Uniqueness of the representation for G-martingales with finite variation

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Abstract

Our purpose is to prove the uniqueness of the representation for G-martingales with finite variation.

Key words: uniqueness; representation theorem; G-martingale; finite variation; G-expectation

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1 Introduction

In [P07b], processes in form of $\int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$, $\eta \in M_G^1(0,T)$ are proved to be G-martingales. However, the uniqueness of the representation remains unresolved. In order to prove the uniqueness, we must find ways to distinguish the two classes of processes in forms of $\int_0^t \eta_s d\langle B\rangle_s$ and $\int_0^t \zeta_s ds$, $\eta, \zeta \in M_G^1(0,T)$.

For a process $\{K_t\}$ with finite variation, motivated by [Song10], we define

$$d(K) := \limsup_{n\to\infty} \hat{E}[\int_0^T \delta_n(s) dK_s],$$

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where, for $n \in N$, $\delta_n(s)$ is defined in the following way:

$$\delta_n(s) = \sum_{i=0}^{n-1} (-1)^i 1_{]\frac{iT}{n}, \frac{(i+1)T}{n}]}(s), \text{ for all } s \in [0,T].$$

We prove that $d(K) = 0$ if $K_t = \int_0^t \zeta_s ds$ for some $\zeta \in M_G^1(0,T)$ and that $d(K) > 0$ if $K_t = \int_0^t \eta_s d\langle B\rangle_s$ for some $\eta \in M_G^1(0,T)$ such that $\hat{E}[\int_0^T |\eta_s| ds] > 0$. By this, we distinguish these two classes of processes completely:

If $\int_0^t \eta_s d\langle B\rangle_s = \int_0^t \zeta_s ds$, for some $\eta, \zeta \in M_G^1(0,T)$, then we have

$$\hat{E}[\int_0^T |\eta_s| ds] = \hat{E}[\int_0^T |\zeta_s| ds] = 0.$$

As an application, we prove the uniqueness of the representation for G-martingales with finite variation.

This article is organized as follows: In section 2, we recall some basic notions and results of G-expectation and the related space of random variables. In section 3, we present the main results and some corollaries. In section 4, we give the proofs to the main results.

2 Preliminaries

We recall some basic notions and results of G-expectation and the related space of random variables. More details of this section can be found in [P07a, P07b, P08, P10].

Definition 2.1 Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.

(b) Constant preserving: $\hat{E}(c) = c$.

(c) Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$.

(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X)$, $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Definition 2.2 Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if $\hat{E}_1[\varphi(X_1)] =$

$\hat{E}_2[\varphi(X_2)]$, $\forall\varphi \in C_{l,Lip}(R^n)$, where $C_{l,Lip}(R^n)$ is the space of real continuous functions defined on R^n such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \forall x, y \in R^n,$$

where k depends only on φ.

Definition 2.3 In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}(\cdot)$, denoted by $Y \perp X$, if for each test function $\varphi \in C_{l,Lip}(R^m \times R^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}]$.

Definition 2.4 (G-normal distribution) A d-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normal distributed if for each $a, b \in R$ we have

$$aX + b\hat{X} \sim \sqrt{a^2 + b^2}X,$$

where $\hat{X}$ is an independent copy of X. Here the letter G denotes the function

$$G(A) := \frac{1}{2}\hat{E}[(AX, X)] : S_d \to R,$$

where S_d denotes the collection of $d \times d$ symmetric matrices.

The function $G(\cdot) : S_d \to R$ is a monotonic, sublinear mapping on S_d and $G(A) = \frac{1}{2}\hat{E}[(AX, X)] \leq \frac{1}{2}|A|\hat{E}[|X|^2] =: \frac{1}{2}|A|\bar{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset S_d^+$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Gamma} Tr(\gamma A). \tag{2.0.1}$$

If there exists some $\beta > 0$ such that $G(A) - G(B) \geq \beta Tr(A - B)$ for any $A \geq B$, we call the G-normal distribution is non-degenerate.

