ALMOST-ISOMETRY BETWEEN TEICHMÜLLER METRIC AND LENGTH-SPECTRA METRIC ON MODULI SPACE

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Abstract. We prove an analogue of Farb-Masur's theorem that the length-spectra metric on moduli space is "almost isometric" to a simple model $\mathcal{V}(S)$ which is induced by the cone metric over the complex of curves. As an application, we know that the Teichmüller metric and the length-spectra metric are "almost isometric" on moduli space, while they are not even quasi-isometric on Teichmüller space.

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1. Introduction

Let $S = S_{g,n}$ be an oriented surface of genus g with n punctures. We assume that $3g - 3 + n \geq 1$. Let $\mathcal{T}(S)$ denote the Teichmüller space of equivalence classes of marked Riemann surfaces on S. A marked Riemann surface is a pair (X, f) where X is a Riemann surface, considered as a surface with either a conformal structure or hyperbolic metric on S, and $f : S \to X$ is an orientation preserving homeomorphism. Two markings $(X_i, f_i), i = 1, 2$ are equivalent if and only if there exists a conformal map $h : X_1 \to X_2$ which is homotopic to $f_2 \circ f_1^{-1}$. In the following, we will often denote $(X, f) \in \mathcal{T}(S)$ by X, without explicit reference to the marking or the equivalence relation.

There are several natural metrics on $\mathcal{T}(S)$. In this paper we will consider the Teichmüller metric d_{Teich} and the length-spectra metric d_{ls}. Both of the two metrics are complete Finsler metrics. For the Teichmüller metric, this is a classical well-known result. The fact that the length spectrum metric is Finsler follows from the fact that this metric is a symmetrization of Thurston's asymmetric metric, which is Finsler, which was proved by Thurston in [17]. In this paper, the Finsler property of the length-spectra metric to used to show that any two points in the Teichmüller metric can be connected by a geodesic.

Recall that for marked conformal structures $X_1, X_2 \in \mathcal{T}(S)$, the Teichmüller metric is defined by

$$d_{\text{Teich}}(X_1, X_2) = \frac{1}{2} \log K$$

where $K \geq 1$ is the least number such that there exists a K-quasiconformal map between the marked structures X_1 and X_2. Teichmüller's Theorem states that there exists a unique extremal quasiconformal map realizing $d_{\text{Teich}}(X_1, X_2)$. See Abikoff [1] for details.

The length-spectra metric (also called the Lipschitz metric [3]) on $\mathcal{T}(S)$ is defined by

$$d_{\text{ls}}(X_1, X_2) = \frac{1}{2} \log \max\{K_1, K_2\}$$

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where $K_1 \geq 1$ is the least number such that there is a global K_1-Lipschitz homeomorphism between the marked hyperbolic metrics X_1 and X_2, and where $K_2 \geq 1$ is the least number such that there is a global K_2-Lipschitz homeomorphism between the marked hyperbolic metrics X_2 and X_1. Note that there are two asymmetric metrics (called Thurston's asymmetric metrics) defined by $d_1(X_1, X_2) = \frac{1}{2}\log K_1$ and $d_2(X_1, X_2) = \frac{1}{2}\log K_2$. Thurston [17] showed that the extremal Lipschitz maps exist.

The mapping class group $\mathrm{Mod}(S)$ is the group of homotopy classes of orientation-preserving homeomorphisms of S. This group acts properly discontinuously and isometrically on $(\mathcal{T}(S), d_{\mathrm{Teich}})$ and $(\mathcal{T}(S), d_{\mathrm{ls}})$, thus inducing two metrics $d_{\mathcal{T}}$ and $d_{\mathcal{L}}$ on the quotient moduli space $\mathcal{M}(S) := \mathcal{T}(S)/\mathrm{Mod}(S)$. Let $\pi : \mathcal{T}(S) \to \mathcal{M}(S)$ be the natural projection.

