\mathcal{D}}_x$ and any $x \in M$. The **set of such lower sections** will be denoted by $\mathcal{X}^-_{\mathcal{D}}$.

The following Proposition gives a relation between integral manifolds and $\mathcal{X}^-_{\mathcal{D}}-$invariant weak distributions:

Proposition 2.8
If a lower locally trivial weak distribution $\mathcal{D}$ (resp. lower locally trivial closed distribution) is integrable, then it is $\mathcal{X}^-_{\mathcal{D}}-$invariant (resp. $\mathcal{X}_{\mathcal{D}}-$invariant)

In this context, we obtain the following version of Stefan-Sussmann Theorem:

Theorem 1 *Let $\mathcal{D}$ be a lower locally trivial weak distribution on a Banach manifold M.*

1. *$\mathcal{D}$ is integrable if and only if it is $\mathcal{X}^-$-invariant.*

2. *if $\mathcal{D}$ is integrable, on M,*
   consider the binary relation

   $x\mathcal{R}y$ iff there exists an integral manifold (N,f) of $\mathcal{D}$ such that $x, y \in f(N)$.

   Then $\mathcal{R}$ is an equivalence relation and the equivalence class $L(x)$ of x has a natural structure of connected Banach manifold modeled on $\tilde{\mathcal{D}}_x$.
   Moreover $(L(x), i_{L(x)})$, is a maximal integral manifold of $\mathcal{D}$ in the following sense: for any integral manifold (N,f) of $\mathcal{D}$, such that $f(N) \cap L(x)$ is not empty then $f(N) \subset L(x)$.

Taking into account Remark 2.2, the property of lower triviality of a weak distribution corresponds to the usual assumptions on the distribution that we find in [ChSt], [Nu], [Gl]. When $\mathcal{D}_x$ is closed (resp. complemented) in T_xM the following Corollary of Theorem I gives exactly the main result of integrability of distributions we can find in [Gl] (resp. [ChSt], [Nu]):

Corollary 2.9
For a lower trivial closed distribution the following propositions are equivalent:

(i) *$\mathcal{D}$ is integrable;*

(ii) *$\mathcal{D}$ invariant;*

(iii) $\mathcal{D}$ *is* $\mathcal{X}^-$*-invariant.*

We end this section with the <u>*proof of Proposition 2.8*</u> :

Consider a lower section $X(y) = \Theta(u, y)$, associated to a lower trivialization $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$. So $\mathrm{Dom}(X) = V$. Fix such such a lower section X and the associated lower trivialization Θ. Denote by $\triangle$ (resp. $\tilde{\triangle}$) the subspace $\mathcal{D}_x$ of $T_xM \equiv E$ (resp. the Banach space $\tilde{\mathcal{D}}_x$). Let be $\triangle_y = \Theta_y(\triangle)$ and $\tilde{\triangle}_y$ the natural Banach structure induced by $\tilde{\mathcal{D}}_y$.

Given any $z \in V$, the map $\Theta'_y = \Theta_y \circ [\tilde{\Theta}_z]^{-1}$ is a smooth field continuous operators from $\tilde{\triangle}_z$ into $T_yM \equiv E \times \{y\}$ and moreover, $\tilde{\Theta'}_y = \tilde{\Theta}_y \circ [\tilde{\Theta}_z]^{-1}$ is an isomorphism between $\tilde{\triangle}_z$ and $\tilde{\triangle}_y$. Of course, if $v = \tilde{\Theta}_z(u)$, we have $X(y) = \Theta'_y(v)$.

Let $f : N \to M$ be an integral manifold of $\mathcal{D}$ passing through some $z \in V$. Then, N is a Banach manifold modeled on the Banach space $\tilde{G} = \tilde{\triangle}_z$. For any open neighbourhood U of z the set $\tilde{U} = f^{-1}(U)$ is an open neigbourhood of $\tilde{z} = f^{-1}(z)$. According to Remark 2.1, without loss of generality, we may assume that N is an open set in $\tilde{G}$ and M is an open set in E.In these identifications, f is the natural inclusion i_N of N in M, that is the restriction to N of the natural inclusion $i : \tilde{G} \to E$. In this context, on $i(N) \subset M$, $y \to \Theta'_y$ is smooth field of continuous linear operator from $\tilde{\triangle}_z \subset \tilde{G}$ into $\mathcal{D}_y \equiv G \times \{y\}$. Moreover, $\tilde{\Theta'}_y$ is an isomorphism between $\tilde{\triangle}_z \subset \tilde{G}$ and $\tilde{\triangle}_y \subset \tilde{G} \times \{y\}$ for any $y \in i(N)$.

Lemma 2.10
With the previous notations, the map $y \mapsto \tilde{\Theta'}_y$ *from* N *to* $L(\tilde{\triangle}_z, \tilde{G})$ *is smooth (for the topology induced by* $\tilde{\triangle}_z$ *on* N*).*

From Lemma 2.10, $\tilde{Y} = \tilde{\Theta'}_y(v)$ is a smooth vector field on the Banach manifold N, and, moreover, and we have $X(i(y)) = T_y i[\tilde{Y})(y)] = (i_*Y)(y)$ on $i(N)$. So the flow ϕ^X_t satisfies the relation

$$\phi^X_t \circ i = i \circ \phi^{\tilde{Y}}_t$$

on a small neighbourhood W of z and for all t such that $\phi^{\tilde{Y}}_t$ is defined on N. Of course for any $y \in W$ and t such that $\phi^{\tilde{Y}}_t(y)$ is defined, we have

$$T_y\phi^X_t(\mathcal{D}_y) = T_y\phi^X_t(i[T_yN]) = i \circ T_y\phi^{\tilde{Y}}_t(T_yN) = i[T_{\phi^{\tilde{Y}}_t(y)}N] = i[\tilde{\mathcal{D}}_{\phi^{\tilde{Y}}_t(y)}] = \mathcal{D}_{\phi^X_t(y)}.$$

Now, consider any $(z, t) \in \Omega_X$. Denote by $]\alpha_z, \beta_z[$ the maximal interval on which $\Phi^X_t(z)$ is defined. Given any $\tau \in]\alpha_z, \beta_z[$, consider the integral curve $\gamma(t) = \phi^X_t(z)$ for $t \in [0, \tau]$. By compactness of $[0, \tau]$ there exists a finite number of integral manifolds $(N_1, f_1), \cdots, (N_r, f_r)$ so that $\gamma([0, \tau])$ is contains in $\cup_{i=1}^r f_i(N_i)$. Using the previous argument, by induction, we obtain:

$$T_z\phi^X_\tau(\mathcal{D}_z) = \mathcal{D}_{\phi^X_\tau(z)}.$$

We deduce that integrability implies $\mathcal{X}^-_{\mathcal{D}}-$invariance.

Now, if moreover $\mathcal{D}$ is closed, given an integral manifold $f : N \to M$ and any local section X of $\mathcal{D}$ whose domain intersects $f(N)$, then X induces, by restriction on $f(N)$, a smooth vector fields on N. So the same arguments used last part of in the previous proof works too. (see [Gl]). $\triangle$

<u>*Proof of Lemma 2.10*</u>

From convenient analysis (see [KrMi]), recall that for a map f from an open set U in a Banach space E_1 to a Banach space E_2 we have the equivalent following properties

(i) f is smooth;

(ii) for any smooth curve $c : \mathbb{R} \to U$ the map $t \mapsto f \circ c(t)$ is smooth;

(iii) the map $t \mapsto < \alpha, f \circ c(t) >$ is smooth for any $\alpha \in E_2^*$;

(iv) for any smooth curve $c : \mathbb{R}^2 \to U$, all partial derivatives of $f \circ c$ exist and are locally bounded.

Fix some $v \in \tilde{\triangle}_z$. Note that, for any $\alpha \in \tilde{G}^*$ we have

$$< \alpha, \tilde{\Theta}'_y(v) >=< [\tilde{\Theta}'_y]^*(\alpha), v > \tag{10}$$

If $i : \tilde{G} \to G$ is the natural inclusion, we have $[\Theta'_y]^* = [\tilde{\Theta}'_y]^* \circ i^*$.

For $y \in i(N) \subset \tilde{\triangle}_z$ and $\alpha \in G^*$ fixed, we consider the map
$h(y) = [\Theta'_y]^*(\alpha) = [\tilde{\Theta}'_y]^*(i^*\alpha))$
Clearly, h is a smooth map from the open $i(N)$ in the normed space $\triangle_z \subset E$ to the Banach $[\tilde{\triangle}_z]^*$. Take any smooth curve $c : \mathbb{R} \to N \subset \tilde{\triangle}_z$. As the inclusion of $\tilde{\triangle}_z$ into $\triangle_z$ is linear continuous, c is also a smooth map from $\mathbb{R}$ to $N \subset \triangle_z$, the map $h \circ c$ is a smooth map from $\mathbb{R}$ to $[\tilde{\triangle}_z]^*$. We conclude that h is a smooth map from $N \subset \tilde{\triangle}_z$ to $[\tilde{\triangle}_z]^*$

So from (10), we see that the map $y \to < i^*\alpha, \tilde{\Theta}'_y(v) >$ is a smooth from $N \subset \tilde{\triangle}_z$ to $\mathbb{R}$, for any $\alpha \in G^*$. As $i^*(G^*)$ is dense in $\tilde{G}^*$, given any $\beta \in \tilde{G}^*$ there exists a sequence $\alpha_n \in G^*$ so that $i^*(\alpha_n)$ converges to β in $\tilde{G}^*$. For simplicity, we set $g(y) = \tilde{\Theta}'_y(v)$. Consider any smooth curve $c : \mathbb{R} \to N \subset \tilde{\triangle}_z$.