Definition 2.5 i) Let $\Omega_T = C_0([0, T]; R^d)$ with the supremum norm, $\mathcal{H}_T^0 := \{\varphi(B_{t_1}, ..., B_{t_n}) | \forall n \geq 1, t_1, ..., t_n \in [0, T], \forall\varphi \in C_{l,Lip}(R^{d\times n})\}$, G-expectation is a sublinear expectation defined by

$$\hat{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$
$$= \tilde{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \cdots, \xi_n$ are identically distributed d-dimensional G-normal distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})$ such that ξ_{i+1} is independent to $(\xi_1, \cdots, \xi_i)$ for each $i = 1, \cdots, m$. $(\Omega_T, \mathcal{H}_T^0, \hat{E})$ is called a G-expectation space.

ii) For $t \in [0,T]$ and $\xi = \varphi(B_{t_1}, ..., B_{t_n}) \in \mathcal{H}_T^0$, the conditional expectation defined by(there is no loss of generality, we assume $t = t_i$)

$$\hat{E}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$

$$= \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\tilde{\varphi}(x_1, \cdots, x_i) = \hat{E}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].$$

Define $\|\xi\|_{p,G} = [\hat{E}(|\xi|^p)]^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \geq 1$. Then $\forall t \in [0,T]$, $\hat{E}_t(\cdot)$ is a continuous mapping on $\mathcal{H}_T^0$ with norm $\|\cdot\|_{1,G}$ and therefore can be extended continuously to the completion $L_G^1(\Omega_T)$ of $\mathcal{H}_T^0$ under norm $\|\cdot\|_{1,G}$.

Let $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) | n \geq 1, t_1, ..., t_n \in [0,T], \varphi \in C_{b,Lip}(R^{d\times n})\}$, where $C_{b,Lip}(R^{d\times n})$ denotes the set of bounded Lipschitz functions on $R^{d\times n}$. [DHP08] proved that the completions of $C_b(\Omega_T)$, $\mathcal{H}_T^0$ and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ are the same and we denote them by $L_G^p(\Omega_T)$.

Definition 2.5 Let $M_G^0(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0,T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{]t_j, t_{j+1}]}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N-1$. For $p \geq 1$ and $\eta \in M_G^0(0,T)$, let $\|\eta\|_{M_G^p} = \{\hat{E}(\int_0^T |\eta_s|^p ds)\}^{1/p}$ and denote by $M_G^p(0,T)$ the completion of $M_G^0(0,T)$ under the norm $\|\cdot\|_{M_G^p}$.

Theorem 2.6([DHP08]) There exists a tight subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ such that

$$\hat{E}(\xi) = \max_{P\in\mathcal{P}} E_P(\xi) \quad \text{for all } \xi \in \mathcal{H}_T^0.$$

$\mathcal{P}$ is called a set that represents $\hat{E}$.

Remark 2.7 Let $(\Omega^0, \{\mathcal{F}_t^0\}, \mathcal{F}^0, P^0)$ be a filtered probability space and $\{W_t\}$ be a d-dimensional Brownian motion under P^0. [DHP08] proved that

$$\mathcal{P}_M := \{P_0 \circ X^{-1} | X_t = \int_0^t h_s dW_s, h \in L_{\mathcal{F}}^2([0,T]; \Gamma^{1/2})\}$$

is a set that represents $\hat{E}$, where $\Gamma^{1/2} := \{\gamma^{1/2} | \gamma \in \Gamma\}$ and Γ is the set in the representation of $G(\cdot)$ in the formula (2.0.1).

3 Main results

In the sequel, we only consider the G-expectation space $(\Omega_T, L_G^1(\Omega_T, \hat{E}))$ with $\Omega_T = C_0([0,T], R)$ and $\overline{\sigma}^2 = \hat{E}(B_1^2) > -\hat{E}(-B_1^2) = \underline{\sigma}^2 \geq 0$.

Proposition 3.1 For each $\eta \in M_G^1(0,T)$, let

$$d(\eta) = \limsup_{n\to\infty} \hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s].$$

Then

$$-\frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\hat{E}[-\int_0^T |\eta_s|ds] \leq d(\eta) \leq \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\hat{E}[\int_0^T |\eta_s|ds]. \qquad (3.0.2)$$

Proof. It suffices to prove the conclusion for $\eta \in M_G^0(0,T)$. Let $\eta_s = \sum_{i=0}^{m-1} \xi_{t_i} 1_{]t_i,t_{i+1}]}(s)$, $\xi_{t_i} \in L_G^1(\Omega_{t_i})$, $i = 0, \cdots, m-1$.

