CHARACTERIZATIONS OF Γ-AG**-GROUPOIDS BY THEIR Γ-IDEALS

Madad Khan, Naveed Ahmad and Inayatur Rehman

COMSATS Institute of Information Technology
Abbottabad Pakistan
madadmath@yahoo.com, naveed.maths@yahoo.com

Department of Mathematics, Quaid-i-Azam University
Islamabad, Pakistan
s_inayat@yahoo.com

Abstract. In this paper we have discusses Γ-left, Γ-right, Γ-bi-, Γ-quasi-, Γ-interior and Γ-ideals in Γ-AG**-groupoids and regular Γ-AG**-groupoids. Moreover we have proved that the set of Γ-ideals in a regular Γ-AG**-groupoid form a semi-lattice structure. Also we have characterized a regular Γ-AG**-groupoid in terms of left ideals.

1. Introduction

Kazim and Naseeruddin [4] have introduced the concept of an LA-semigroup. This structure is the generalization of a commutative semigroup. It is closely related with a commutative semigroup and commutative groups because if an LA-semigroup contains right identity then it becomes a commutative semigroup and if a new binary operation is defined on a commutative group which gives an LA-semigroup [9]. The connection of the class of LA-semigroups with the class of vector spaces over finite fields and fields has been given as: Let W be a sub-space of a vector space V over a field F of cardinal $2r$ such that $r > 1$. Many authors have generalized some useful results of semigroup theory.

In 1981, the notion of Γ-semigroups was introduced by M. K. Sen [6] and [7].

T. Shah and I. Rehman [14] defined Γ-AG-groupoids analogous to Γ-semigroups and then they introduce the notion of Γ-ideals and Γ-bi-ideals in Γ-AG-groupoids. It is easy to see that Γ-ideals and Γ-bi-ideals in Γ-AG-groupoids are infect a generalization of ideals and bi-ideals in AG-groupoids (for a suitable choice of Γ).

In this paper we define Γ-quasi-ideals and Γ-interior ideals in Γ-AG**-groupoids and generalize some results. Also we have proved that Γ-AG-groupoids with left identity and AG-groupoids with left identity coincide.

Let G and Γ be two non-empty sets. G is said to be a Γ-AG-groupoid if there exist a mapping $G \times \Gamma \times G \to G$, written (a, γ, b) as $a\gamma b$, such that G satisfies the identity $(a\gamma b)\,\delta c = (c\gamma b)\,\delta a$, for all $a, b, c \in G$ and $\gamma, \delta \in \Gamma$ [14].

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Definition 1. *An element $e \in S$ is called a left identity of* Γ-*AG-groupoid if* $e\gamma a = a$ *for all* $a \in S$ *and* $\gamma \in \Gamma$.

Lemma 1. *If a* Γ-*AG-groupoid contains left identity, then it becomes an AG-groupoid with left identity.*

Proof. Let G be a Γ-AG-groupoid and e be the left identity of G and let $a, b \in G$ and $\alpha, \beta \in \Gamma$ therefore we have

$$a\alpha b = a\alpha(e\beta b) = e\alpha(a\beta b) = a\beta b.$$

Hence Γ-AG-groupoid with left identity becomes and an AG-groupoid with left identity. □

Remark 1. *From Lemma 1, it is easy to see that all the results given in* [14] *and* [15] *for a* Γ-*AG-groupoid with left identity is identical to the results given in* [10] *and* [11].

Definition 2. *A* Γ-*AG-groupoid is called a* Γ-*AG****-groupoid if it satisfies the following law*

$$a\alpha(b\beta c) = b\alpha(a\beta c), \text{ for all } a, b, c \in S \text{ and } \alpha, \beta \in \Gamma.$$

The following results and definition from definition 3 to lemma 3 have been taken from [14].