Kerckhoff [7] discovered an elegant and useful formula to compute the Teichmüller metric in terms of extremal length:

$$d_{\mathrm{Teich}}(X_1, X_2) = \frac{1}{2}\log\sup_{\gamma}\frac{\mathrm{Ext}_{X_1}(\gamma)}{\mathrm{Ext}_{X_2}(\gamma)}$$

where the supremum is taken over all isotopy classes of essential (neither homotopic to a point nor to a puncture) simple closed curves on S. The extremal length of γ in X, denoted by $\mathrm{Ext}_X(\gamma)$, is defined by

$$\mathrm{Ext}_X(\gamma) := \sup_{\rho}\frac{L_\rho(\gamma)^2}{\mathrm{Area}_\rho}$$

where the supremum is taken over all conformal metrics ρ on X of finite positive area.

On the other hand, it was shown by Thurston [17] that the minimal Lipschitz constant is given by the ratios of hyperbolic length:

$$K_1 = \sup_{\gamma}\frac{l_{X_2}(\gamma)}{l_{X_1}(\gamma)}, K_2 = \sup_{\gamma}\frac{l_{X_1}(\gamma)}{l_{X_2}(\gamma)}$$

where the supremum is taken over all isotopy classes of essential simple closed curves on S. Thus the length-spectra metric is given by

$$d_{\mathrm{ls}}(X_1, X_2) = \max\{\frac{1}{2}\log\sup_{\gamma}\frac{l_{X_2}(\gamma)}{l_{X_1}(\gamma)}, \frac{1}{2}\log\sup_{\gamma}\frac{l_{X_1}(\gamma)}{l_{X_2}(\gamma)}\}. \tag{1}$$

It is of interest to study the relation between the metrics d_{Teich} and d_{ls}. The following lemma of Wolpert [18] implies that $d_{\mathrm{ls}}(X_1, X_2) \leq d_{\mathrm{Teich}}(X_1, X_2)$.

Lemma 1.1. *For any K-quasiconformal mapping f from X_1 to X_2 and any simple closed curves γ, we have*

$$\frac{l_{X_2}(f(\gamma))}{l_{X_1}(\gamma)} \leq K.$$

It was shown by Li [9] that d_{Teich} and d_{ls} induce the same topology on $\mathcal{T}(S)$. Moreover, $d_{\mathrm{ls}}(X_1, X_2) \leq d_{\mathrm{Teich}}(X_1, X_2) \leq 2d_{\mathrm{ls}}(X_1, X_2) + C(X_1)$, where $C(X_1)$ is a constant depending on X_1. The proof can be shown by considering the ratios of extremal length and square of hyperbolic length $\frac{\mathrm{Ext}_X(\gamma)}{l_X^2(\gamma)}$, which defines a function on the compact space of projective measured foliations.

However, Li [10] also proved that d_{Teich} and d_{ls} are not metrically equivalent; that is, there is no constant $C > 0$ such that $d_{\mathrm{ls}}(X_1, X_2) \leq d_{\mathrm{Teich}}(X_1, X_2) \leq C d_{\mathrm{ls}}(X_1, X_2)$ for any X_1 and X_2 in $\mathcal{T}(S)$. In particular, Choi and Rafi [3] (see also Liu, Sun and Wei [12]) showed that although the two metrics are quasi-isometric to each other on the thick part of $\mathcal{T}(S)$, there are sequences $X_n, Y_n, n = 1, 2, \cdots$ in the thin part of $\mathcal{T}(S)$, such that $d_{\mathrm{ls}}(X_n, Y_n) \to 0$, while $d_{\mathrm{Teich}}(X_n, Y_n) \to \infty$.