$$\begin{aligned} & \hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s] - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\hat{E}[\int_0^T |\eta_s|ds] \\ = & \hat{E}[\sum_{i=0}^{m-1} |\xi_{t_i}| \int_{t_i}^{t_{i+1}} \delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s] - \hat{E}[\sum_{i=0}^{m-1} |\xi_{t_i}| \int_{t_i}^{t_{i+1}} \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}ds] \\ \leq & \sum_{i=0}^{m-1} \hat{E}[|\xi_{t_i}|(\int_{t_i}^{t_{i+1}} \delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s - \int_{t_i}^{t_{i+1}} \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}ds)] \to 0 \end{aligned}$$

as n goes to infinity. So

$$d(\eta) \leq \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\hat{E}[\int_0^T |\eta_s|ds].$$

On the other hand,

$$\begin{aligned} & \hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s] + \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\hat{E}[-\int_0^T |\eta_s|ds] \\ = & \hat{E}[\sum_{i=0}^{m-1} |\xi_{t_i}| \int_{t_i}^{t_{i+1}} \delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s] + \hat{E}[\sum_{i=0}^{m-1} (-|\xi_{t_i}|) \int_{t_i}^{t_{i+1}} \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}ds] \\ \geq & \hat{E}[\sum_{i=0}^{m-1} |\xi_{t_i}|(\int_{t_i}^{t_{i+1}} \delta_n(s)\mathrm{sgn}(\xi_{t_i})d\langle B\rangle_s - \int_{t_i}^{t_{i+1}} \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}ds)] \\ \geq & \sum_{i=0}^{m-1} [-\hat{E}(|\xi_{t_i}|)a_i(n)], \end{aligned}$$

where $a_i(n) = \max\{|\hat{E}(\int_{t_i}^{t_{i+1}} \delta_n(s)d\langle B\rangle_s - \int_{t_i}^{t_{i+1}} \frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds)|, |\hat{E}(-\int_{t_i}^{t_{i+1}} \delta_n(s)d\langle B\rangle_s - \int_{t_i}^{t_{i+1}} \frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}ds)|\} \to 0$ as n goes to infinity. So

$$-\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\hat{E}[-\int_0^T |\eta_s|ds] \leq d(\eta).$$

□

Remark 3.2 (i) A straightforward corollary of Proposition 3.1 is that if $\int_0^T |\eta_s|ds$ is symmetric (i.e., $\hat{E}[\int_0^T |\eta_s|ds] = -\hat{E}[-\int_0^T |\eta_s|ds]$), the equality $d(\eta) = \frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\hat{E}[\int_0^T |\eta_s|ds]$ holds.

(ii) By Lemma 3.1, we could not conclude that $d(\eta) > 0$ whenever $\hat{E}[\int_0^T |\eta_s|ds] > 0$, which is the conclusion of Theorem 3.3 below.

(iii) The inequalities in (3.0.2) may be strict:

Let $\eta_s = \langle B\rangle_{T/2} 1_{]T/2,T]}(s) + a1_{[0,T/2]}(s)$, $a = T(\overline{\sigma}^2 - \underline{\sigma}^2)/4$.

Then

$$d(\eta) = \lim_{n\to\infty} \hat{E}[\int_0^T \delta_{2n}(s)\eta_s d\langle B\rangle_s] = a\overline{\sigma}^2 T/2,$$

$$\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\hat{E}[\int_0^T |\eta_s|ds] = a^2 + a\overline{\sigma}^2 T/2,$$

$$-\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\hat{E}[-\int_0^T |\eta_s|ds] = -a^2 + a\overline{\sigma}^2 T/2.$$

□

Now, we shall state the main result of this article, whose proof is postponed to Section 4.

Theorem 3.3 For $\eta \in M_G^1(0,T)$ with $\hat{E}[\int_0^T |\eta_s|ds] > 0$, we have

$$d(\eta) = \limsup_{n\to\infty} \hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s] > 0.$$

Theorem 3.4 Let $\eta \in M_G^1(0,T)$. Then $\lim_{n\to\infty} \hat{E}[\int_0^T \delta_n(s)\eta_s ds] = 0$.