Definition 3. *Let G be a* Γ-*AG-groupoid, a non-empty subset S of G is called sub* Γ-*AG-groupoid if* $a\gamma b \in S$ *for all* $a, b \in S$ *and* $\gamma \in \Gamma$ *or S is called sub* Γ-*AG-groupoid if* $S\Gamma S \subseteq S$.

Definition 4. *A subset I of a* Γ-*AG-groupoid G is called left(right)* Γ-*ideal of G if* $G\Gamma I \subseteq I$ $(I\Gamma G \subseteq I)$ *and I is called* Γ-*ideal of G if it is both left and right* Γ-*ideal.*

Definition 5. *An element a of a* Γ-*AG-groupoid G is called regular if there exist* $x \in G$ *and* $\beta, \gamma \in \Gamma$ *such that* $a = (a\beta x)\gamma a$. *G is called regular* Γ-*AG-groupoid if all elements of G are regular.*

Definition 6. *A sub* Γ-*AG-groupoid B of a* Γ-*AG-groupoid G is called* Γ-*bi-ideal of G if* $(B\Gamma G)\Gamma B \subseteq B$.

Definition 7. *Let G and Γ be any non-empty sets. If there exists a mapping* $G \times \Gamma \times G \to G$, *written* (x, γ, y) *as* $x\gamma y$, *G is called a* Γ-*medial if it satisfies* $(x\alpha y)\beta(l\gamma m) = (x\alpha l)\beta(y\gamma m)$, *and called* Γ-*paramedial if it satisfies* $(x\alpha y)\beta(l\gamma m) = (m\alpha l)\beta(y\gamma x)$ *for all* $x, y, l, m \in G$ *and* $\alpha, \beta, \gamma \in \Gamma$.

Lemma 2. *If A and B are any* Γ-*ideals of a regular* Γ-*AG-groupoid G then* $A\Gamma B = B\Gamma A$.

Definition 8. *A* Γ-*ideal P of a* Γ-*AG-groupoid G is called* Γ-*prime(*Γ-*semiprime) if for any* Γ-*ideals A and B,* $A\Gamma B \subseteq P$ $(A\Gamma A \subseteq P)$ *implies either* $A \subseteq P$ *or* $B \subseteq P$ $(A \subseteq P)$.

Lemma 3. *Any* Γ-*ideal A of a regular* Γ-*AG-groupoid is a* Γ-*idempotent that is* $A\Gamma A = A$.

It is important to note that every Γ-AG-groupoid G is Γ-medial and every Γ-AG**-groupoid G is Γ-paramedial because for any $x, y, l, m \in G$ and $\alpha, \beta, \gamma \in \Gamma$, we have

$$(x\alpha y)\beta(l\gamma m) = ((l\gamma m)\alpha y)\beta x = ((y\gamma m)\alpha l)\beta x = (x\alpha l)\beta(y\gamma m).$$

We call it as Γ-medial law.

Theorem 1. *If L and R are left and right Γ-ideals of a Γ-AG^{**}-groupoid G then $L\cup L\Gamma G$ and $R\cup G\Gamma R$ are Γ-ideals of G.*

Proof. Let L be a left Γ-ideal of G then we have

$$\begin{aligned}(L\cup L\Gamma G)\,\Gamma G &= (L\Gamma G)\cup(L\Gamma G)\,\Gamma G=(L\Gamma G)\cup(G\Gamma G)\,\Gamma L\\ &\subseteq L\Gamma G\cup(G\Gamma L)\subseteq L\Gamma G\cup L=L\cup L\Gamma G \text{ and}\\ G\Gamma\,(L\cup L\Gamma G) &= G\Gamma L\cup G\Gamma\,(L\Gamma G)\subseteq L\cup L\Gamma\,(G\Gamma G)=L\cup L\Gamma G.\end{aligned}$$

