THE BOUNDED SPHERICAL FUNCTIONS FOR THE FREE TWO-STEP NILPOTENT LIE GROUP

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ABSTRACT. In this paper, we give the expressions for the bounded spherical functions, or equivalently the spherical functions of positive type, for the free two-step nilpotent Lie groups endowed with the actions of orthogonal groups or their special subgroups. Next we deduce some results about the (Kohn) sub-Laplacian, and we compute the radial Plancherel measure.

1. Introduction

A (connected, simply connected) nilpotent Lie group which forms with a compact Lie group a Gelfand pair, is at most of step two, and the bounded spherical functions are the spherical functions of positive type [1]. The cases of the Heisenberg group with some subgroups of the unitary matrix group are well known [2], and the bounded spherical functions are then explicit. In this paper, we are interested by the Gelfand pair formed by the free two-step nilpotent Lie groups with the actions of orthogonal groups. The expressions of some of the corresponding bounded spherical functions were given in [11, Section 6] with a sketched proof. Here we give the expression of all such functions with a complete proof, and we obtain the corresponding eigenvalues for the sub-Laplacian and the radial Plancherel measure.

This paper is organized as follows. After recalling some definitions and setting some notations, we give in the second section the statement of the main results: the expressions of the bounded spherical functions on the free two-step nilpotent Lie groups. In the third section, we recall a few facts about spherical functions and representations, which allow to construct our bounded spherical functions in the following section. We also give an equivalent method of construction from which we obtain some properties of the sub-Laplacian and the radial Plancherel formula in the fifth section.
We shall omit some computations and the proof for the case of the special orthogonal group. We refer the interested reader to the French thesis of the author [5].

2. The Free Two-step Nilpotent Lie Groups

Here we give definitions and notations for the free two-step nilpotent Lie groups and algebras; we also present the action of orthogonal groups.

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First Definition. Let $\mathcal{N}_p$ be the (unique up to isomorphism) free two-step nilpotent Lie algebra with p generators. The definition using the universal property of the free nilpotent Lie algebra can be found in [7, Chapter V §4]. Roughly speaking, $\mathcal{N}_p$ is a (nilpotent) Lie algebra with p generators $X_1, \ldots, X_p$, such that the vectors $X_1, \ldots, X_p$ and $X_{i,j} = [X_i, X_j], i < j$ form a basis; we call this basis the canonical basis of $\mathcal{N}_p$.
We denote by $\mathcal{V}$ and $\mathcal{Z}$, the vector spaces generated by the families of vectors $X_1, \ldots, X_p$ and $X_{i,j} := [X_i, X_j], 1 \leq i < j \leq p$ respectively; these families become the canonical base of $\mathcal{V}$ and $\mathcal{Z}$. Thus $\mathcal{N}_p = \mathcal{V} \oplus \mathcal{Z}$, and $\mathcal{Z}$ is the center of $\mathcal{N}_p$. With the canonical basis, the vector space $\mathcal{Z}$ can be identified with the vector space of antisymmetric $p \times p$-matrices $\mathcal{A}_p$. Let $z = \dim \mathcal{Z} = p(p-1)/2$.
The connected simply connected nilpotent Lie group which corresponds to $\mathcal{N}_p$ is called the free two-step nilpotent Lie group and is denoted N_p. We denote by $\exp : \mathcal{N}_p \to N_p$ the exponential map.
In the following, we use the notations $X + A \in \mathcal{N}$, $\exp(X + A) \in N$ when $X \in \mathcal{V}, A \in \mathcal{Z}$. We write $p = 2p'$ or $2p' + 1$.
A Realization of $\boldsymbol{\mathcal{N}_p}$. We now present here a realization of $\mathcal{N}_p$, which will be helpful to define more naturally the action of the orthogonal group and representations of N_p.
Let $(\mathcal{V}, <,>)$ be an Euclidean space with dimension p. Let $O(\mathcal{V})$ be the group of orthogonal transformations of $\mathcal{V}$, and $SO(\mathcal{V})$ its special subgroup. Their common Lie algebra denoted by $\mathcal{Z}$, is identified with the vector space of antisymmetric transformations of $\mathcal{V}$. Let $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ be the exterior direct sum of the vector spaces $\mathcal{V}$ and $\mathcal{Z}$.
Let $[,] : \mathcal{V} \times \mathcal{V} \to \mathcal{Z}$ be the bilinear application given by :

$$[X, Y].(V) = <X, V> Y - <Y, V> X \quad X, Y, V \in \mathcal{V} \quad .$$

We also denote by $[,]$ the bilinear application extended to $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$ by:

$$[.,.]_{\mathcal{N} \times \mathcal{Z}} = [.,.]_{\mathcal{Z} \times \mathcal{N}} = 0 \quad .$$