Now on any compact $K \subset \mathbb{R}$, and for any $p \in \mathbb{N}$ we have:
$| < \beta, (g \circ c)^{(p)}(t) > - < i^*(\alpha_n), (g \circ c)^{(p)}(t) > | \leq ||\beta - i^*(\alpha_n)|| \sup_{t \in K} |(g \circ c)^{(p)}(t)|$

So the map $< i^*\alpha_n, (g \circ c)^{(p)} >$ converges uniformly to $< \beta, (g \circ c)^{(p)} >$ on K. It follows that $< \beta, g \circ c >$ is a smooth map for any $\beta \in \tilde{G}^*$. On one hand, we have proved that the map $y \to \tilde{\Theta}'_y(v)$is smooth for any $v \in \tilde{\triangle}_z$. On the other hand, we know that $\tilde{\Theta}'_y$ is a continuous operator from $\tilde{\triangle}_z$ to $\tilde{G}$. I It follows from (iv) that the map $y \to \tilde{\Theta}'_y$ is a smooth map from $N \subset \tilde{\triangle}_z$ into $L(\tilde{\triangle}_z, \tilde{G})$.

$\triangle$

2.3 Proof of Theorem 1

Proof of Part (1)
At first, according to the Proposition 2.8, integrability implies $\mathcal{X}_{\mathcal{D}}^-$−invariance.
So we have to prove the converse. In fact, this proof is an adaption of arguments of Chillingworth and Stefan used in [ChSt].

Given $x \in M$, we may assume that M is an open set of E and $TM \equiv E \times M$. We denote by $\triangle$ (resp. $\tilde{\triangle}$) the normed space (resp. the Banach space) $\mathcal{D}_x$ (resp. $\tilde{\mathcal{D}}_x$). From the property of lower local triviality, by restricting this open if necessary, we have a smooth fields of operators Θ_y of continuous operators from $\tilde{\triangle}$ to E. Consider the family $\{X_u(y) = \Theta_y(u),\ u \in \tilde{\triangle}\}$ of smooth vector fields on M. By standard argument (see [ChSt] proof of Corollary 4.2), we can choose an open ball $B(0, r) \subset \tilde{D}$ so that the flow $\phi_t^{X_u}$ is defined on an open neighbourhood W of x for all $|t| \leq 1$. We set $\Phi(t, y, u) = \phi_t^{X_u}(y)$, $t \in [0, 1]$, $y \in W$ and $u \in B \equiv B(0, r) \subset \tilde{\triangle}$

Lemma 2.11
For any smooth map $\Phi : \mathbb{R} \times W \times B \to E$ we denote by $D_t\Phi(t, y, u)$ (resp. $D_y\Phi(t, y, u)$, resp. $D_u\Phi(t, y, u)$ the partial derivative of Φ according to the first (resp. the second (resp. the third)

variable, at point $(t,y,u)\in\mathbb{R}\times,W\times B$.
With these notations, $u\to\Phi(t,y,u)$ *is smooth . Moreover assume that* $T_x\phi_t^{X_u}[\mathcal{D}_x]=\mathcal{D}_{\phi_t^{X_u}(x)}$ *for all* t *such that* $(x,t)\in\Omega_{X_u}$ *and all* $u\in B$*, then we have:*

$$D_u\phi(t,x,u)(\triangle)\subset\mathcal{D}_{x(t)} \tag{11}$$

where $x(t)=\phi(t,y,u)$.

Proof
At first, we fix $y\in W$ and $u\in B$, and we set :
$y(t)=\phi(t,y,u)$ (the integral curve of X_u though y);
$X(t,y,u)=X_u(y(t,u))$;
$A(t)=D_yX(t,y,u)$;
$B(t)=D_uX(t,y,u)$.

Of course, A (resp. B) is a smooth field of operators in $L(E,E)$ (resp. $L(\tilde{\triangle},E)$). In fact, we have $B(t)=\Theta_{y(t)}$. So, in the Banach space $L(\tilde{\triangle},E)$), the linear differential equation

$$\dot{\Sigma}=A\circ\Sigma+B$$

as an unique solution with initial condition $U(0)=0$ given by

$$\Sigma(t,u)=\Gamma_t\int_0^t(\Gamma_s)^{-1}\circ\Theta_{y(s)}ds \tag{12}$$

where Γ_s is the solution of the differential equation

$$\dot{\Gamma}=A\circ\Gamma$$

with initial condition $\Gamma_0=Id_E$
From (10.7.3) and (10.7.4) of [Di], we obtain that ϕ is smooth in the third variable and we have

$$D_u\phi(t,y,u)=\Sigma(t,u). \tag{13}$$

We now look for the integral curve $x(t)$ through x. In this case, Γ_s is in fact the $t\to\phi_t^{X_u}(x)$ (see[Di] (10.8.5)), from our assumption of invariance by $\phi_t^{X_u}(x)$, we have:

$$\Gamma_s(\mathcal{D}_x)=\mathcal{D}_{x(s)} \tag{14}$$

On the other hand, from the assumption of lower triviality, we have $\Theta_{x(s)}(\mathcal{D}_x)\subset\mathcal{D}_{s(s)}$. So, we get

$$(\Gamma_s)^{-1}\circ\Theta_{x(s)}(\mathcal{D}_x)\subset\mathcal{D}_x$$

and moreover by integration we also have

$$\int_0^t(\Gamma_s)^{-1}\circ\Theta_{x(s)}(\mathcal{D}_x)\subset\mathcal{D}_x$$

Finally, using (14), (12) and (12), we obtain the announced result.

△

We are now in situation to give a sufficient condition of the existence of an integral manifold through $x\in M$:

Proposition 2.12
Consider the map:

$$\Phi : B \to M \text{ defined by } \Phi(u) = \Phi(1, x, u) = \phi_1^{X_u}(x), \text{ for } u \in B \equiv B(0, r) \subset \tilde{\triangle}. \tag{15}$$

There exists $\delta > 0$ such that $\Phi : B(0, \delta) \to M$ is a weak submanifold of M. Moreover, if we have $T_x\phi_t^{X_u}[\mathcal{D}_x] = \mathcal{D}_{\phi_t^{X_u}(x)}$ for all t such that $(x, t) \in \Omega_{X_u}$ and all $u \in B$, then, for $\delta > 0$ small enough, $(B(0, \delta), \Phi)$ is an integral manifold of $\mathcal{D}$ through x

It is clear that Proposition 2.12, **ends the proof of part (1) of Theorem 1**.

We now end this subsection with the proof of the previous Proposition.

Proof

According to Lemma 2.7, it follows that, for $\delta > 0$ small enough, $(B(0, \delta), \Phi)$ is a weak submanifold of M.

Assume now that $T_x\phi_t^{X_u}[\mathcal{D}_x] = \mathcal{D}_{\phi_t^{X_u}(x)}$ for all t such that $(x, t) \in \Omega_{X_u}$ and all $u \in B$. From Lemma 2.11, for any $u \in B \subset \tilde{\triangle}$, we have:

$$D_u\Phi(\tilde{\triangle}) \subset \mathcal{D}_{\Phi(u)}.$$

So, it follows that

$$T_u\Phi(\tilde{\triangle}) \subset \mathcal{D}_{\Phi(u)}$$

for all $u \in B$.
Now, from the assumption of invariance, we have

$$[T_x\phi_1^{X_u}]^{-1} \circ T_u\Phi(\tilde{\triangle}) \subset [T_x\phi_1^{X_u}]^{-1}(\mathcal{D}_{\Phi(u)}) = \mathcal{D}_x \equiv F \tag{16}$$

We set $\Lambda_u = [T_x\phi_1^{X_u}]^{-1} \circ T_u\Phi$ for $u \in B$. In particular, Λ_u is a continuous operator from the Banach space $\tilde{\triangle}$ to the normed space $\triangle$. The part (1) will be a consequence of the following Lemma:

Lemma 2.13
Let be E_1 (resp. E_2) a Banach space (resp. a normed space). Suppose that the set $L_s(E_1, E_2)$ of surjective operators in $L(E_1, E_2)$ is non empty. Then, $L_s(E_1, E_2)$ is an open set.

Proof

The first part of this proof is an adaptation of an argument which can be found in [QuZu].