Again let R be a right Γ-ideal of G then we have

$$\begin{aligned}(R\cup G\Gamma R)\,\Gamma G &= R\Gamma G\cup(G\Gamma R)\,\Gamma G\subseteq R\cup(G\Gamma R)\,\Gamma\,(G\Gamma G)\\ &= R\cup(G\Gamma G)\,\Gamma\,(R\Gamma G)\subseteq R\cup G\Gamma R, \text{ and}\\ G\Gamma\,(R\cup G\Gamma R) &= G\Gamma R\cup G\Gamma\,(G\Gamma R)=G\Gamma R\cup(G\Gamma G)\,\Gamma\,(G\Gamma R)\\ &= G\Gamma R\cup(R\Gamma G)\,\Gamma\,(G\Gamma G)\subseteq G\Gamma R\cup R\Gamma G\\ &\subseteq G\Gamma R\cup R=R\cup G\Gamma R.\end{aligned}$$

□

Lemma 4. *Right identity in a Γ-AG-groupoid G becomes identity of G and hence G becomes commutative Γ-semigroup.*

Proof. Let e be the right identity of G , $g\in G$, α and $\beta\in\Gamma$, then

$$e\alpha g=(e\beta e)\,\alpha g=(g\beta e)\,\alpha e=g\alpha e=g.$$

Again for a, b, $c\in G$ and α, $\beta\in\Gamma$ we have

$$a\gamma b=(e\alpha a)\,\gamma b=(e\alpha a)\,\gamma(e\alpha b)=(b\alpha e)\gamma\,(a\alpha e)=b\gamma a.$$

Now

$$\begin{aligned}(a\alpha b)\,\beta c &= (a\alpha b)\,\beta\,(e\alpha c)=(a\alpha e)\,\beta\,(b\alpha c)=e\alpha\,((a\alpha e)\,\beta\,(b\alpha c))\\ &= (a\alpha e)\,\alpha\,(e\beta\,(b\alpha c))=a\alpha\,(e\beta\,(b\alpha c))=a\alpha\,(b\beta\,(e\alpha c))\\ &= a\alpha\,(b\beta c)\,.\end{aligned}$$

□

Definition 9. *A sub Γ-AG-groupoid Q of a Γ-AG-groupoid G is called a quasi-ideal of G if $G\Gamma Q\cap Q\Gamma G\subseteq Q$.*

Definition 10. *A sub Γ-AG-groupoid I of a Γ-AG-groupoid G is called a Γ-interior ideal of G if $(G\Gamma I)\,\Gamma G\subseteq I$.*

Lemma 5. *Every one sided (left or right) Γ-ideal of a Γ-AG-groupoid G is a Γ-quasi ideal of G.*

Proof. Let L be a left Γ-ideal of G then we have

$$L\Gamma G\cap G\Gamma L\subseteq G\Gamma L\subseteq L.$$

Which implies L is a Γ-quasi ideal of G. Similarly if R is a right Γ-ideal of G then it is a Γ-quasi ideal of G. □

Lemma 6. *Every right Γ-ideal and left Γ-ideal of a Γ-AG-groupoid G is a Γ-bi-ideal of G.*

Proof. Let R be a right Γ-ideal of G then we have

$$(R\Gamma G)\,\Gamma R \subseteq R\Gamma R \subseteq R\Gamma G \subseteq R.$$

Again let L be a left Γ-ideal of G then we have

$$(L\Gamma G)\,\Gamma L \subseteq (G\Gamma G)\,\Gamma L \subseteq G\Gamma L \subseteq L.$$

□

Corollary 1. *Every* Γ*-ideal of a* Γ*-AG-groupoid G is a* Γ*-bi-ideal of G.*

Proof. It follows from lemma 6. □

Lemma 7. *If B_1 and B_2 are* Γ*-bi-ideals of a* Γ*-AG**-groupoid G then $B_1\Gamma B_2$ is also a* Γ*-bi-ideals of G.*