In fact, all the well-known examples that illustrate the divergence of d_{Teich} and d_{ls} are constructed by Dehn twists. Choi and Rafi [3] also noticed the fact that for any two points X_1, X_2 in the thin part of $\mathcal{T}(S)$, if they have no short curves in common, then $d_{\mathrm{Teich}}(X_1, X_2)$ is comparable to $d_{\mathrm{ls}}(X_1, X_2)$. These give evidences that d_{Teich} and d_{ls} may be quasi-isometric on the moduli space $\mathcal{M}(S)$.

For more recent progress on the length-spectra metric and Teichmüller metric in Teichmüller space, please see [13], [14] and [15].

Recently, Farb and Masur [5] studied the large-scale geometry of moduli space. They built an "almost isometric" simplicial model for $\mathcal{M}(S)$ with the Teichmüller metric, from which they determine the tangent cone at infinity of $\mathcal{M}(S)$. Their result can be seen as a step in providing a "reduction theory" for the action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$, in analogy with the case of locally symmetric spaces. See Farb and Masur [5], Leuzinger [8] for details.

Let $\mathcal{C}(S)$ be the complex of curves on S. This was introduced by Harvey [6] as an analogue in the context of Teichmüller space of the Tits building associated to an arithmetic group.The vertices of $\mathcal{C}(S)$ are the free isotopy classes of essential simple closed curves on S, and a k-simplex consist of $k+1$ isotopy classes of mutually disjoint essential simple closed curves. Note that $\mathcal{C}(S)$ is a simplicial complex of dimension $(d-1)$, where $d = 3g-3+n$. A $(d-1)$-simplex is called a maximal simplex. Every simplex is the face of a maximal simplex. While $\mathcal{C}(S)$ is locally infinite, the mapping class group $\mathrm{Mod}(S)$ acts on $\mathcal{C}(S)$ and the quotient $\mathcal{C}(S)/\mathrm{Mod}(S)$ is a finite orbi-complex. See [6] for reference.

Denote $\widetilde{\mathcal{V}}(S)$ to be the topological cone

$$\widetilde{\mathcal{V}}(S) := \frac{[0,\infty) \times \mathcal{C}(S)}{\{0\} \times \mathcal{C}(S)}.$$

For each maximal simplex σ of $\mathcal{C}(S)$, we will think of the cone over σ as an orthant with coordinates $(x_1, \cdots, x_d) \in \mathbb{R}^d_{\geq 0}$. This orthant is endowed with the standard sup metric:

$$d((x_1, \cdots, x_d), (y_1, \cdots, y_d)) := \frac{1}{2} \max_{1 \leq i \leq d} |x_i - y_i|.$$

Since the metrics on the cones on any two such maximal simplices agree on (the cone on) any common face, we can endow $\widetilde{\mathcal{V}}(S)$ with the corresponding path metric. The natural action of $\mathrm{Mod}(S)$ on $\mathcal{C}(S)$ induces an isometric action on $\widetilde{\mathcal{V}}(S)$, thus induces a well-defined metric $d_{\mathcal{V}}$ on the quotient

$$\mathcal{V}(S) := \widetilde{\mathcal{V}}(S)/\mathrm{Mod}(S).$$

Given $C \geq 0$ and $\lambda \geq 1$, a map $f : X \to Y$ is called a (λ, C) quasi-isometry if

$$\frac{1}{\lambda} d_X(x,y) - C \leq d_Y(f(x), f(y)) \leq \lambda d_X(x,y) + C$$

for any $x, y \in X$, and the C-neighborhood of $f(X)$ in Y is all of Y. A $(1, C)$ quasi-isometry is called an "almost isometry".

The following theorem is the main result of Farb and Masur [5], which provides a simple and geometric model for $\mathcal{M}(S)$.