Recall that an operator, $T \in L(E_1, E_2)$ is almost open, if for any open ball $B(0, r)$ in E_2, there exists an open ball $\tilde{B}(0, \rho) \subset E_1$ such that :

$$B(0, r) \subset \overline{T(\tilde{B}(0, \rho))}$$

Given $\alpha \in]0, 1[$, there exists $\rho > 0$ such that, for any $y \in B(0, 1)$ we can find $x_1 \in \tilde{B}(0, \rho)$ such that $||y - T(x_1)|| \leq \alpha$. So, $\dfrac{1}{\alpha}||y - T(x_1)|| \leq 1$, and then, there exists $x_2 \in \tilde{B}(0, \rho)$ such that

$$||\frac{1}{\alpha}(y - T(x_1)) - T(x_2)|| \leq \alpha \text{ i. e. } ||y - T(x_1) - \alpha T(x_2)|| \leq \alpha^2.$$

By induction, we can build a sequence (x_n) such that $x_n \in \tilde{B}(0, \rho)$ and also

$$||y - T(x_1 + \alpha x_2 + \cdots + \alpha^{n-1}x_n)|| \leq \alpha^n.$$

In the Banach space E_1, the series of general term $||\alpha^{n-1}x_n||$ converges. So, there exists $z \in E_1$ such that $z = \sum_{n=1}^{\infty} \alpha^{n-1}x_n$, with $||z|| \leq \frac{\rho}{1-\alpha}$ and, of course, $y = T(z)$. It follows that T must be surjective. On the other hand, the set of almost open operator in $L(E_1, E_2)$ is an open set (see [Ha], [HaMb]), so the Lemma is proved.

$\triangle$

Coming back to the proof of part(1), the map $T_0\Phi$ is the inclusion map of $\tilde{F}$ in F and $[T_x\phi_1^{X_0}] = Id_E$ so Λ_0 is surjective. From Lemma 2.13, for $\delta > 0$ small enough, Λ_u is surjective for all $u \in B(0, \delta)$; in particular, we get an equality

$$[T_x\phi_1^{X_u}]^{-1} \circ T_u\Phi(\tilde{F}) = [T_x\phi_1^{X_u}]^{-1}(\mathcal{D}_{\Phi(u)})$$

in (16) which ends the proof of Proposition 2.12.

$\triangle$

Proof of Part (2)

In this subsection, we will use the notations introduced in the previous one. In particular, for any $x \in M$, we associate an integral manifold $(B(0, \delta), \Phi)$ build in Proposition 2.12. Such an integral manifold will be called a **slice** through x.

At first, we must prove that the relation $\mathcal{R}$ is transitive. This fact is a direct consequence of the following Lemma:

Lemma 2.14

1. *Given any integral manifold (N, f) of $\mathcal{D}$ through $x \in M$, there exists a slice $(B(0, \delta), \Phi)$ such that $\Phi(0) = x$ and $f^{-1}[\Phi(B(0, \triangle)]$ is an open set in N*
2. *For any two integral manifolds (N, f) and (N', f') through $x \in M$, then $f^{-1}[f(N) \cap f'(N')]$ (resp. ${f'}^{-1}[f(N) \cap f'(N')]$) is open in N (resp. N'). Moreover, $L = f(N) \cup f'(N') \subset M$ has a natural structure of Banach manifold modeled on $\tilde{\mathcal{D}}_x$ and $(L, i_L)$is an integral manifold of $\mathcal{D}$ through x, where i_L is the natural inclusion of L in M.*

Proof

We fix any $x \in f(N)$. Note that N is a connected Banach manifold modelled on $\tilde{\triangle} \equiv \tilde{\mathcal{D}}_x$. As the problem is local, according to Remark 2.1, we can assume that N is an open subset of $\tilde{\triangle}$, M is an open subset of $E \equiv T_xM$ and f is the natural inclusion i of $\tilde{\triangle}$ into E (restricted to N). Consider a lower trivialization $\Theta : \tilde{\triangle} \times V \to M$ around x. Given any $u \in \tilde{\triangle}$, according to the arguments used in the proof of Proposition 2.8, (with Θ instead of Θ'), we get that the restriction of $X_u = \Theta(u, \ )$ to $i(N)$ induces a vector field $\tilde{Y_u}$ on $i(N)$ relative to its natural Banach manifold structure. It follows that the integral curve $t \to \Phi_t^{X_u}(x)$ of X_u through x lies in $i(N)$. So, for δ small enough, $\Phi[B(0, \delta)]$ is contained in $i(N) \subset i(\tilde{\triangle}) \subset E$. But as sets, we have $i(\tilde{\triangle}) = \tilde{\triangle} = \triangle$. So using the same arguments used in the proof of part (1) of Theorem 1, but in the Banach space $\tilde{\triangle}$, we can prove that Φ is a local diffeomorphism of $B(0, \delta)$ into N for δ small enough. In particular $L = \Phi[B(0, \delta)]$ is an open subset for the topology of the Banach structure on $i(N)$ which ends the proof of part (1).

Let be (N, f) and (N', f') integral manifolds through $x \in M$. Note that N and N' are connected Banach manifold modelled on $\tilde{\triangle} \equiv \tilde{\mathcal{D}}_x$. Applying part (1) for any $z \inf(N) \cap f'(N')$ to the integral manifold (N, f) (resp. (N', f') we obtain that $f^{-1}[f(N) \cap f'(N')]$ (resp. ${f'}^{-1}[f(N) \cap f'(N')]$) is open in N (resp. N').

Consider $L = f(N) \cup f'(N') \subset M$. It is clear that L is connected. From part(1), For any $z \in L$ there exists a slice $(B(0, \delta), \Phi)$ such that $\Phi(0) = z$ so we get a covering of L by slices. On the other hand, if we have two slices $(B(0, \delta), \Phi$ and $(B(0, \delta'), \Phi')$ so that $\Phi(B(0, \delta)) \cap \Phi'(B(0, \delta'))$ is not empty, than from part (1), the restriction of $\Phi^{-1} \circ \Phi'$ to ${\Phi'}^{-1}[\Phi(B(0, \delta)) \cap \Phi'(B(0, \delta'))]$ is a diffeomorphim on $\Phi^{-1}[\Phi(B(0, \delta)) \cap \Phi'(B(0, \delta'))]$. So we get a structure of connected Banach manifold on L, modelled on $\tilde{\triangle}$. As each slice is an integral manifold of $\mathcal{D}$ modelled on $\tilde{\triangle}$, It is clear that (L, i_L) is an integral manifold of $\mathcal{D}$, where i_L is the natural inclusion of L in M.

$\triangle$

It remains to show that any equivalent class $L(x)$ of $x \in M$ carries a natural structure of connected Banach manifold modelled on $\tilde{\mathcal{D}}_x$. Note that $L(x)$ is the union of all the subset $f(N)$ where (N, f) any integral manifold through x. So $L(x)$ is connected. Moreover, as in the proof of part(2) Lemma 2.14, we can cover $L(x)$ by slices and this gives rise to a natural structure of connected Banach manifold on $L(x)$. Again, $(L(x), i_{L(x)})$ is an integral manifold of $\mathcal{D}$ through x, which is maximal by construction.

3 Integrability and Lie invariance

3.1 Case of lower trivial weak distribution

Let $\mathcal{D}$ be a lower trivial weak distribution on M and U the domain of a chart around $x \in M$. Consider a local vector field X defined on a chart domain V and let $\gamma : [\alpha, \beta] \to V$ be an integral curve of X. As in [St] and [ChSt], we define :

Definition 3.1
On the chart domain V, we consider:

1. *An upper trivialization of $\mathcal{D}$ over γ is a a smooth map $\psi : [\alpha, \beta] \to L(G, TM)$ such that $\psi(t) \in L(G, TM)$, for some Banach space G.*
2. *Given an upper trivialization ψ as in (1) , the Lie derivative of ψ by X along γ is defined by:*

$$L_X\psi(\gamma(t)[v] = \dot{\psi}(t)[v] - DX(\gamma(t))[\psi(t)[v]] \tag{17}$$

Remark 3.2 *Definition 3.1 is independent of the choice of the chart and so (17) can be defined along any integral curve not necessary contained in a chart domain.*

Definition 3.3
Let $\mathcal{D}$ be a lower trivial weak distribution.

1. *Let $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$, a lower trivialization around x and $X_u = \Theta(u, \ )$ a lower section on V.*
   The weak distribution $\mathcal{D}$ is called **Lie invariant by** X_u *if for any $y \in V$, there exists $\varepsilon > 0$, such that, for all $0 < |\tau| < \varepsilon$, there exists:*
   - *an upper trivialization $\psi : [-\tau, \tau] \to L(G, TM)$ of $\mathcal{D}$ over $\gamma(t) = \phi_t^{X_u}(y)$ for $t \in [-\tau, \tau]$,*
   - *a smooth field of operator $\Lambda : [-\tau, \tau] \to L(G, G)$*

   such that

$$L_{X_u}\psi = \psi \circ \Lambda \tag{18}$$

2. *The weak distribution $\mathcal{D}$ is called* **Lie invariant** *if for any $x \in M$ there exists a lower trivialization $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$ such that, for any $u \in \mathcal{D}_x$, $\mathcal{D}$ is Lie invariant by $X_u = \Theta(u, \ )$.*

As in [ChSt], we have the following Theorem but without the assumption of closeness and existence of a complement for all subspaces $\mathcal{D}_x$

Theorem 2
Let $\mathcal{D}$ be a lower trivial weak distribution. The following properties are equivalent:

1. $\mathcal{D}$ *is integrable;*

2. $\mathcal{D}$ *is Lie invariant;*

3. $\mathcal{D}$ *is* $\mathcal{X}_{\mathcal{D}}^{-}$*-invariant.*

Proof of Theorem 2.