Proof. Let B_1 and B_2 be Γ-bi-ideals of G then we have

$$\begin{aligned}
((B_1\Gamma B_2)\,\Gamma G)\,\Gamma\,(B_1\Gamma B_2) &= ((B_1\Gamma B_2)\,\Gamma\,(G\Gamma G))\,\Gamma\,(B_1\Gamma B_2)\\
&= ((B_1\Gamma G)\,\Gamma\,(B_2\Gamma G))\,\Gamma\,(B_1\Gamma B_2)\\
&= ((B_1\Gamma G)\,\Gamma B_1)\,\Gamma\,((B_2\Gamma G)\,\Gamma B_2)\\
&\subseteq B_1\Gamma B_2.
\end{aligned}$$

□

Lemma 8. *Every* Γ*-idempotent quasi-ideal of a* Γ*-AG-groupoid G is a* Γ*-bi-ideal of G.*

Proof. Let Q be an Γ-idempotent quasi-ideal of G. Now

$$\begin{aligned}
(Q\Gamma G)\,\Gamma Q &\subseteq (G\Gamma G)\,\Gamma Q \subseteq G\Gamma Q, \text{ and}\\
(Q\Gamma G)\,\Gamma Q &= (Q\Gamma G)\,\Gamma\,(Q\Gamma Q) = (Q\Gamma Q)\,\Gamma\,(G\Gamma Q) = Q\Gamma\,(G\Gamma Q)\\
&\subseteq Q\Gamma\,(G\Gamma G) \subseteq Q\Gamma G, \text{ which implies that}\\
(Q\Gamma G)\,\Gamma Q &\subseteq G\Gamma Q \cap Q\Gamma G \subseteq Q.
\end{aligned}$$

□

Lemma 9. *Every* Γ*-ideal of a* Γ*-AG-groupoid G is a* Γ*-interior ideal of G.*

Proof. Let I be a Γ-ideal of G then we have

$$(G\Gamma I)\,\Gamma G \subseteq I\Gamma G = I.$$

□

Lemma 10. *A subset I of a* Γ*-AG**-groupoid G is a* Γ*-interior ideal if and only if it is right* Γ*-ideal.*

Proof. Let I be a right Γ-ideal G then it becomes a left Γ-ideal so is Γ-ideal and by lemma 9 it is Γ-interior ideal.

Conversely assume that I is a Γ-interior ideal of G. Using Γ-paramedial law, we have

$$\begin{aligned}
I\Gamma G &= I\Gamma\,(G\Gamma G) = G\Gamma\,(I\Gamma G) = (G\Gamma G)\,\Gamma\,(I\Gamma G)\\
&= (G\Gamma I)\,\Gamma\,(G\Gamma G) \subseteq (G\Gamma I)\,\Gamma G \subseteq G.
\end{aligned}$$

Which shows that I is a right Γ-ideal of G. □

Example 1. *Let $G = \{1, 2, 3, 4, 5\}$ with binary operation "$\cdot$" given in the following Cayley's table, an AG-groupoid with left identity* 4.

$\cdot$	1	2	3	4	5
1	4	5	1	2	3
2	3	4	5	1	2
3	2	3	4	5	1
4	1	2	3	4	5
5	5	1	2	3	4

It is easy to observe that G is a simple AG-groupoid that is there is no left or right ideal of G. Now let $\Gamma = \{\alpha,\ \beta,\ \gamma\}$ defined as

α	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1

β	1	2	3	4	5
1	2	2	2	2	2
2	2	2	2	2	2
3	2	2	2	2	2
4	2	2	2	2	2
5	2	2	2	2	2

γ	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	3	3

It is easy to prove that G is a Γ-AG-groupoid because $(a\pi b)\,\psi c = (c\pi b)\,\psi a$ for all $a,\ b,\ c \in G$ and $\pi,\ \psi \in \Gamma$ also G is non-associative because $(1\alpha 2)\,\beta 3 \neq 1\alpha\,(2\beta 3)$. This Γ-AG-groupoid does not contain left identity because $4\alpha 5 \neq 5$, $4\beta 5 \neq 5$ and $4\gamma 5 \neq 5$. It is easy to see that every AG-groupoid with left identity not necessarily implies Γ-AG-groupoid with left identity. Clearly $A = \{1, 2, 3\}$ is a Γ-ideal of G. $B = \{1, 2, 4\}$ is a right Γ-ideal but is not a left Γ-ideal. A and B both are Γ-bi-ideals of G. $C = \{1,\ 2,\ 3,\ 4\}$ is a Γ-interior ideal of G.