Proof. For $\eta \in M_G^0(0,T)$, the claim is obvious. For $\eta \in M_G^1(0,T)$, there exists a sequence of $\{\eta^m\} \subset M_G^0(0,T)$ such that $\hat{E}[\int_0^T |\eta_s^m - \eta_s|ds] \to 0$. Then $|\hat{E}[\int_0^T \delta_n(s)\eta_s ds]| \leq |\hat{E}[\int_0^T \delta_n(s)\eta_s^m ds]| + \hat{E}[\int_0^T |\eta_s^m - \eta_s|ds]$. First let $n \to \infty$, then let $m \to \infty$, and we get the desired result. □

Remark 3.5 Let $(\Omega, F, \mathcal{F}, P)$ be a filtered probability space. We recall that for any progressively measurable process η such that $E[\int_0^T |\eta_s|ds] < \infty$, we have

$$\lim_{n\to\infty}\hat{E}[\int_0^T \delta_n(s)\eta_s ds]=0.$$

Therefore, Theorem 3.3 presents a particular property of G-expectation space relative to probability space.

Corollary 3.6 Let $\zeta,\eta\in M_G^1(0,T)$. If $\int_0^t \eta_s d\langle B\rangle_s=\int_0^t\zeta_s ds$ for all $t\in[0,T]$, then $E[\int_0^T|\eta_s|ds]=\hat{E}[\int_0^T|\zeta_s|ds]=0$.

Proof. By Theorem 3.4, we have

$$\limsup_{n\to\infty}\hat{E}[\int_0^T \delta_n(s)\eta_s d\langle B\rangle_s]=\lim_{n\to\infty}\hat{E}[\int_0^T \delta_n(s)\zeta_s ds]=0.$$

By Theorem 3.3, we have $\hat{E}[\int_0^T|\eta_s|ds]=0$, which leads to $\hat{E}[\int_0^T|\zeta_s|ds]=0$. □

The following corollary is about the uniqueness of representation for G-martingales with finite variation.

Corollary 3.7 Let $\zeta,\eta\in M_G^1(0,T)$. If for all $t\in[0,T]$,

$$\int_0^t\eta_s d\langle B\rangle_s-\int_0^t 2G(\eta_s)ds=\int_0^t\zeta_s d\langle B\rangle_s-\int_0^t 2G(\zeta_s)ds, \tag{3.0.3}$$

we have $\hat{E}[\int_0^T|\eta_s-\zeta_s|ds]=0$.

Proof. By the assumption, we have

$$\int_0^t(\eta_s-\zeta_s)d\langle B\rangle_s=\int_0^t 2[G(\eta_s)-G(\zeta_s)]ds,\ \text{for all } t\in[0,T].$$

Since $\eta-\zeta, 2[G(\eta)-G(\zeta)]\in M_G^1(0,T)$, we have $\hat{E}[\int_0^T|\eta_s-\zeta_s|ds]=0$ by Corollary 3.6. □

Remark 3.8(i) In the setting considered in this article, $G(a)=\frac{1}{2}(\overline{\sigma}^2a^+-\underline{\sigma}^2a^-)$. For $\varepsilon\in(0,\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2})$, [HuP10] defined G_ε in the following way:

$$G_\varepsilon(a)=G(a)-\frac{\varepsilon}{2}|a|,\ \text{for all } a\in R.$$

Indeed, Proof to Theorem 3.3 in the next section leads to the following conclusion:

$$d(\eta)\geq\varepsilon\hat{E}_{G_\varepsilon}[\int_0^T|\eta_s|ds]. \tag{3.0.4}$$

(ii) For $\eta \in M_G^1(0,T)$, let $K_t = \int_0^T \eta_s d\langle B\rangle_s - \int_0^T 2G(\eta_s)ds$. Then, by Theorem 3.4, we have

$$\hat{E}(-K_T) \geq \limsup_{n\to\infty} \hat{E}(\int_0^T \delta_n(s)dK_s) = d(\eta). \tag{3.0.5}$$

This, combined with (3.0.4), leads to the following estimate:

$$\hat{E}[-K_T] \geq \varepsilon \hat{E}_{G_\varepsilon}[\int_0^T |\eta_s|ds],$$

which was already proved in [HuP10]. Then for $\eta, \zeta \in M_G^1(0,T)$ such that (3.0.3) and

$$\int_0^t 2[G(\eta_s) - G(\zeta_s)]ds = \int_0^t 2[G(\eta_s - \zeta_s)]ds \text{ for all } t \in [0,T] \tag{3.0.6}$$

hold, we have $\hat{E}[\int_0^T |\eta_s - \zeta_s|ds] = 0$. However, (3.0.6) does not hold generally since the nonlinearity of G, which is the main difficulty to deal with such questions. □

4 Proof to Theorem 3.3

In order to prove Theorem 3.3, we first introduce two lemmas.