According to Theorem 1, we have only to prove the equivalence (2) $\Longleftrightarrow$ (3).

Assume that $\mathcal{D}$ is $\mathcal{X}_{\mathcal{D}}^{-}$-invariant. Let $x \in M$ be a fixed point and choose a lower trivialization $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$. Consider a lower section $X_u = \Theta(u, \ )$ and $y \in V$. Note that there exists $\varepsilon > 0$ such that the integral curve $t \mapsto \phi_t^{X_u}(y)$ of X_u through y is defined for all $0 < |t| < \varepsilon$. Choose any $0 < |\tau| < \varepsilon$ and set $\gamma(t) = \phi_t^{X_u}(y)$ for $t \in [-\tau, \tau]$. From our assumption, we have $T_y\phi_t^{X_u}(\mathcal{D}_y) = \mathcal{D}_{\gamma(t)}$. If $i_y : \tilde{\mathcal{D}}_y \to \mathcal{D}_y$ is the natural inclusion, denote by $\psi(t) = T_y\phi_t^{X_u} \circ i_y$. Set $\Gamma(t) = T_y\phi_t^{X_u}$. It is clear that ψ is an upper trivialization of $\mathcal{D}$ over γ. On the other hand, we have:

$$L_{X_u}\psi = [\dot{\Gamma} - DX_u(\gamma(t)) \circ \Gamma] \circ i_y$$

But, we have $\dot{\Gamma} = DX_u(\gamma(t)) \circ \Gamma$ (see proof of Lemma 2.11). So we have $L_{X_u}\psi = 0$ on $[-\tau, \tau]$.

For the converse, as in [St] [ChSt] and [Nu], we need the following result whose proof is somewhat different (each space $\mathcal{D}_x$ can be not closed here)

Lemma 3.4
Let X be a local vector field and ψ an upper trivialization of $\mathcal{D}$ defined over an integral curve $\gamma :]-\varepsilon, \varepsilon[\to V =dom(X)$. Moreover we assume that, for any $0 < |\tau| < \varepsilon$ there exists a smooth field $\Lambda : [-\tau, \tau] \to L(G, G)$ such that

$$L_X\psi = \psi \circ \Lambda.$$

Then, there exists $\varepsilon > 0$ such that $T_y\phi_t^X[\mathcal{D}_y] = \mathcal{D}_{\phi_t^X(y)}$ for all $0 < |t| < \varepsilon$

Now, assume that $\mathcal{D}$ is Lie invariant by X_u; let us fix some $y \in V = \mathrm{Dom}(X_u)$, and take a maximal integral curve $\gamma :]\alpha_y, \beta_y[\to V$ of X_u. Consider the set $I = \{t \in]\alpha_y, \beta_y[: T_y\phi_t^{X_u}[\mathcal{D}_y] = \mathcal{D}_{\phi_t^{X_u}(y)}\}$. This set is clearly closed and also open from Lemma 3.4. So, we have $I =]\alpha_y, \beta_y[$ and finally we deduce that $\mathcal{D}$ is invariant by X_u $\triangle$

Proof of Lemma 3.4

Let $\psi : [-\tau, \tau] \to L(G, TM)$ be an upper trivialization of $\mathcal{D}$ over an integral curve γ of X such that $\gamma(0) = y \in V$. Consider any smooth field of operators $\sigma : [-\tau, \tau] \to L(G, G)$ and set $\tilde{\psi} = \psi \circ \sigma$. On a chart domain, we have

$$L_X\tilde{\psi} = \dot{\tilde{\psi}} - DX \circ \tilde{\psi} = \dot{\psi} \circ \sigma + \psi \circ \dot{\sigma} - DX \circ \psi \circ \sigma = L_X\psi \circ \sigma + \psi \circ \dot{\sigma} \quad (19)$$

Assume that $L_X\psi = \psi \circ \Lambda$ for some smooth field of operators $\Lambda : [-\tau, \tau] \to L(G, G)$. Then we have:

$$L_X\tilde{\psi} = \psi \circ \Lambda \circ \sigma + \psi \circ \dot{\sigma} = \psi \circ [\Lambda \circ \sigma + \dot{\sigma}] \quad (20)$$

Consider the solution (*again denoted by* σ) of the linear equation $\dot{\sigma} = (-\Lambda) \circ \sigma$ with initial condition $\sigma(0) = Id_G$. So σ is a smooth field of isomorphisms of G and in particular, for this choice of σ, we have $\tilde{\psi}(t)[G] = \mathcal{D}_\gamma(t)$ for any $t \in [-\tau, \tau]$. Moreover, from(20), we have $L_X\tilde{\psi} = 0$.

Now, we can assume that there exists an upper trivialization $\psi : [-\tau, \tau] \to L(G, TM)$ such that $L_{X_u}\psi = 0$ on γ. Again we set $\Gamma(t) = T_y\phi_t^X$. Then $\Sigma(t) = [\Gamma(t)]^{-1} \circ \psi(t)$ is a smooth field of continuous operators from G to $E \equiv T_xM$ defined on $[-\tau, \tau]$.

On a chart domain we have

$$\dot{\psi} = \dot{\Gamma} \circ \Sigma + \Gamma \circ \dot{\Sigma} = DX_u(\gamma) \circ \Gamma \circ \Gamma^{-1} \circ \psi + \Gamma \circ \dot{\Sigma} = DX_u \circ \psi + \Gamma \circ \dot{\Sigma}$$

It follows that, on $[-\tau, \tau]$ we have
$\dot{\psi} = \dot{\Gamma} \circ \Sigma + \Gamma \circ \dot{\Sigma} = DX_u(\gamma) \circ \Gamma \circ \psi + \Gamma \circ \dot{\Sigma} = DX_u \circ \psi + \Gamma \circ \dot{\Sigma}$
So, it follows that, on $[-\tau, \tau]$ we have
$L_X \psi = \Gamma \circ \dot{\Sigma}$.
From our assumption, as, $\Gamma(t)$ is an isomorphism, we must have $\Sigma(t) = \Sigma(0) = \psi(0)$. We conclude that, for any $t \in [-\tau, \tau]$, we have $[\Gamma(t)]^{-1} \circ \psi(t)[G] = \psi(0)[G] = \mathcal{D}_y$ and finally

$$T_y \phi_t^X[\mathcal{D}_y] = \psi(t)[G] = \mathcal{D}_{\gamma(t)}. \tag{21}$$

Now, from the assumption in this Lemma, there exists $\varepsilon > 0$ such that, we are in the previous situation for any interval $[-\tau, \tau]$ with $0 < |\tau| < \varepsilon$ so that (21) is true for any $0 < |t| < \varepsilon$ △

3.2 Case of strong upper trivial weak distribution

Let $\mathcal{D}$ be a strong upper trivial weak distribution on M (see Definition 2.5). By analogy with lower sections (see subsection 2.2), for any strong upper trivialization $\Psi : F \times V \to TM$ such that the associated lower trivialization $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$, an **upper section** is a local vector field on M defined by

$$Z(y) = \Psi(u, y) \text{ for any } u \in F \tag{22}$$

The set $\mathcal{X}^+_{\mathcal{D}}$ of upper sections generates $\mathcal{D}$. Note that any lower section of $\mathcal{D}$ can be written $\Theta(\Psi(u, x), \ )$. So the set $\mathcal{X}^-_{\mathcal{D}}$ of lower sections coming from a lower trivialization associated to any strong upper trivialization is a subset of $\mathcal{X}^+_{\mathcal{D}}$.

Let $\mathcal{D}$ be a strong upper trivial weak distribution on M. Let V be the domain of a chart around $x \in M$. Consider a strong upper trivialization $\Psi : F \times V \to TM$ and $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$ the associated lower section. Given any smooth function $\sigma : V \to F$, let $Z_\sigma = \Psi(\sigma, \ )$ be the associated vector field on V. Consider $\gamma : [-\tau, \tau] \to V$ an integral curve of Z_σ, then, $\Psi_{\gamma(\ )}$ is an upper trivialization of $\mathcal{D}$ along γ. According to Definition 3.1 , the Lie derivative of Ψ along γ is $L_{Z_\sigma} \Psi_{\gamma(\ )}$ which we simply denoted by $L_{Z_\sigma} \Psi$

Remark 3.5 *If $\Psi : F \times V \to TM$ is an upper trivialization we have (see [St]):*

$$L_{Z_\sigma} \Psi(v, \gamma(t)) = [Z_\sigma, Z_v](\gamma(t)) \textit{ for any } Z_v = \Psi(v, \ ) \tag{23}$$

Definition 3.6
A strong upper trivial weak distribution $\mathcal{D}$ is called Lie bracket invariant if, for any $x \in M$, there exists an upper trivialization $\Psi : F \times V \to TM$ such that for any $u \in F$, there exists $\varepsilon > 0$, such that, for all $0 < |\tau| < \varepsilon$, there exists a smooth field of operator $\Lambda : [-\tau, \tau] \to L(F, F)$ with the following property

$$L_{X_u} \Psi = \Psi \circ \Lambda \tag{24}$$

along the integral curve $t \mapsto \phi_t^{X_u}(x)$ on $[-\tau, \tau]$ of any lower section $X_u = \Theta(\Psi(u, x), \ )$.