Lemma 11. *For a regular Γ-AG-groupoid G $A\Gamma G = A$ and $G\Gamma B = B$ for every right Γ-ideal A and for every left Γ-ideal B.*

Proof. Let A be a right Γ- ideal of G then $A\Gamma G \subseteq A$. Let $a \in A$, since G is regular so there exist $x \in G$ and $\alpha,\ \gamma \in \Gamma$ such that

$$a = (a\alpha x)\,\gamma a \in (A\Gamma G)\,\Gamma A \subseteq (A\Gamma G)\,\Gamma G \subseteq A\Gamma G.$$

Now again let B be a left Γ-ideal of G then $G\Gamma B \subseteq B$. Let $b \in B$, also G is regular so there exist $t \in G$ and $\pi,\ \sigma \in \Gamma$ such that

$$b = (b\pi t)\,\sigma b \in (B\Gamma G)\,\Gamma B \subseteq (G\Gamma G)\,\Gamma B \subseteq G\Gamma B.$$

□

Lemma 12. *If G is a Γ-AG**-groupoid then $g\Gamma G$ and $G\Gamma g$ are Γ-bi-ideals for all $g \in G$.*

Proof. Using the definition of Γ-AG**-groupoid we have

$$\begin{aligned}((g\Gamma G)\,\Gamma G)\,\Gamma\,(g\Gamma G) &= ((G\Gamma G)\,\Gamma g)\,\Gamma\,(g\Gamma G) \subseteq (G\Gamma g)\,\Gamma\,(g\Gamma G)\\ &= g\Gamma\,((G\Gamma g)\,\Gamma G) \subseteq g\Gamma\,((G\Gamma G)\,\Gamma G) \subseteq g\Gamma\,(G\Gamma G)\\ &\subseteq g\Gamma G.\end{aligned}$$

Again using Γ-paramedial law we have

$$\begin{aligned}((G\Gamma g)\,\Gamma G)\,\Gamma\,(G\Gamma g) &= (((G\Gamma g)\,\Gamma g)\,\Gamma G)\,\Gamma G = (((g\Gamma g)\,\Gamma G)\,\Gamma G)\,\Gamma G\\ &= ((G\Gamma G)\,\Gamma G)\,\Gamma\,(g\Gamma g) \subseteq (G\Gamma G)\,\Gamma\,(g\Gamma g)\\ &= (g\Gamma g)\,\Gamma\,(G\Gamma G) \subseteq (g\Gamma g)\,\Gamma G = (G\Gamma g)\,\Gamma g\\ &\subseteq (G\Gamma G)\,\Gamma g \subseteq G\Gamma g.\end{aligned}$$

□

Corollary 2. *If G is a regular Γ-AG^{**}-groupoid then $a\Gamma G$ is a Γ-bi-ideal in G, for all $a \in G$.*