Let $\Omega_T = C_b([0,T];R)$ be endowed with the supremum norm and let $\sigma : [0,T] \times \Omega_T \to R$ be a measurable mapping satisfying

i) σ is bounded;

ii) There exists $C > 0$ such that $|\sigma(s,x) - \sigma(s,y)| \leq C\|x - y\|$ for any $s \in [0,T]$ and $x, y \in C_b([0,T];R)$;

iii)For $t \in [0,T]$, $\sigma(t,\cdot)$ is $\mathcal{B}_t(\Omega_T)$ measurable.

Then the following lemma is easy.

Lemma 4.1 Let $(\Omega, F, \mathcal{F}, P)$ be a filtered probability space and let M be a continuous F-martingale with $\langle M\rangle_t - \langle M\rangle_s \leq C(t-s)$ for some $C > 0$ and any $0 \leq s < t \leq T$. Let F^X be the augmented filtration generated by X. Then for any $Y_0 \in \mathcal{F}_0^X$, there exists a unique F-adapted continuous process with $E[\sup_{t\in[0,T]} |Y_t|^2] < \infty$ such that $Y_t = Y_0 + \int_0^t \sigma(s,Y)dX_s$. Moreover, Y is F^X-adapted. □

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{W_t\}$ be a standard 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. Let F^W be the augmented filtration generated by W.

Denote by $\mathcal{A}^0([c,C])$, for some $0 < c \le C < \infty$, the collection of F^W adapted processes in the following form

$$h_s = \sum_{i=0}^{m-1} \xi_i 1_{]\frac{iT}{m},\frac{(i+1)T}{m}]}(s),$$

where $\xi_i = \psi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)$, $\psi_i \in C_{b,lip}(R^i)$, $c \le |\psi_i| \le C$. Denote by $\mathcal{A}([c,C])$ the collection of F^W adapted processes such that $c \le |h_s| \le C$.

Lemma 4.2 $\mathcal{A}^0([c,C])$ is dense in $\mathcal{A}([c,C])$ under the norm

$$\|h\|_2 = [E(\int_0^T |h_s|^2 ds)]^{1/2}.$$

Proof. Let $h_s = \sum_{i=0}^{m-1} \xi_i 1_{]\frac{iT}{m},\frac{(i+1)T}{m}]}(s)$, where

$$\xi_i = \varphi_i(W_{\frac{iT}{m}} - W_{\frac{(i-1)T}{m}}, \cdots, W_{\frac{T}{m}} - W_0),$$

$$\varphi_i \in C_{b,lip}(R^i), c \le |\varphi_i| \le C.$$

Then $\sigma(s,x) = h_s^{-1}(x)$ is a bounded Lipschitz function. Let $X_t := \int_0^t h_s dW_s$. Since $W_t = \int_0^t \sigma(s,W) dX_s$, we conclude, by Lemma 4.1, that W is F^X-adapted.

For a process $\{X_t\}$, we denote the vector $(X_T - X_{\frac{(m-1)T}{m}}, \cdots, X_{\frac{T}{m}} - X_0)$ by $X^m_{[0,T]}$.

For arbitrary $\varepsilon_i > 0$, $i = 0, \cdots, m-1$, there exists $\psi_i \in C_{b,lip}(R^{in_i})$ with the Lipschitz constant L_i such that $E[|\xi_i - \widetilde{\xi}_i|^2] < \varepsilon_i^2$. Here $\widetilde{\xi}_i = \psi_i(X^{in_i}_{[0,\frac{iT}{m}]})$, $c \le |\psi_i| \le C$. Without loss of generality, we assume that there exists $K_{ji} \in N$ such that $n_j = K_{ji} n_i$ for $m-1 \ge i > j \ge 0$.