This application is a Lie bracket. It endows $\mathcal{N}$ with the structure of a two-step nilpotent Lie algebra.
As the elements $[X, Y]$, $X, Y \in \mathcal{V}$ generate the vector space $\mathcal{Z}$, we also define a scalar product $<,>$ on $\mathcal{Z}$ by:

$$<[X, Y], [X', Y']> = <X, X'><Y, Y'> - <X, Y'><X', Y>, \quad ,$$

where $X, Y, X', Y' \in \mathcal{V}$.
It is easy to see $\mathcal{N}$ as a realization of $\mathcal{N}_p$ when an orthonormal basis $X_1, \ldots, X_p$ of $(\mathcal{V}, <,>)$ is fixed.
We remark that $<[X, Y], [X', Y']> = <[X, Y]X', Y'>$, and so we have for an antisymmetric transformation $A \in \mathcal{Z}$, and for $X, Y \in \mathcal{V}$:

$$(1) \qquad <A, [X, Y]> = <A.X, Y> \quad .$$

This equality can also be proved directly using the canonical basis of $\mathcal{N}_p$.
Actions of Orthogonal Groups. We denote by $O(\mathcal{V})$ the group of orthogonal linear maps of $(\mathcal{V}, <,>)$, and by O_p the group of orthogonal $p \times p$-matrices.

On $\boldsymbol{\mathcal{N}_p}$ and $\boldsymbol{N_p}$. The group $O(\mathcal{V})$ acts on the one hand by automorphisms on $\mathcal{V}$, on the other hand by the adjoint representation $\mathrm{Ad}_{\mathcal{Z}}$ on $\mathcal{Z}$. We obtain an action of $O(\mathcal{V})$ on $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$. Let us prove that this action respects the Lie bracket of $\mathcal{N}$. It suffices to show for $X, Y, V \in \mathcal{V}$ and $k \in O(\mathcal{V})$:

$$\begin{aligned}
[k.X, k.Y](V) &= \; < k.X, V > k.Y - < k.Y, V > k.X \\
&= \; k.\left(< X,^t k.V > Y - < Y,^t k.V > X\right) \\
&= \; k.[X, Y](k^{-1}.V) \; = \; Ad_{\mathcal{Z}} k.[X, Y] \quad .
\end{aligned}$$

We then obtain that the group $O(\mathcal{V})$ (and also its special subgroup $SO(\mathcal{V})$) acts by automorphism on the Lie algebra $\mathcal{N}$, and finally on the Lie group N.
Suppose an orthonormal basis $X_1, \dots, X_p$ of $(\mathcal{V}, <,>)$ is fixed; then the vectors $X_{i,j} := [X_i, X_j], 1 \leq i < j \leq p$, form an orthonormal basis of $\mathcal{Z}$ and we can identify:

- the vector space $\mathcal{Z}$ with $\mathcal{A}_p$,
- the group $O(\mathcal{V})$ with O_p,
- the adjoint representation $\mathrm{Ad}_{\mathcal{Z}}$ with the conjugate action of O_p on $\mathcal{A}_p$: $k.A = kAk^{-1}$, where $k \in O_p$, $A \in \mathcal{A}_p$.

Thus the group $O_p \sim O(\mathcal{V})$ acts on $\mathcal{V} \sim \mathbb{R}^p$ and $\mathcal{Z} \sim \mathcal{A}_p$, and consequently on $\mathcal{N}_p$. Those actions can be directly defined; and the equality $[k.X, k.Y] = k.[X, Y]$, $k \in O_p$, $X, Y \in \mathcal{V}$, can then be computed.
On $\boldsymbol{\mathcal{A}_p}$. Now we describe the orbits of the conjugate actions of O_p and SO_p on $\mathcal{A}_p$. An arbitrary antisymmetric matrix $A \in \mathcal{A}_p$ is O_p-conjugated to an antisymmetric matrix $D_2(\Lambda)$ where $\Lambda = (\lambda_1, \dots, \lambda_{p'}) \in \mathbb{R}^{p'}$ and:

$$(2) \qquad D_2(\Lambda) := \begin{bmatrix} \lambda_1 J & & & \\ & & & \\ 0 & \ddots & 0 & \\ & & \lambda_{p'} J & \\ & & & (0) \end{bmatrix} \qquad \text{where} \quad J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad .$$

((0) means that a zero appears only in the case $p = 2p' + 1$.) Furthermore, we can assume that Λ is in $\bar{\mathcal{L}}$, where we denote by $\bar{\mathcal{L}}$ the set of $\Lambda = (\lambda_1, \dots, \lambda_{p'}) \in \mathbb{R}^{p'}$, such that $\lambda_1 \geq \ldots \geq \lambda_{p'} \geq 0$.
An arbitrary antisymmetric matrix $A \in \mathcal{A}_p$ is SO_p-conjugated to

$$D_2^\epsilon(\Lambda) := D_2(\lambda_1, \dots, \lambda_{p'-1}, \epsilon\lambda_{p'}) \quad ,$$

where $\epsilon = \pm 1$ and $\Lambda = (\lambda_1, \dots, \lambda_{p'}) \in \bar{\mathcal{L}}$.