Proof. Let G be a regular Γ-AG-groupoid then for every $a \in G$ there exist $x \in G$ and α, $\beta \in \Gamma$ such that $a = ((a\alpha x)\,\beta a)$ therefore we have

$$\begin{aligned}((a\Gamma G)\,\Gamma G)\,\Gamma\,(a\Gamma G) &= ((((a\alpha x)\,\beta a)\,\Gamma G)\,\Gamma G)\,\Gamma\,(a\Gamma G)\\ &= ((G\Gamma G)\,\Gamma\,((a\alpha x)\,\beta a))\,\Gamma\,(a\Gamma G)\\ &\subseteq (G\Gamma\,((a\alpha x)\,\beta a))\,\Gamma\,(a\Gamma G) = ((a\alpha x)\,\Gamma\,(G\beta a))\,\Gamma\,(a\Gamma G)\\ &\subseteq ((a\alpha x)\,\Gamma\,(G\beta G))\,\Gamma\,(G\Gamma G) \subseteq ((a\alpha x)\,\Gamma G)\,\Gamma G\\ &= (G\Gamma G)\,\Gamma\,(a\alpha x) \subseteq G\Gamma\,(a\alpha x) = a\Gamma\,(G\alpha x) \subseteq a\Gamma\,(G\Gamma G)\\ &\subseteq a\Gamma G.\end{aligned}$$

□

Lemma 13. *For a Γ-bi-ideal B in a regular Γ-AG-groupoid G, $(B\Gamma G)\,\Gamma B = B$.*

Proof. Let B be a Γ-bi-ideal in G then $(B\Gamma G)\,\Gamma B \subseteq B$. Let $x \in B$, since G is a regular Γ-AG-groupoid therefore there exist $a \in G$ and α, $\beta \in \Gamma$ such that

$$x = (x\alpha a)\,\beta x \in (B\Gamma G)\,\Gamma B.$$

Which implies that $B \subseteq (B\Gamma G)\,\Gamma B$. □

Lemma 14. *If G is a regular Γ-AG-groupoid then, $G\Gamma G = G$.*

Proof. Since $G\Gamma G \subseteq G$. Let $x \in G$, since G is a regular Γ-AG-groupoid therefore there exist $a \in G$ and α, $\beta \in \Gamma$ such that

$$x = (x\alpha a)\,\beta x \in (G\Gamma G)\,\Gamma G \subseteq G\Gamma G.$$

Which implies that $G \subseteq G\Gamma G$. □

Lemma 15. *A subset I of a regular Γ-AG^{**}-groupoid G is a left Γ-ideal if and only if it is a right Γ-ideal of G.*

Proof. Let I be a left Γ-ideal of G then $G\Gamma I \subseteq I$. Let $i\gamma g \in I\Gamma G$ for $g \in G$, $i \in I$ and $\gamma \in \Gamma$, also G is a regular Γ-AG-groupoid therefore there exist x, $y \in G$ and α, β, γ, δ, $\pi \in \Gamma$ such that

$$\begin{aligned}i\gamma g &= ((i\alpha x)\,\beta i)\,\gamma\,((g\delta y)\,\pi g) = ((i\alpha x)\,\beta\,(g\delta y))\,\gamma\,(i\pi g)\\ &= ((((i\alpha x)\,\beta i)\,\alpha x)\,\beta\,(g\delta y))\,\gamma\,(i\pi g) = ((y\alpha g)\,\beta\,((i\beta\,(i\alpha x))\,\delta x))\,\gamma\,(i\pi g)\\ &= (i\beta\,(((y\alpha g)\,\beta\,(i\alpha x))\,\delta x))\,\gamma\,(i\pi g) = ((i\pi g)\,\beta\,(((y\alpha g)\,\beta\,(i\alpha x))\,\delta x))\,\gamma i\\ &\in (G\Gamma I) \subseteq I.\end{aligned}$$

Conversely let I be a right Γ-ideal then there exist $x \in G$ and α, $\beta \in \Gamma$ such that

$$g\gamma i = ((g\alpha x)\,\beta g)\,\gamma i = (i\beta g)\,\gamma\,(g\alpha x) \in (I\Gamma G)\,\Gamma G \subseteq I\Gamma G \subseteq I.$$

□

Theorem 2. *for a* Γ*-*AG^{**}*-groupoid* G*, following statements are equivalent.*
(i) G *is regular* Γ*-*AG*-groupoid.*
(ii) *Every left* Γ*-ideal of* G *is* Γ*-idempotent.*