Define $\widehat{\xi}_i$ in the following way:

$\widehat{\xi}_0 = \widetilde{\xi}_0$,

For $s \in]0, \frac{T}{m}]$, $\widehat{h}_s = \widehat{\xi}_0$,

Assume that we have defined $\widehat{h}_s$ for all $s \in [0, \frac{iT}{m}]$, $0 \le i \le m-1$,

Define $\widehat{X}_t := \int_0^t \widehat{h}_s dW_s$, for $t \in [0, \frac{iT}{m}]$,

$\widehat{\xi}_i = \psi_i(\widehat{X}^{in_i}_{[0,\frac{iT}{m}]})$,

For $s \in]\frac{iT}{m}, \frac{(i+1)T}{m}]$, $\widehat{h}_s = \widehat{\xi}_i$.

We claim that for any $m-1 \geq i \geq 1$,

$$\hat{E}[|\widehat{\xi}_i - \widetilde{\xi}_i|^2] \leq \sum_{j=0}^{i-1} A_j^i \varepsilon_j^2, \tag{4.0.7}$$

where $A_j^i = 2TL_i^2(\sum_{k=j+1}^{i-1} A_j^k + 1)$, for $i \geq j+2$, $A_{i-1}^i = 2TL_i^2$, which shows that A_j^i depends only on $L_{j+1}, \cdots, L_i$ and T.

Indeed, $E[|\widehat{\xi}_1 - \widetilde{\xi}_1|^2] \leq L_1^2 E[|\widehat{\xi}_0 - \xi_0|^2|]E[|W^{n_1}_{[0,\frac{T}{m}]}|^2] = \frac{T}{m} L_1^2 \varepsilon_0^2 \leq A_0^1 \varepsilon_0^2$. Assume (4.0.7) holds for $1 \leq i \leq l$. For $i = l+1$,

$$\begin{aligned}
& E[|\widehat{\xi}_{l+1} - \widetilde{\xi}_{l+1}|^2] \\
\leq\ & L_{l+1}^2 \sum_{i=0}^{l} E[|\widehat{\xi}_i - \xi_i|^2] E[|W^{n_{l+1}}_{[\frac{iT}{m}, \frac{(i+1)T}{m}]}|^2] \\
\leq\ & 2TL_{l+1}^2 \sum_{i=0}^{l} E[(|\widehat{\xi}_i - \widetilde{\xi}_i|^2 + |\widetilde{\xi}_i - \xi_i|^2)] \\
\leq\ & 2TL_{l+1}^2 (\sum_{i=0}^{l} \varepsilon_i^2 + \sum_{i=1}^{l} \sum_{j=0}^{i-1} A_j^i \varepsilon_j^2) \\
=\ & 2TL_{l+1}^2 [\sum_{j=0}^{l-1} (\sum_{i=j+1}^{l} A_j^i + 1) \varepsilon_j^2 + \varepsilon_l^2] \\
=\ & \sum_{j=0}^{l} A_j^{l+1} \varepsilon_j^2.
\end{aligned}$$

Then

$$\begin{aligned}
& E[|\widehat{\xi}_i - \xi_i|^2] \\
\leq\ & 2(E[|\widehat{\xi}_i - \widetilde{\xi}_i|^2] + E[|\widetilde{\xi}_i - \xi_i|^2]) \\
\leq\ & 2\varepsilon_i^2 + 2\sum_{j=0}^{i-1} A_j^i \varepsilon_j^2 \\
=:\ & \sum_{j=0}^{i} B_j^i \varepsilon_j^2,
\end{aligned}$$

which shows that B_j^i depends only on $L_{j+1}, \cdots, L_i$ and T. So for any $\varepsilon > 0$, we can choose $\widehat{\xi}_i$, $i = 0, \cdots, m-1$ defined above such that $E[|\widehat{\xi}_i - \xi_i|^2] < \varepsilon$ for all $i = 0, \cdots, m-1$. Then

$$E[\int_0^T |h_s - \widehat{h}_s|^2] < T\varepsilon.$$

□

Proof to Theorem 3.3. For $\eta \in M_G^1(0,T)$ with $\hat{E}[\int_0^T |\eta_s|ds] > 0$, by Theorem 2.6 and Remark 2.7, there exists $\varepsilon > 0$ and $P \in \mathcal{P}_M$ such that $E_P[\int_0^T |\eta_s|ds] =: A > 0$ and for any $0 \le s < t \le T$

$$(\underline{\sigma}^2 + \varepsilon)(t-s) \le \langle B\rangle_t - \langle B\rangle_s \le (\overline{\sigma}^2 - \varepsilon)(t-s), \;\; P\text{-}a.s..$$