Remark 3.7 *According to Remark 3.5, the property (24) is equivalent to*

$$[X_u, Z_v](\gamma(t)) = \Psi(\Lambda(t)[v], \gamma(t)) \textit{ for any } Z_v = \Psi(v, \ ) \tag{25}$$

along $\gamma(t) = \phi_t^{X_u}(x)$.
(25) justifies the term "Lie bracket invariant" in Definition 3.6.

With these definitions we have:

Theorem 3
Let $\mathcal{D}$ be a strong upper trivial weak distribution. The following propositions are equivalent:

1. $\mathcal{D}$ *is integrable;*
2. $\mathcal{D}$ *is Lie bracket invariant;*
3. $\mathcal{D}$ *is* $\mathcal{X}_{\mathcal{D}}^{-}$*-invariant.*

Coming back to the context of Corollary 2.6, let $\Pi : \mathcal{F} \to M$ be a Banach fiber bundle over M with typical fiber F, $\Psi : \mathcal{F} \to TM$ a morphism of bundle whose kernel is complemented in each fiber. We denote by $\mathcal{S}(\mathcal{F})$ the set of local sections of $\Pi : \mathcal{F} \to M$, that is smooth maps $\sigma : U \subset M \to \mathcal{F}$ such that $\Pi \circ \sigma = Id_U$ where U is an open set of M. A subset $\mathcal{S}$ of $\mathcal{S}(\mathcal{F})$ is called a **generating upper set** of $\mathcal{D}$ if for any $x \in M$, the set $\mathcal{X}_{\mathcal{S}} = \{\Psi \circ \sigma,\ \sigma \in \mathcal{S}\}$ contains $\mathcal{X}_{\mathcal{D}}^{+}$. Of course $\mathcal{S}(\mathcal{F})$ is a maximal generating upper set. From Theorem 3 we get the following Theorem

Theorem 4
Let $\Pi : \mathcal{F} \to M$ be a Banach fiber bundle over M with typical fiber F and $\Psi : \mathcal{F} \to TM$ a morphism of bundles such that the kernel of Ψ is complemented in each fiber and denote $\mathcal{D} = Im\ \Psi$.

1. *Then $\mathcal{D}$ is an integrable distribution if and only there exists a generating upper set $\mathcal{S}$ such that:*

   (LB) For any local section $\sigma \in \mathcal{S}$ there exists an open set $V \subset M$ on which $\mathcal{F}$ is trivializable and σ is defined on V such that, for any $x \in V$ we have the following property: there exists $\varepsilon > 0$ such that for any integral curve $\gamma :]-\varepsilon, \varepsilon[\to V$ of $X = \Psi \circ \sigma$ with $\gamma(0) = x$, there exists a smooth field $\Lambda :]-\varepsilon, \varepsilon[\to L(\mathcal{F}_x, \mathcal{F}_x)$ we have

   $$[\Psi \circ \sigma, \Psi(u,\ )](\gamma(t)) = \Psi(\Lambda(t)[u], \gamma(t)) \text{ for any } t \in]-\varepsilon, \varepsilon[\text{ for any} u \in \mathcal{F}_x \tag{26}$$

   Moreover, if (LB) is true and if S_x fulfills $\mathcal{F}_x = \ker \Psi_x \oplus S_x$, there exists $\Lambda :]-\varepsilon, \varepsilon[\to L(\mathcal{F}_x, S_x)$ which satisfies (26)

2. *If $\mathcal{D}$ is a closed distribution, then this distribution is integrable if and only if (LB) is satisfied where (26) can be replaced by*

   $$[\Psi \circ \sigma, \Psi(u,\ )](\gamma(t)) \in \Psi_{\gamma(t)}(S_x) \text{ for any } t \in]-\varepsilon, \varepsilon[\text{ for any} u \in \mathcal{F}_x \tag{27}$$

Remark 3.8

1. *The assumption "the kernel of Ψ is complemented in each fiber" is automatically satisfied if the kernel of Ψ is finite dimensional or finite codimensional in each fiber, or in the context of Hilbert manifold.*
2. *When M is a finite dimensional manifold, Theorem 4 gives exactly Theorem 4.7 of [Ba]. So we obtain a generalization of classical Stefan-Sussman Theorem for locally finitely generated distribution which takes into account some remarks of [Ba] about the original proofs ([Su],[St]).*
3. *When $\mathcal{F}$ is a subbundle of TM, Theorem 4 is equivalent to the version of a Frobenius Theorem which can be found in [Gl] see also Example 4.3 (2).*

3.3 Proofs of Theorem 3 and Theorem 4

Proof of Theorem 3

According to Theorem 1, we have only to prove (1) $\Longleftrightarrow$ (2).

From Lemma 3.4, assumption (2) implies that for any $x \in M$, we have $T_x\phi_t^{X_u}(\mathcal{D}_x) = \mathcal{D}_{\phi_t^{X_u}(x)}$ for all t such that $(x,t) \in \Omega_{X_u}$. From Proposition 2.12, $(B(0,\delta),\Phi)$ is an integral manifold through x. So (2) $\Longrightarrow$ (1).

For the converse, we will use the following Lemma:

Lemma 3.9
Let $\Psi : F \times V \to TM$ be a strong upper trivialization, and $\sigma : V \to F$ a smooth map and let $X = \Psi(\sigma, \ )$ be the associated vector field on V. Consider any integral curve $\gamma : [-\tau,\tau] \to V$ of X such that $\gamma(0) = x$. Then there exists a smooth field $\Lambda : [-\tau,\tau] \to L(F,F)$ such that :

$$L_X\Psi(v,\gamma(t)) = \Psi(\Lambda(t)[v],\gamma(t))$$

So, for $\sigma(y) = (u,y)$, with $u \in S$, the vector field Z_σ is the lower section X_u for $u \in S$ et clearly Lemma 3.9 proves (1) $\Longrightarrow$ (2).

$\triangle$

Proof of Lemma 3.9

Assume that $\mathcal{D}$ is integrable and fix $x \in M$. Take a strong upper trivialization $\Psi : F \times V \to TM$ around x an let be $\Theta : \tilde{\mathcal{D}}_x \times V \to TM$ the associated lower trivialization. We can choose V such that $TM_{|V} \equiv E \times V$. Recall that we have the decomposition $F = \ker\Psi_x \oplus S$, and $\Theta = (\theta_y \circ [\theta_x]^{-1}, \ )$ where θ_y is the restriction to S of Ψ_y (see the proof of Proposition 2.4). At first note that any lower section $X_u = \Theta(\Psi(u,x), \ )$ for some $u \in F$ can be written $X_u = \theta(u, \ )$ but with $u \in S$. On the other hand, from Lemma 2.14, $(B(0,\delta),\Phi)$ is an integral manifold of $\mathcal{D}$ through x (associated to Θ). In particular, $\tilde{\theta}_y$ is an isomorphism from S to $\tilde{\mathcal{D}}_y$. Given $u \in F$, there exists an unique $v \in S$ such that $\Psi_y(u) = \theta_y(v)$ so $u \in \ker\Psi_y \oplus S$. It follows that $F = \ker\Psi_y \oplus S$.