Proof. $(i) \Rightarrow (ii)$
Let G be a regular Γ-AG-groupoid then by lemma 3 every Γ-ideal of G is Γ-idempotent.
$(ii) \Rightarrow (i)$
Let every left Γ-ideal of a Γ-AG**-groupoid G is Γ-idempotent, since $G\Gamma a$ is a left Γ-ideal of G for all $a \in G$ [14], so is Γ-idempotent and by Γ-paramedial law, lemma **??** and Γ-medial law, we have, $a \in G\Gamma a$ implies

$$\begin{aligned} a &\in (G\Gamma a)\,\Gamma\,(G\Gamma a) = ((G\Gamma a)\,\Gamma a)\,\Gamma G = ((a\Gamma a)\,\Gamma G)\,\Gamma G \\ &= ((a\Gamma a)\,\Gamma\,(G\Gamma G))\,\Gamma G = ((G\Gamma G)\,\Gamma\,(a\Gamma a))\,\Gamma G \\ &= (a\Gamma\,((G\Gamma G)\,\Gamma a))\,\Gamma G = (G\Gamma\,((G\Gamma G)\,\Gamma a))\,\Gamma a \\ &= (G\Gamma\,(G\Gamma a))\,\Gamma a = (G\Gamma\,((G\Gamma a)\,\Gamma\,(G\Gamma a)))\,\Gamma a \\ &= (G\Gamma\,((a\Gamma G)\,\Gamma\,(a\Gamma G)))\,\Gamma a = ((G\Gamma G)\,\Gamma\,((a\Gamma G)\,\Gamma\,(a\Gamma G)))\,\Gamma a \\ &= ((G\Gamma\,(a\Gamma G))\,\Gamma\,(G\Gamma\,(a\Gamma G)))\,\Gamma a = (((a\Gamma G)\,\Gamma G)\,\Gamma\,((a\Gamma G)\,\Gamma G))\,\Gamma a \\ &= ((((a\Gamma G)\,\Gamma G)\,\Gamma G)\,\Gamma\,(a\Gamma G))\,\Gamma a = (a\Gamma\,((((a\Gamma G)\,\Gamma G)\,\Gamma G)\,\Gamma G))\,\Gamma a \\ &\subseteq (a\Gamma G)\,\Gamma a. \end{aligned}$$

Which shows that G is a regular Γ-AG**-groupoid. □

Lemma 16. *Any* Γ*-ideal of a regular* Γ*-*AG*-groupoid* G *is* Γ*-semiprime.*

Proof. It is an easy consequence of lemma 3. □

Theorem 3. *Set of all* Γ*-ideals in a regular* Γ*-*AG*-groupoid* G *with forms a semilattice* $(G, \circ)$ *where* $A \circ B = A\Gamma B$*, for all* Γ*-ideals* A *and* B *of* G*.*

Proof. Let A and B be any Γ-ideals in G, then by Γ-medial law we have

$$\begin{aligned} (A\Gamma B)\,\Gamma G &= (A\Gamma B)\,\Gamma\,(G\Gamma G) = (A\Gamma G)\,\Gamma\,(B\Gamma G) \subseteq A\Gamma B. \text{ And} \\ G\Gamma\,(A\Gamma B) &= (G\Gamma G)\,\Gamma\,(A\Gamma B) = (G\Gamma A)\,\Gamma\,(G\Gamma B) \subseteq A\Gamma B. \end{aligned}$$

Also by lemma 2, we have $A\Gamma B = B\Gamma A$ which implies that

$$(A\Gamma B)\,\Gamma C = C\Gamma\,(A\Gamma B) = A\Gamma\,(C\Gamma B) = A\Gamma\,(B\Gamma C)\,.$$

And by lemma 3, $A\Gamma A = A$.

□

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