For any $\frac{A\varepsilon}{(\overline{\sigma}^2+\varepsilon)} > \delta > 0$, there exists $\zeta \in M_G^0(0,T)$ such that

$$\hat{E}[\int_0^T |\eta_s - \zeta_s|ds] < \delta.$$

Let $(\Omega^0, F = \{\mathcal{F}_t^0\}, \mathcal{F}^0, P^0)$ be a filtered probability space, and $\{W_t\}$ be a d-dimensional Brownian motion under P^0. By Remark 2.7, there exists an F adapted process h with $\underline{\sigma}^2 + \varepsilon \le h_s^2 \le \overline{\sigma}^2 - \varepsilon$ such that $P = P^0 \circ (\int_0^\cdot h_s dW_s)^{-1}$.

Without loss of generality, by Lemma 4.2, we assume that there exists $m \in N$ such that

$$\zeta_s = \sum_{i=0}^{m-1} \xi_{\frac{iT}{m}} 1_{]\frac{iT}{m}, \frac{(i+1)T}{m}]}(s)$$

where $\xi_{\frac{iT}{m}} = \varphi_i(B_{\frac{iT}{m}} - B_{\frac{(i-1)T}{m}}, \cdots, B_{\frac{T}{m}} - B_0)$, $\varphi_i \in C_{b,lip}(R^i)$, for all $0 \le i \le m-1$;

$$h_s = \sum_{i=0}^{m-1} a_{\frac{iT}{m}} 1_{]\frac{iT}{m}, \frac{(i+1)T}{m}]}(s)$$

where $a_{\frac{iT}{m}} = \psi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)$, $\underline{\sigma}^2 + \varepsilon \le |\psi_i|^2 \le \overline{\sigma}^2 - \varepsilon$, $\psi_i \in C_{b,lip}(R^i)$, for all $0 \le i \le m-1$.

1. Define $H^i : [\underline{\sigma}^2 + \varepsilon, \overline{\sigma}^2 - \varepsilon] \to [\underline{\sigma}, \overline{\sigma}]$, i=1, -1 in the following way:

$$H^1(x)^2 = \overline{\sigma}^2 1_{[x \ge \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{2}]} + (2x - \underline{\sigma}^2) 1_{[x < \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{2}]};$$

$$H^{-1}(x)^2 = (2x - \overline{\sigma}^2) 1_{[x \ge \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{2}]} + \underline{\sigma}^2 1_{[x < \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{2}]}.$$

It's easily seen that $H^1(x)^2 + H^{-1}(x)^2 = 2x$ and $H^1(x)^2 - H^{-1}(x)^2 \ge 2\varepsilon$.

For $n \in N$, define $H_n^i : [0, 1/m] \times [\underline{\sigma}^2 + \varepsilon, \overline{\sigma}^2 - \varepsilon] \to [\underline{\sigma}, \overline{\sigma}]$, $i = 1, -1$ by

$$H_n^i(s,x) = \sum_{j=0}^{2n-1} 1_{]\frac{jT}{2mn}, \frac{(j+1)T}{2mn}]}(s) H^{(-1)^j i}(x).$$

2. Fix $n \in N$.

$a_0^n = a_0$, $\xi_0^n = \xi_0$,

For $s \in]0, \frac{T}{m}]$, $h_s^n = H_n^{\mathrm{sgn}(\xi_0^n)}(s, (a_0^n)^2)$;

Assume that we have defined h_s^n for all $s \in [0, \frac{iT}{m}]$, $0 \le i \le m-1$.

$a_{\frac{iT}{m}}^n = \psi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s^n dW_s, \cdots, \int_0^{\frac{T}{m}} h_s^n dW_s)$,

$\xi_{\frac{iT}{m}}^n = \varphi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s^n dW_s, \cdots, \int_0^{\frac{T}{m}} h_s^n dW_s)$,

For $s \in]\frac{iT}{m}, \frac{(i+1)T}{m}]$, $h_s^n = H_n^{\mathrm{sgn}(\xi_{\frac{iT}{m}}^n)}(s - \frac{iT}{m}, (a_{\frac{iT}{m}}^n)^2)$.