Theorem 1.2. *There is a map $\Psi : (\mathcal{V}(S), d_{\mathcal{V}}) \to (\mathcal{M}(S), d_{\mathcal{T}})$ which is an almost isometry. That is, there is a constant D that depends on S such that:*

- $|d_{\mathcal{V}}(x,y) - d_{\mathcal{T}}(\Psi(x), \Psi(y))| \leq D$ *for each* $x, y \in \mathcal{V}(S)$, *and*
- *the D-neighborhood of $\Psi(\mathcal{V}(S))$ in $(\mathcal{M}(S), d_{\mathcal{T}})$ is all of $\mathcal{M}(S)$.*

As a corollary of Theorem 1.2, the tangent cone at infinity of $\mathcal{M}(S)$ with the Teichmüller metric is isometric to $\mathcal{V}(S)$ and has dimension d.

The main goal of our article is to show that:

Theorem 1.3. *Endow $\mathcal{M}(S)$ with the length-spectrum metric $d_{\mathcal{L}}$, then the map $\Psi : \mathcal{V}(S) \to \mathcal{M}(S)$ in Theorem 1.2 is an almost isometry. That is, there is a constant D' that depends on S such that:*

- $|d_{\mathcal{V}}(x,y) - d_{\mathcal{L}}(\Psi(x),\Psi(y))| \leq D'$ *for each* $x, y \in \mathcal{V}(S)$, *and*
- *the* D'*-neighborhood of* $\Psi(\mathcal{V}(S))$ *in* $\mathcal{M}(S)$ *is all of* $\mathcal{M}(S)$.

As a result, the tangent cone at infinity of $\mathcal{M}(S)$ with the length-spectrum metric is also isometric to $\mathcal{V}(S)$. Since quasi-isometry is an equivalence relation between metric spaces, it is clear that Theorem 1.2 and Theorem 1.3 together imply that

Theorem 1.4. *The Teichmüller metric and the length-spectrum metric are almost isometric on $\mathcal{M}(S)$. That is, there is a constant D'' that depends on S, such that*

$$d_{\mathcal{L}}(X_1, X_2) \leq d_{\mathcal{T}}(X_1, X_2) \leq d_{\mathcal{L}}(X_1, X_2) + D''$$

for any $X_1, X_2 \in \mathcal{M}(S)$.

We will give a proof of Theorem 1.3 in Section 3. The method of the proof of the theorem relies on Minsky's product theorem [16]. Another ingredient of the proof is that any two points of $\Psi(\mathcal{V}(S))$ can be joined by a quasi-geodesic that lies in $\Psi(\mathcal{V}(S))$ and enters each simplex of $\Psi(\mathcal{V}(S))$ at most once, as observed by Farb and Masur [5].

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2. The map Ψ

We will define the map $\Psi : \mathcal{V}(S) \to \mathcal{M}(S)$ as constructed by Farb-Masur [5].

Let us first fix some notations. Given a maximal simplex $\sigma \in \mathcal{C}(S)$, the cone over σ is denoted by $\bar{\Delta}(\sigma)$ and the open cone over σ (with no $x_i = 0$) is denoted by $\Delta(\sigma)$. Let $\mathrm{Mod}(\sigma)$ be the subgroup of $\mathrm{Mod}(S)$ that fixes σ. It acts on $\Delta(\sigma)$ with finite orbit. Let $\Lambda(\sigma)$ be a sector inside $\Delta(\sigma)$ which is a fundamental domain for the action of $\mathrm{Mod}(\sigma)$.

Now fix a maximal simplex σ of $\mathcal{C}(S)$. σ is represented by a maximal collection of disjoint simple closed curves $\{\alpha_1, \cdots, \alpha_d\}$. The choice of curves determines a set of Fenchel-Nielsen coordinates on $\mathcal{T}(S)$, where a point $X \in \mathcal{T}(S)$ is given by coordinates as following:

$$X \to (l_X(\alpha_1), \cdots, l_X(\alpha_d), \theta_1(X), \cdots, \theta_d(X))$$

where $l_X(\alpha_i)$ is the length of α_i with respect to the hyperbolic metric X, and θ_i are the so-called "twist coordinates".