Set $N = \Phi(B(0,\delta)) \subset M$ endowed with its Banach manifold structure. Without loss of generality, we can identify S with $\tilde{\theta}_x(S) = \tilde{\mathcal{D}}_x$ so that, N is an open set of $\tilde{\mathcal{D}}_x$ and denote by $i : \tilde{\mathcal{D}}_x \to T_xM \equiv E$ the canonical inclusion. We have $T_yN \equiv S \times \{y\}$. On N, each $\tilde{\Psi}_y$ can be considered as an element of $L(F,S)$. By analog arguments as the ones used in the proof of Lemma 2.10 we can show that $y \mapsto \tilde{\Psi}_y$ is a smooth field of operators in $L(F,S)$. So, $y \mapsto \tilde{\theta}_y$ is a smooth field of isomorphisms of S. We get a smooth field $\pi_y = [\tilde{\theta}_y]^{-1} \circ \tilde{\Psi}_y$ of operators in $L(F,S)$ with the following properties:

$$\Psi_y = \theta_y \circ \pi_y \tag{28}$$

$$\ker\pi_y = \ker\tilde{\Psi}_y = \ker\Psi_y \tag{29}$$

$$\pi_y(u) = u \text{ for all } u \in S \tag{30}$$

Take any smooth map $\sigma : V \to F$. Then $Z_\sigma(y) = \Psi_y \circ \sigma(y)$ for $y \in V$ (resp. $\tilde{Z}_\sigma(y) = \tilde{\Psi}_y \circ \sigma(y)$ for $y \in N$) is a smooth vector field on V (resp. on N) and we have the relations:

$$\Psi(\sigma(y),y) = Z_\sigma(i(y)) = i[\tilde{Z}_\sigma(y)] = i \circ \tilde{\theta}_{i(y)} \circ \pi_y(\sigma(y)) = \theta_{i(y)} \circ \pi_y(\sigma(y)) = \theta(\pi_y(\sigma(y)),y) \tag{31}$$

Consider the integral curves $\gamma(t) = \phi_t^{Z_\sigma}(x)$ and $\tilde{\gamma}(t) = \phi_t^{\tilde{Z}_\sigma}(x)$ for $t \in [-\tau,\tau]$. Of course we have $\gamma(t) = i \circ \tilde{\gamma}(t)$. For simplicity, we set:
$\sigma(\gamma(t)) = \sigma(t)$ and $\sigma(\tilde{\gamma}(t)) = \tilde{\sigma}(t)$
$P(t) = \pi_{\tilde{\gamma}(t)}$.
Note that , from (31) we have

$$\Psi(v\ ,\gamma(t)) = \theta(P(t)[v],\gamma(t)) \tag{32}$$

Now, from (23), for any $v \in S$, we have:

$$L_{Z_\sigma}\Psi(v,\gamma(t)) = [Z_\sigma, X_v](\gamma(t)) = L_{Z_\sigma}\theta(v,\gamma(t)). \tag{33}$$

As we have $Z_\sigma = i_*\tilde{Z}_\sigma$ and $X_v = i_*\tilde{X}_v$, on the Banach manifold N, we get

$$[\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t)) \in T_{\tilde{\gamma}(t)}N \equiv S \times \{\tilde{\gamma}(t)\}.$$

Note that we have $[Z_\sigma, X_v] = i_*[\tilde{Z}_\sigma, \tilde{X}_v]$. It follows that we have:

$$L_{Z_\sigma}\theta(v, \gamma(t)) = i_*[\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t)) \tag{34}$$

Without loss of generality, we can choose $\delta > 0$ small enough such that on $N = \Phi(B(0, \delta))$ we have :

$$||\tilde{\theta}(\ , y)|| \leq K \text{ and } ||D_2\tilde{\theta}(\ , y)[\ ]|| \leq K \text{ for all } y \in N. \tag{35}$$

On the other hand, we have:

$$[\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t)) = D_2\tilde{\theta}(v, \tilde{\gamma}(t))[\tilde{\theta}(P(t)[\tilde{\sigma}(t)], \tilde{\gamma}(t))] - D_2\tilde{\theta}(\tilde{\theta}(P(t)[\tilde{\sigma}(t)], \tilde{\gamma}(t))[\tilde{\theta}(v, \tilde{\gamma}(t))]. \tag{36}$$

From (35), we have :

$$\begin{aligned} ||D_2\tilde{\theta}(v, \tilde{\gamma}(t))[\tilde{\theta}(P(t)[\tilde{\sigma}(t)], \tilde{\gamma}(t))]|| \leq K||v||.||\tilde{\theta}(P(t)[\tilde{\sigma}(t)], \tilde{\gamma}(t)|| &\leq K^2||v||.||P(t)||.||\tilde{\sigma}(t)|| \\ &\leq K^2.||v||.||P||_\infty||\tilde{\sigma}||_\infty. \end{aligned} \tag{37}$$

In the second member of (36), the same majoration is true for

$$||D_2\tilde{\theta}(\tilde{\theta}(P(t)[\tilde{\sigma}(t)], \tilde{\gamma}(t))[\tilde{\theta}(v, \tilde{\gamma}(t))||.$$

So we get

$$||[\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t))|| \leq 2K^2||P||_\infty||\tilde{\sigma}||_\infty||v||$$

It follows that , for each $t \in [-\tau, \tau]$, the map $v \to [\tilde{Z}_\sigma, \tilde{X}_v](\tilde{\gamma}(t))$ is linear continuous map $\tilde{\Lambda}_{\tilde{\gamma}(t)}$ from S to $S \times \{\tilde{\gamma}(t)\}$. We set

$$\bar{\Lambda}(t) = [\tilde{\theta}_{\tilde{\gamma}(t)}]^{-1} \circ \tilde{\Lambda}_{\tilde{\gamma}(t)}$$

Clearly, $t \mapsto \overline{\Lambda}(t)$ is a smooth field of endomorphisms of S and taking into account (34) we get

$$L_{Z_\sigma}\theta(v, \gamma(t)) = \theta(\bar{\Lambda}(t)[v], \gamma(t)) \tag{38}$$

Now from (32), with the same argument (19) used in the proof of Lemma 3.4), we get:

$$L_{Z_\sigma}\Psi_{\gamma(\ )} = L_{Z_\sigma}\{\theta_{\gamma(\ )} \circ P\} + \theta_{\gamma(\ )} \circ \dot{P} = \theta_{\gamma(\ )} \circ P \circ \bar{\Lambda} + \theta_{\gamma(\ )} \circ \dot{P} \tag{39}$$

But, from the definition of P and (30) we have $P \circ \dot{P} = \dot{P}$. So from (39), we get:

$$L_{Z_\sigma}\Psi = \Psi(\bar{\Lambda}[\ ] + \dot{P}[\ ],\ )$$

which ends the proof of Lemma 3.9 by setting $\Lambda = \bar{\Lambda} + \dot{P}$.

△

Proof of Theorem 4

(1) According to the context of the proof of Corollary 2.6, for any given $x \in M$ we consider a local trivialization of $\mathcal{F}$ on an open set V around x, so that the morphism Ψ can be identified, as a map $\Psi : \mathcal{F}_x \times V \to TM$ and $\mathcal{F}_x = \ker \Psi_x \oplus S_x$ and let $\Theta : S_x \times V \to TM$ be the associated lower trivialization. In this context, taking any $\sigma(y) = (u, x)$, for any $u \in S_x$ in (LB) for any $x \in M$, we get property (2) of Theorem 3 so (LB) is a sufficient condition for integrability of $\mathcal{D}$.
Assume now that $\mathcal{D}$ is integrable and consider an upper generating set $\mathcal{S}$ of $\mathcal{D}$ and any section $\sigma \in \mathcal{S}$ defined on an open set U. Fix any $x \in U$. From Lemma 3.9 we get (26) with $\Lambda :]-\varepsilon, \varepsilon[\to L(\mathcal{F}_x, S_x)$ So in particular, if (LB) is true for some $\Lambda :]-\varepsilon, \varepsilon[\to L(\mathcal{F}_x, \mathcal{F}_x)$, then $\mathcal{D}$ is integrable and by Lemma

3.9 we can find $\Lambda' :]-\varepsilon,\varepsilon[\to L(\mathcal{F}_x, S_x)$ which satisfies (26) .

(2) If now $\mathcal{D}$ is a closed distribution, then in each fiber $\mathcal{F}_x = \ker\Psi_x \oplus S_x$, θ_x is a continuous bijective morphism between both Banach space S_x and $\mathcal{D}_x$ so θ_x is an isomorphism. In particular, $\tilde{\mathcal{D}}_x$ and $\mathcal{D}_x$ are equivalent as Banach spaces. Coming back to the previous local context of the strong upper trivialization $\Psi : \mathcal{F}_x \times V \to TM$, the map $y \mapsto \theta_y$ is a smooth field of isomorphisms from S_x to $\Psi_{\gamma(t)}(S_x)$.
If $\mathcal{D}$ is integrable, from (26) and the properties of Λ we obtain (27). For the converse, it is sufficient to set

$$\Lambda(t)[u] = [\theta_{\gamma(t)}]^{-1}[Z_\sigma, Z_u](\gamma(t))$$

to get (26).

△

4 Applications

4.1 Banach Lie Algebroid

The concept of Lie algebroid was first introduced by J. Pradines [Pr], in relation with Lie groupoids. The theory of algebroids was developped by A. Weinstein ([We]) and, independently, by M. Karasev ([Ka]), in view of the symplectization of Poisson manifolds and applications to quantization (see also [Ma]). This notion admits a straightforward generalization to the infinite dimensional case on Banach manifold ([An]) and also when the base manifold is the infinite jet bundle over a fiber bundle, which gives a framework for evolutionary PDE. Given a Hamiltonian operator, we then get an involutive weak distribution spanned by hamiltonian evolutionary vector fields (cf. [KiVa]). We give here the definition of a Lie algebroid on a Banach manifold M in the following way:

Definition 4.1 *[An]:*
A Banach Lie algebroid structure on a Banach bundle $\Pi : \mathcal{A} \to M$ is a quadruple $(\mathcal{A}, \Psi, M, \{\ \})$ such that

1. *$\{\ ,\ \}$ is a composition law $(\sigma_1, \sigma_2) \mapsto \{\sigma_1, \sigma_2\}$ on the set of (global) sections $\Sigma(\mathcal{A})$ of $\Pi : \mathcal{A} \to M$ such that $(\Sigma(\mathcal{A}), \{\ ,\ \})$ has a Lie algebra structure;*
2. *$\Psi : \mathcal{A} \to TM$ is a smooth vector bundle map with satisfies the following two properties:*
   - **(i)** *the map $s \mapsto \Psi \circ s$ is a Lie algebra homomorphism;*
   - **(ii)** *for any smooth function f defined on M and any pair of sections (σ_1, σ_2) we have :*

   $$\{\sigma_1, f\sigma_2\} = f\{\sigma_1, \sigma_2\} + df[Z_{\sigma_1}]\sigma_2 \text{ (Leibniz formula)}$$

   where $Z_{\sigma_1} = \Psi \circ \sigma_1$ is the vector field associated to σ_1.