3. $E_P[\int_0^T |\zeta_s| ds] = E_{P_{h^n}}[\int_0^T |\zeta_s| ds]$.

In fact,

$$\begin{aligned} & E_P[\int_0^T |\zeta_s| ds] \\ = \; & \frac{T}{m} E_{P_0}[\sum_{i=0}^{m-1} |\varphi_i(\int_{\frac{(i-1)T}{m}}^{\frac{iT}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)|] \\ = \; & : E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{T} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)] \end{aligned}$$

and

$$E_{P_{h^n}}[\int_0^T |\zeta_s| ds] = E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{T} h_s^n dW_s, \cdots, \int_0^{\frac{T}{m}} h_s^n dW_s)].$$

Let $x = (x_{m-1}, \cdots, x_1)$. Noting that

$$\begin{aligned} & \Phi_{m-1}(x) \\ := \; & E_{P_0}\{\Phi(\int_{\frac{(m-1)T}{m}}^{T} H_n^{\mathrm{sgn}(\varphi_{m-1}(x))}(s - \frac{(m-1)T}{m}, \psi_{m-1}(x)^2) dW_s, x)\} \\ = \; & E_{P_0}\{\Phi(\int_{\frac{(m-1)T}{m}}^{T} \psi_{m-1}(x) dW_s, x)\}, \end{aligned}$$

we have

$$E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{T} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)] = E_{P_0}[\Phi_{m-1}(\int_{\frac{(m-2)T}{m}}^{\frac{(m-1)T}{m}} h_s dW_s, \cdots, \int_0^{\frac{T}{m}} h_s dW_s)]$$

$$E_{P_0}[\Phi(\int_{\frac{(m-1)T}{m}}^{T} h_s^n dW_s, \cdots, \int_0^{\frac{T}{m}} h_s^n dW_s)] = E_{P_0}[\Phi_{m-1}(\int_{\frac{(m-2)T}{m}}^{\frac{(m-1)T}{m}} h_s^n dW_s, \cdots, \int_0^{\frac{T}{m}} h_s^n dW_s)].$$

By induction on m, we get the desired result.

4.

$$
\begin{aligned}
& \hat{E}[\int_0^T \delta_{2mn}(s)\eta_s d\langle B\rangle_s] \\
\geq\; & \hat{E}[\int_0^T \delta_{2mn}(s)\zeta_s d\langle B\rangle_s] - \hat{E}[\int_0^T |\eta_s-\zeta_s| d\langle B\rangle_s] \\
\geq\; & E_{P_{h^n}}[\int_0^T \delta_{2mn}(s)\zeta_s d\langle B\rangle_s] - \overline{\sigma}^2\delta \\
=\; & E_{P_{h^n}}[\sum_{i=0}^{m-1}\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} \delta_{2mn}(s)\zeta_s d\langle B\rangle_s] - \overline{\sigma}^2\delta \\
=\; & E_{P_{h^n}}[\sum_{i=0}^{m-1}\xi_{\frac{iT}{m}}\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} \delta_{2mn}(s) d\langle B\rangle_s] - \overline{\sigma}^2\delta \\
\geq\; & \frac{T}{m}\varepsilon E_{P_{h^n}}[\sum_{i=0}^{m-1}|\xi_{\frac{iT}{m}}|] - \overline{\sigma}^2\delta \\
=\; & \varepsilon E_{P_{h^n}}[\int_0^T |\zeta_s| ds] - \overline{\sigma}^2\delta \\
=\; & \varepsilon E_P[\int_0^T |\zeta_s| ds] - \overline{\sigma}^2\delta \\
\geq\; & \varepsilon E_P[\int_0^T |\eta_s| ds] - \varepsilon\delta - \overline{\sigma}^2\delta \\
\geq\; & A\varepsilon - \varepsilon\delta - \overline{\sigma}^2\delta > 0.
\end{aligned}
$$

Since A, ε, δ do not depend on n, we have $d(\eta) \geq A\varepsilon - \varepsilon\delta - \overline{\sigma}^2\delta > 0$. The proof is completed. □

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