We now define a map $\widetilde{\Psi} : \widetilde{\mathcal{V}}(S) \to \mathcal{M}(S)$ in the following way. First $\widetilde{\Psi}$ is restricted to the sector $\Lambda(\sigma)$ to be:

$$\widetilde{\Psi}(x_1, \cdots, x_d) = \pi(X) \tag{2}$$

where X is a point of $\mathcal{T}(S)$ with $\ell_X(\alpha_i) = \epsilon_0 e^{-x_i}$ and with twist coordinates all equal to 0. Here the constant $\epsilon_0 = \epsilon_0(S)$ is a sufficiently small constant such that for any hyperbolic surface X homeomorphic to S, any two simple closed curves α, β with length not larger than ϵ_0 are disjoint. Note that $\epsilon_0 e^{-x_i} \leq \epsilon_0$ and the image of $\widetilde{\Psi}$ lies on $\Omega_\sigma(\epsilon_0)$, where

$$\Omega_\sigma(\epsilon_0) = \{X \in \mathcal{T}(S) : \ell_X(\alpha_i) < \epsilon, \text{ for each } i = 1, \cdots, d\}.$$

We use the action of $\mathrm{Mod}(\sigma)$ to extend $\widetilde{\Psi}$ from $\Lambda(\sigma)$ to $\Delta(\sigma)$. Since there are finitely many collection of maximal simplices that represent all combinatorial types, we define the map $\widetilde{\Psi}$ for each maximal cone in the finite collection. Then we use the action of Mod(S) to extend $\widetilde{\Psi}$ to the open cones on all maximal simplices by having it be constant on orbits.

To define the map $\widetilde{\Psi}$ on $\widetilde{\mathcal{V}}(S)$, we have to define it on the cone over any simplex τ of $\mathcal{C}(S)$ that is not maximal. Since a simplex is a face of some maximal simplex (maybe not unique), we can choose some maximal simplex σ that containing τ. The cone over τ is given by the coordinates $(x_1, \cdots, x_d)$ for the cone over σ. We define the map $\widetilde{\Psi}$ on the cone over τ via the equation (2) by setting the coordinates x_i corresponding curves in $\sigma - \tau$ to be 0. It follows that $\widetilde{\Psi}$ is Mod(S)-invariant and thus induces a map $\Psi : \mathcal{V}(S) \to \mathcal{M}(S)$.

It is noticed that Ψ is not continuous in general because of the choices made at a face of a maximal simplex. Nevetheless we know that the jump in the function at any face is uniformly bounded. Such an argument for the Teichmüller metric is proved by Farb and Masur [5]. Moreover, they proved that the map Ψ satisfies the condition of Theorem 1.2, that is, $\Psi : (\mathcal{V}(S), d_{\mathcal{V}}) \to (\mathcal{M}(S), d_{\mathcal{T}})$ is almost onto and almost isometric. To prove an analogue for $(\mathcal{M}(S), d_{\mathcal{L}})$, we will show that the propositions that were used in the proof of Theorem 1.2 are also applied to the length-spectra metric.

Let $\sigma = \{\alpha_1, \cdots, \alpha_d\}$ be a maximal simplex. Following Minsky [16], we change the Fenchel-Nielsen coordinates to

$$X \to (\theta_1(X), \frac{1}{l_X(\alpha_1)}, \cdots, \theta_d(X), \frac{1}{l_X(\alpha_d)}) \in (\mathbf{H}^2)^d.$$

We give $\mathbf{H}^2$ the Poincaré metric $ds^2 = \frac{dx^2+dy^2}{4y^2}$ and endow $(\mathbf{H}^2)^d$ with the sup metric.

We need the following lemma, which is a special case of the product region theorem of Minsky [16].