The quadruplet $(\mathcal{A}, \Psi, M, \{\ ,\ \})$ is called a Banach Lie algebroid, $\{\ ,\ \}$ (resp. Ψ) is called the **Lie bracket** *on $\mathcal{A}$, (resp.* **anchor morphism***).*

In this context, Ψ gives rise to a Lie algebra morphism from $\Sigma(\mathcal{A})$ into $\mathcal{X}_L(M)$

$$[\Psi \circ \sigma_1, \Psi \circ \sigma_2] = \Psi \circ \{\sigma_1, \sigma_2\} \tag{40}$$

Given some open set U in M, we denote by $\mathcal{A}_U$ the restriction of the Banach bundle $\Pi : \mathcal{A} \to \mathcal{M}$ to the Banach manifold U: $\mathcal{A}_U = \Pi^{-1}(U)$; the set of sections of $\mathcal{A}_U$ will be denote by $\Sigma(\mathcal{A}_U)$.

We will say that the Lie bracket $\{\ ,\ \}$ is **localizable** if for any open set U of M, there exists a unique Lie bracket $\{\ ,\ \}_U$ on the space of sections $\Sigma(\mathcal{A}_U)$ such that , for any σ_1 and σ_2 in $\Sigma(\mathcal{A})$, we have:

$$\{\sigma_{1|U}, \sigma_{1|U}\}_U = \{\sigma_1, \sigma_2\}_{|U}$$

In this case, we will say that the Lie bracket $\{\ ,\ \}$ is **compatible with its restriction** to U. When the Lie bracket $\{\ ,\ \}$ is localizable, then , $(\mathcal{A}_U, \Psi_{|U}, U, \{\ ,\ \}_U)$ is a Banach Lie algebroid for any open U. By same arguments as in finite dimension, when M is paracompact, we can prove that **any** Banach Lie algebroid $(\mathcal{A}, \Psi, M, \{\ ,\ \})$ has a localizable Lie bracket. Note that if M is a paracompact Banach manifold modelled on E, this Banach space needs to have a partition of unity (see for instance [Ar], [Ku], [Ll], [Va]). Moreover, any paracompact manifold modelled on a separable Hilbert space is paracompact ([Io]).

Definition 4.2
We say that a Banach bundle $\Pi : \mathcal{A} \to M$ has a structure of local Banach Lie algebroid, if there exists a Banach morphism bundle $\Psi : \mathcal{A} \to TM$ such that, for any $x \in M$, there exists an open neighbourdhood U of x such that, for any open set $V \subset U$ with $V \ni x$, we have a Banach Lie algebroid $(\mathcal{A}_V, \Psi_V, V, \{\ ,\ \}_V)$ compatible with the restriction to V of $(\mathcal{A}_U, \Psi_U, U, \{\ ,\ \}_U)$.

Of course a Banach Lie algebroid whose Lie bracket is localizable gives naturally rise to local Banach Lie algebroid structure on it . For instance, if M is paracompact, this situation always occurs. On the other hand, we have no example for which a local Banach Lie algebroid does not give rise to a Banach Lie algebroid structure.

From Theorem 4 we get:

Theorem 5
If a Banach bundle $\Pi : \mathcal{A} \to M$ has a structure of local Banach Lie algebroid associated to an anchor $\Psi : \mathcal{A} \to TM$ and if the kernel of Ψ is complemented in each fiber, then $\mathcal{D} = \Psi(\mathcal{A})$ is an integrable weak distribution.
In particular, for any Banach Lie algebroid $(\mathcal{A}, \Psi, M, \{\ ,\ \})$ whose Lie bracket is localizable, if the kernel of Ψ is complemented in each fiber, then $\mathcal{D} = \Psi(\mathcal{A})$ is an integrable weak distribution.

Example 4.3

1. *Consider a smooth right action $\psi : M \times G \to M$ of a connected Banach Lie group G over a Banach manifold M. Denote by $\mathcal{G}$ the Lie algebra of G. We have a natural morphism Ψ from the trivial Banach bundle $M \times \mathcal{G}$ into TM which is defined by*

$$\Psi(x, X) = T_{(x,e)}\psi(0, X)$$

For any X and Y in $\mathcal{G}$, we have:

$$\Psi(\{X, Y\}) = [\Psi(X), \Psi(Y)]$$

where $\{\ ,\ \}$ denote the Lie algebra bracket on $\mathcal{G}$ (see for instance [KrMi] chap. VIII, 36.12 or [Bo]).

It follows that $(M \times \mathcal{G}, \Psi, M, \{\ ,\ \})$ has a Banach Lie algebroid structure on M. Moreover, from the triviality of $M \times \mathcal{G}$, we get a localizable Lie bracket.
Denote by G_x the closed subgroup of isotropy of a point $x \in M$ and $\mathcal{G}_x \subset \mathcal{G}$ its Lie subalgebra. Of course, we have $\ker \Psi_x = \mathcal{G}_x$. According to Theorem 4, if $\mathcal{G}_x$ is complemented in $\mathcal{G}$ for any $x \in M$, the weak distribution $\mathcal{D} = \Psi(M \times \mathcal{G})$ is integrable. In fact the leaf through x is its orbit $\psi(x, G)$.

2. *Let $\Pi : \mathcal{A} \to M$ be a Banach bundle such that $\mathcal{A}$ is a subset of TM , Π is the restriction to $\mathcal{A}$ of the canonical projection of TM onto M and the inclusion $i : \mathcal{A} \to TM$ is a morphism*

bundle. Any local section of $\Pi : \mathcal{A} \to M$ induces a (local) vector field on M. If , for any $x \in M$, there exists an open $U \ni x$, such that the Lie bracket of vector fields induces a structure of Lie algebra on the set $\Sigma(\mathcal{A}_U)$, we get a natural structure of local Banach Lie algebroid on $\mathcal{A}$. So it follows from Theorem 5 that $\mathcal{D} = i(\mathcal{A})$ is an integrable distribution. When $\Pi : \mathcal{A} \to M$ is a Banach subbunble of TM we get a version of Frobenius Theorem as we can find in [Gl]. In the general situation we can also consider this result as an appropriate version of **Frobenius Theorem**.

3. *Let $\Pi : \mathcal{A} \to M$ be a Banach bundle and $\Psi : \mathcal{A} \to TM$ an injective morphism bundle. If $\mathcal{D} = \Psi\mathcal{A}$ satisfies the condition (LB) of Theorem 4 then $\mathcal{D}$ is integrable. From the injectivity of Ψ, it follows that, for any open set U of M that we can define a Lie algebra structure on the sections of $\Pi_{|U} : \mathcal{A}_U \to U$, by:*

$$\{s_1, s_2\}_U = \Psi^{-1}([\Psi(s_1), \Psi(s_2)]$$

So, we get a local Banach Lie algebroid structure on $\mathcal{A}$.

Proof of Theorem 5

We will show that the property (LB) of Theorem 4 is satisfied in our context. Of course the set $\mathcal{S}(\mathcal{A})$ of local sections of $\mathcal{A}$ is a generating upper set. Suppose that we have a structure of local Banach Lie algebroid struture on $\mathcal{A}$. As (LB) is a local property, we may assume that M is an open set of E and $\mathcal{A} \equiv F \times M$ if F is the typical fiber of $\mathcal{A}$. So we adopt the (local) notation used in the proof of Theorem 4.

Consider any section $\sigma \in \mathcal{S}(\mathcal{A})$ and fix some $x \in V$. Again, we set $Z_\sigma = \Psi \circ \sigma$ and $Z_u = \Psi(u,)$ an upper section. Given an integral curve $\gamma(t) = \phi_t^{Z_\sigma}(x)$ on $]-\varepsilon, \varepsilon[$, from (40), we have

$$[Z_\sigma, Z_u](\gamma(t)) = \Psi(\{\sigma, s_u\}(\gamma(t))) \text{ where } s_u(x) = (u, x).$$

But, using the same arguments as the ones used in the proof of Lemma 3.9, we can show that the map

$$t \mapsto \{\sigma, s_u\}(\gamma(t))$$

is a smooth field of endomorphisms of F. It follows that $\mathcal{D}$ satisfies (LB), and then, $\mathcal{D}$ is integrable.

$\triangle$

4.2 Banach Lie-Poisson manifold

We first recall the context of Banach Lie-Poisson manifold studied these last years (see for example [OdRa2]). In particular, we will prove in a large context the existence of weak symplectic leaves.