Lemma 2.1. *With the notations above, there exists a constant C depending on ϵ_0, such that for any $X, Y \in \Omega_\sigma(\epsilon_0)$.*

$$|d_{\text{Teich}}(X,Y) - \sup_{i=1,\cdots,d} \{d_{\boldsymbol{H}^2}((\theta_i(X), \frac{1}{l_X(\alpha_i)}), (\theta_i(X), \frac{1}{l_X(\alpha_i)}))\}| \leq C.$$

The following proposition is due to Farb and Masur [5], which shows that Ψ is almost onto. We include a proof here for completeness.

Proposition 2.2. *There is a constant $C_1 = C_1(S)$, such that for any $X \in \mathcal{M}(S)$, there exist a $Z \in \Psi(\mathcal{V}(S))$ such that $d_{\mathcal{T}}(X, Z) \leq C_1$.*

Proof. By a theorem of Bers, there is a constant $c = c(S)$ such that every $X \in \mathcal{M}(S)$ has a pants decomposition corresponding to a maximal simplex σ such that every curve of σ has length at most c on X. With respect to these pants curves, each of the twist coordinates is bounded by $2\pi c$, modulo the action of Dehn twist about the curves in σ.

Now for a given X, we choose a point of $\Psi(\mathcal{V}(S))$ whose corresponding simplex has the topological type of σ. For each curves α in σ whose length is at most ϵ_0, we choose the corresponding Fenchel-Nielsen coordinate of a point in $\Psi(\mathcal{V}(S))$ to be $l_X(\alpha)$. For each curves β in σ whose length is between ϵ_0 and c, we choose the corresponding Fenchel-Nielsen coordinates of a point of $\Psi(\mathcal{V}(S))$ to be ϵ_0. In this way we have chosen all the coordinates which determine a point Z in $\Psi(\mathcal{V}(S))$.

Since X and Z have bounded ratios of hyperbolic lengths and bounded differences in twist coordinates, both X and Z either project to a given thick part of moduli space or to the thin part. In the first case, this implies the ratios of extremal lengths are proportional to the ratios of hyperbolic lengths, and so one can use Kerckhoff's distance formula to show that $d_{\mathcal{T}}(X, Z)$ are bounded by some constant. In the second case one can use Minsky's product theorem, noting that the constant C in his formula is universal, depending only on the topological type of S. $\square$

3. The proof of Theorem 1.3

Since $d_{\mathcal{L}}(X_1, X_2) \leq d_{\mathcal{T}}(X_1, X_2)$, by Proposition 2.2, we have shown that $\Psi : (\mathcal{V}(S), d_{\mathcal{V}}) \to (\mathcal{M}(S), d_{\mathcal{L}})$ is almost onto . By Theorem 1.2, there is a constant D such that

$$d_{\mathcal{L}}(\Psi(x), \Psi(y)) \leq d_{\mathcal{T}}(\Psi(x), \Psi(y)) \leq d_{\mathcal{V}}(x, y) + D.$$

For the proof of Theorem 1.3, it remains to show the opposite inequality

$$d_{\mathcal{V}}(x, y) \leq d_{\mathcal{L}}(\Psi(x), \Psi(y)) + D'. \tag{3}$$

Let σ be a maximal simplex and P be the quotient map from $\mathcal{C}(S)$ to $\mathcal{C}(S)/\mathrm{Mod}(S)$. Denote the cone over $P(\sigma)$ by $\Delta(P(\sigma))$.

Proposition 3.1. *For any $Z_1, Z_2 \in \Psi(\Delta(P(\sigma)))$, there is a constant C, such that*

$$d_{\mathcal{T}}(Z_1, Z_2) \leq d_{\mathcal{L}}(Z_1, Z_2) + C.$$

Proof. Since $Z_i \in \Omega_\sigma(\epsilon_0)$ and the " twist coordinates " of Z_i are all vanishing, by Lemma 2.1, we have

$$|d_{\mathcal{T}}(Z_1, Z_2) - \frac{1}{2}\log \sup_{\alpha_1, \cdots, \alpha_d} \frac{l_{Z_2}(\alpha_i)}{l_{Z_1}(\alpha_i)}| \leq C,$$

By the the length-spectra metric equation (1), we have

$$\begin{aligned} d_{\mathcal{T}}(Z_1, Z_2) &\leq \frac{1}{2}\log \sup_{\alpha_1, \cdots, \alpha_d} \frac{l_{Z_2}(\alpha_i)}{l_{Z_1}(\alpha_i)} + C \\ &\leq d_{\mathcal{L}}(Z_1, Z_2) + C. \end{aligned}$$

$\square$

We need the following technical lemma.

Lemma 3.2. *There is a constant $C' = C'(S)$ such that any two points of $\Psi(\mathcal{V}(S))$ can be joined by a $(1, C')$ quasi-geodesic in the metric $d_{\mathcal{L}}$ that stays in $\Psi(\mathcal{V}(S))$ and enters each simplex of $\Psi(\mathcal{V}(S))$ at most once.*

Proof. Denote the length of a path η in $\mathcal{M}(S)$ with respect to the Teichmüller metric and the length-spectra metric by $\|\eta\|_{\mathcal{T}}$ and $\|\eta\|_{\mathcal{L}}$.

As shown by Lemma 8 of Farb and Masur [5], there is a constant C'' such that if $\Psi(x), \Psi(y)$ lie in the same simplex $\Psi(\Delta(P(\sigma)))$ of $\mathcal{M}(S)$, then there is a $(1, C'')$ quasi-geodesic $\rho(x, y)$ in the metric $d_{\mathcal{T}}$ joining $\Psi(x)$ and $\Psi(y)$ that stays in $\Psi(\Delta(P(\sigma)))$. By Proposition 3.1,

$$\begin{aligned} \|\rho(x, y)\|_{\mathcal{L}} &\leq \|\rho(x, y)\|_{\mathcal{T}} \\ &\leq d_{\mathcal{T}}(\Psi(x), \Psi(y)) + C'' \\ &\leq d_{\mathcal{L}}(\Psi(x), \Psi(y)) + C'' + C. \end{aligned}$$

As a result, $\rho(x, y)$ is a $(1, C'' + C)$ quasi-geodesic in the metric $d_{\mathcal{L}}$.

Now suppose that $\Psi(x) \in \Psi(\Delta(P(\sigma_1)))$ and $\Psi(y) \in \Psi(\Delta(P(\sigma_2)))$. If $\Psi(y) \in \Psi(\Delta(P(\sigma_1)))$ then we are done by the argument above. Thus we can assume that $\Psi(y) \notin \Psi(\Delta(\sigma_1))$. Suppose that ρ is a geodesic about the length-spectrum metric from $\Psi(x)$ to $\Psi(y)$ (the existence of geodesics about the length-spectrum metric was shown by Thurston [17], though maybe not unique). Suppose ρ leaves $\Psi(\Delta(P(\sigma_1)))$ and returns to it for a last time at some $\Psi(z) \in \Psi(\Delta(P(\sigma_1))) \bigcap \Psi(\Delta(P(\sigma_3)))$ for some simplex $\Psi(\Delta(P(\sigma_3)))$. Then we can replace ρ by a quasi-geodesic ρ' that stays in $\Psi(\Delta(P(\sigma_1)))$ from $\Psi(x)$ to $\Psi(z)$ and then follows ρ from $\Psi(z)$ to $\Psi(y)$ never returning to $\Psi(\Delta(P(\sigma_1)))$. Note that the length $\|\rho'\|_{\mathcal{L}}$ is less than $\|\rho\|_{\mathcal{L}} + C + C''$.

Continue with the above method, we now find the last point $\Psi(w)$ that lies in $\Psi(\Delta(P(\sigma_3)))$ and replace a segment of ρ' with the one that stays in $\Psi(\Delta(P(\sigma_3)))$ and never returns again to $\Psi(\Delta((\sigma_3)))$. Then we get a quasi-geodesic with length-spectra length less than $\|\rho\|_{\mathcal{L}} + 2(C + C'')$.