An **Lie bracket** on $C^\infty(M)$ is $\mathbb{R}$-bilinear antisymmetric pairing $\{.,\}$ on $C^\infty(M)$ which satisfies the Leibniz rule: $\{fg, h\} = f\{g, h\} + g\{f, h\}$ and the Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ for any $f, g, h \in C^\infty(M)$.

An **Poisson morphism** on M is a bundle morphism $\Psi : T^*M \to TM$ which is antisymmetric (i.e. such that $< \alpha, \Psi\beta >= - < \beta, \Psi\alpha >$ for any $\alpha, \beta \in T^*M$).

We can associate to a such morphism a $\mathbb{R}$- bilinear antisymmetric pairing $\{., .\}$ on the set $\mathcal{A}^1(M)$ of 1-form on M defined by:

$$\{\alpha, \beta\}_\Psi =< \beta, \Psi\alpha >$$

Moreover, for any $f \in C^\infty(M)$ we have:

$$\{f\alpha, \beta\}_\Psi =< \beta, f\Psi\alpha >= f\{\alpha, \beta\}$$

So, we get a bracket $\{., .\}_\Psi$ on $C^\infty(M)$ defined by

$$\{f, g\} = \{df, dg\}_\Psi$$

As in finite dimension, $\{.,.\}_\Psi$ satisfies the Jacobi identity if and only if the the Schouten-Nijenhuis bracket $[P,P]$ of P is identically zero (see for instance [MaRa]). In this case, $C^\infty(M)$ has a structure of Lie algebra and $(M,\{.,.\}_\Psi)$ is called a **Banach Lie Poisson manifold** (see for instance [OdRa1] or [OdRa2]).

From now, the Poisson morphism Ψ is fixed and for simplicity we denote $\{.,.\}$ the Lie bracket associated to Ψ.

Given such a Banach Lie-Poisson manifold, the distribution $\mathcal{D} = \Psi(T^*M)$ is called the **characteristic distribution**. Of course, in general $\mathcal{D}$ is not a closed distribution but it is a weak distribution.
Associated to $\{\ ,\ \}$, on T^*M, we have a natural skew-symmetric bilinear form ω defined as follows: for any α and β in T^*_xM, we have $\omega(\alpha,\beta) = \{f,g\}$ if f and g are smooth functions defined on a neighbourhood of x and such that $df(x) = \alpha$ and $dg(x) = \beta$ (this definition is independent of the choice of f and g).
For each x, on the quotient $T^*_xM/\ker\Psi_x$ we get a skew-symmetric bilinear form $\hat{\omega}_x$. On the other hand, let $\hat{\Psi}_x : T^*_xM/\ker\Psi_x \to \tilde{\mathcal{D}}_x$ be the canonical isomorphism associated to Ψ_x between Banach spaces. Finally we get a skew-symmetric bilinear form $\tilde{\omega}_x$ on $\tilde{\mathcal{D}}_x$ such that :

$$[\hat{\Psi}_x]^*\tilde{\omega}_x = \hat{\omega}_x$$

According to [OdRa2], a **symplectic leaf** of $\mathcal{D}$ is a weak submanifold $(\mathcal{L}, i)$ where $\mathcal{L} \subset M$ and $i : \mathcal{L} \to M$ is the natural inclusion with the following properties:

(i) $(\mathcal{L}, i)$ is a maximal integral manifold of $\mathcal{D}$ (in the sense of Theorem 1 part (2));

(ii) on $\mathcal{L}$ we have a weak symplectic form $\omega_\mathcal{L}$ such that $(\omega_\mathcal{L})_x = \tilde{\omega}_x$ for all $x \in \mathcal{L}$

Remark 4.4
As in the context of Lie Banach algebroid, we will say that a Lie bracket $\{\ ,\ \}$ on $C^\infty(M)$ is **localizable** *if for any open set U of M, there exists an unique Lie bracket $\{\ ,\ \}_U$ on $C^\infty(U)$ such that, for any f_1 and f_2 in $C^\infty(M)$, we have:*

$$\{f_{1|U}, f_{1|U}\}_U = \{f_1, f_2\}_{|U}$$

From our definition of Banach Lie-Poisson manifold, the Lie bracket associated to a Poisson morphism Ψ is always localizable. On the other hand, given any Lie bracket $\{.,.\}$ on $C^\infty(M)$, when M is paracompact, we can prove that $\{.,.\}$ is localizable and then we can associate a morphism $\Psi : T^*M \to T^{**}M$ naturally associated. If moreover, $\Psi(T^*M) \subset TM$, then we get the previous definition of Banach Lie-Poisson manifold (see for instance [OdRa1] or [OdRa2]).

Theorem 6
Let be $\Psi : T^*M \to TM$ a Poisson morphism. If the kernel of Ψ is complemented in each fiber, then the associated characteristic distribution $\mathcal{D}$ is integrable. Moreover, each maximal integral manifold has a natural structure of weak symplectic leaf.

For an illustration of this result, the reader will find many examples of Banach Lie-Poisson manifolds in [OdRa1] and [OdRa2] and in some references contained in these papers.

Proof of Theorem 6

At first, we can observe that the set

$$\mathcal{S} = \{\Psi(df) : f \in C^\infty(U), U \text{ any open set in } M\}$$

is an upper generating set for $\mathcal{D}$: given any $x \in M$, modulo any local chart around x, we can suppose that M is an open subset of E and $T^*M \equiv E^* \times M$; for any $\alpha \in E^*$ the function $f_\alpha(x) =< \alpha, x >$

is a smooth map on M such that $df_\alpha(y) = \alpha$ for any $y \in M$; so $Z_\alpha = \Psi(\alpha, y) = \Psi(df_\alpha(y))$ is an upper section.
For any smooth local function $f : U \to \mathbb{R}$, we set $Z_f = \Psi(df, \ )$. From the Jacobi identity in $C^\infty(M)$ we have

$$[Z_f, Z_g] = \Psi(d\{f, g\}, \ ) \text{ for any } f, g \in C^\infty(M) \tag{41}$$

According to Theorem 4, to prove the integrability of $\mathcal{D}$, we have only to prove (LB) for the generating upper set $\mathcal{S}$. As (LB) is a local property, again we assume that M is an open set in E. So fix some smooth function $f : M \to \mathbb{R}$ and consider an integral curve $\gamma(t) = \phi_t^{Z_f}(x)$ through $x \in M$ defined on $]-\varepsilon, \varepsilon[$. For any $\alpha \in E^*$, from (41), we have:

$$[Z_f, Z_\alpha](\gamma(t)) = [\Psi(df), \Psi(df_\alpha)](\gamma(t)) = \Psi(d\{f, f_\alpha\}(\gamma(t)))$$

But, using the same arguments as the ones used in the proof of Lemma 3.9 we can show that the map

$$y \mapsto [\alpha \mapsto d\{f, f_\alpha\}(y)]$$

is a smooth field of continuous operators from E^* to E^*. It follows that $\mathcal{D}$ satisfies (LB), and then, $\mathcal{D}$ is integrable.

Assume now that $\mathcal{D}$ is integrable and choose any maximal leaf $\mathcal{L}$. As $T_x\mathcal{L} = \tilde{\mathcal{D}}_x$, on $T_x\mathcal{L}$ we have the skew-symmetric bilinear form $\tilde{\omega}_x$ previously defined. We will show that $\tilde{\omega}_x$ defines a closed 2-form $\omega_\mathcal{L}$ on $\mathcal{L}$, which is a weak symplectic form.

Fix $x \in \mathcal{L}$. We have $T_x^*M = \ker \Psi_x \oplus S_x$. So $\mathcal{L}$ is a Banach manifold modeled on S_x. From the definition of $\tilde{\omega}_x$, we have

$$\tilde{\omega}_x(\tilde{\theta}_x(\alpha), \tilde{\theta}_x(\beta)) =< \alpha, \tilde{\theta}_x(\beta) > . \tag{42}$$

As we know that $\tilde{\theta}_x$ is an isomorphism from S_x to $T_x\mathcal{L}$ it follows that $\tilde{\omega}_x$ is a weak 2-symplectic form on the Banach space $T_x\mathcal{L}$. On the other hand, locally, from Lemma 2.10, it follows that $\omega_\mathcal{L}$ defined by $(\omega_\mathcal{L})_x = \tilde{\omega}_x$ is a smooth differential 2-form on $\mathcal{L}$. On the other hand, for any smooth function f defined on an open set $U \subset M$, we set $\tilde{f} = f \circ i$. So for any smooth functions f, g and h defined on U, the Jacobi identity is satisfied for $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ on the open set $i^{-1}(U) \subset \mathcal{L}$. So, by classical arguments of Poisson bracket (see for instance [LiMa], [OdRa1], [OdRa2]), we get: $d\omega_\mathcal{L}(i_*Z_f, i_*Z_g, i_*Z_h) = 0$ for any choice of functions f, g and h. So $\omega_\mathcal{L}$ is closed and the proof of Theorem 6 is completed.

△

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