Since there are only a finite number of maximal simplices in $\mathcal{C}(S)/\mathrm{Mod}(S)$, we can repeat the above operation in a uniformly finite step. Then we get a path with length-spectra length larger $d_{\mathcal{L}}(\Psi(x), \Psi(y))$ by an additive constant, which is a $(1, C')$ quasi-geodesic. This prove the lemma.

$\square$

We now continue with the final step in the proof of inequality (3). Let ρ be the $(1, C')$-quasi-geodesic in the length-spectra metric as in Lemma 3.2. Suppose that $\rho = \bigcup_{i=1,\cdots,n} \rho_i$ such that each $\rho_i \subset \Psi(\Delta(P(\sigma_i)))$. By Theorem 1.2, we have

$$\begin{aligned} d_{\mathcal{V}(S)}(x, y) &\leq d_{\mathcal{T}}(\Psi(x), \Psi(y)) + D \\ &\leq \|\rho\|_{\mathcal{T}} + D \\ &\leq \sum_{i=1,\cdots,n} \|\rho_i\|_{\mathcal{T}} + D \end{aligned}$$

It follows from the proof of Lemma 3.2 that each ρ_i is also a Teichmüller $(1, C'')$ quasi-geodesic. As a result, we know that the length $\|\rho_i\|_{\mathcal{T}}$ is less than the Teichmüller distance between the two endpoints of ρ_i. By Proposition 3.1, we have

$$\|\rho_i\|_{\mathcal{T}} \leq \|\rho_i\|_{\mathcal{L}} + C + C''.$$

Therefore,

$$\begin{aligned} d_{\mathcal{V}(S)}(x, y) &\leq \sum_{i=1,\cdots,n} \|\rho_i\|_{\mathcal{L}} + D + n(C + C'') \\ &= \|\rho\|_{\mathcal{L}} + D + n(C + C'') \\ &\leq d_{\mathcal{L}}(\Psi(x), \Psi(y)) + C' + D + n(C + C''). \end{aligned}$$

As a result, we have proved the inequality (3).

Remark 3.3. Combining the known results, we can see that the Teichmüller metric, the length-spectra metric, Bergman metric, Carathéodory mtric, McMullen metric, Ricci metric and perturbed Ricci metric (see Liu, Sun and Yau [11]) are quasi-isometric to each other on moduli space. Note that Leuzinger [6], Farb and Weinberger [3] proved that, while $\mathcal{M}(S)$ admits a metric of positive scalar curvature for most S(when $g > 2$), it admits no metric of positive scalar curvature with the same quasi-isometry type as the Teichmuller metric on $\mathcal{M}(S)$.

There is a natural question that whether the length-spectra metric and the Teichmüller metric are bi-Lipschitz on moduli space. That is, is there a constant $K = K(S)$ such that $d_{\mathcal{L}}(X_1, X_2) \leq d_{\mathcal{T}}(X_1, X_2) \leq K d_{\mathcal{L}}(X_1, X_2)$ for any $X_1, X_2 \in \mathcal{M}(S)$?

Here the left side inequality is trivial. If $d_{\mathcal{L}}(X_1, X_2)$ is sufficiently large, the additive constant in Theorem 1.4 can be absorbed into multiplicative constant to conclude that $d_{\mathcal{T}} \leq K d_{\mathcal{L}}(X_1, X_2)$. Thus this problem concerns mainly the local comparison. In a preprint [2], we have shown that such a constant K depends on the injective radius of X_1, X_2. There exists sequence $\{X_n, Y_n\}$ in the thin part of the Teichmüller space (or moduli space) such that the ratio $d_{\mathrm{Teich}}(X_n, Y_n)/d_{\mathrm{ls}}(X_n, Y_n)$ tends to infinity. As a result, the two metrics are not bi-Lipschitz in general.

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