3. Notations and Main Results

We give here the notations for special functions that will be used to present the main results of this paper. First, we recall the definitions of the Bessel and Laguerre functions, and we set some notations for parameters. We give then the expression of the bounded spherical functions.
Notations for special functions. We will use the following well known functions :

- the Gamma function Γ,
- the Laguerre polynomial of type α and degree n: L_n^α [12, §5.1],
- the Bessel function of type α: J_α [12, §1.71], [4, ch. II, I.1].

Let us now define the normalized Laguerre function $\bar{\mathcal{L}}_{n,\alpha} = \mathcal{L}_{n,\alpha}/C^n_{n+\alpha}$ where:

$$\mathcal{L}_{n,\alpha}(x) := L^\alpha_n(x)e^{-\frac{x}{2}} \quad \text{and} \quad C^n_{n+\alpha} = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \quad ,$$

and the reduced Bessel function $\mathcal{J}_\alpha$ by:

$$\mathcal{J}_\alpha(z) := \Gamma(\alpha+1)(z/2)^{-\alpha}J_\alpha(z) \quad .$$

Let $n = 1, 2, \ldots$, and let dk denote the Haar probability measure of the compact group $K = O_n$ or SO_n. If $<,>$ denotes the Euclidean scalar product, and $|.|$ the Euclidean norm of $\mathbb{R}^n$, we recall for any fixed $x_0 \in \mathbb{R}^n$ such that $|x_0| = 1$ [4, ch.II, I.1]:

$$\mathcal{J}_{\frac{n-2}{2}}(|x|) \ = \ \int_K e^{i<k.x,x_0>}dk \quad . \tag{3}$$

Parameters. To each $\Lambda \in \bar{\mathcal{L}}$, we associate: p_0 the number of $\lambda_i \neq 0$, p_1 the number of distinct $\lambda_i \neq 0$, and $\mu_1, \ldots, \mu_{p_1}$ such that:

$$\{\mu_1 > \mu_2 > \ldots > \mu_{p_1} > 0\} = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{p_0} > 0\} \quad . \tag{4}$$

We denote by m_j the number of λ_i such that $\lambda_i = \mu_j$, and we put:

$$m_0 := m'_0 := 0 \quad \text{and for } j = 1, \ldots p_1 \quad m'_j := m_1 + \ldots + m_j \quad . \tag{5}$$

For $j = 1, \ldots p_1$, let pr_j be the orthogonal projection of $\mathcal{V}$ onto the space generated by the vectors X_{2i-1}, X_{2i}, for $i = m'_{j-1} + 1, \ldots, m'_j$.
Let $\mathcal{M}$ be the set of (r, Λ) where $\Lambda \in \mathcal{L}$, and $r \geq 0$, such that $r = 0$ if $2p_0 = p$.
Expression of the bounded spherical functions. The bounded spherical functions of (N_p, K) for $K = O_p$ or SO_p, are parameterized by

- $(r, \Lambda) \in \mathcal{M}$ (with the previous notations p_0, p_1, μ_i, pr_j associated to Λ),
- $l \in \mathbb{N}^{p_1}$ if $\Lambda \neq 0$, otherwise $\emptyset$,
- $\epsilon = \pm 1$ if $K = SO_p$, otherwise $\emptyset$.

Let (r, Λ), l and ϵ be such parameters. If $\Lambda \neq 0$, we define the function $\phi^{r,\Lambda,l,\epsilon}$ by:

$$\phi^{r,\Lambda,l,\epsilon}(n) \ = \ \int_K \Theta^{r,\Lambda,l,\epsilon}(k.n)dk, \quad n \in N_p \quad , \tag{6}$$

where $\Theta^{r,\Lambda,l,\epsilon}$ is given by:

$$\Theta^{r,\Lambda,l,\epsilon}(\exp(X+A)) \ = \ e^{i<rX_p,X>}e^{i<D^\epsilon_2(\Lambda),A>}\prod_{j=1}^{p_1}\bar{\mathcal{L}}_{l_j,m_j-1}(\frac{\mu_j}{2}|\mathrm{pr}_j(X)|^2) \quad . \tag{7}$$

If $\Lambda = 0$, we define the function $\phi^{r,0}$ by:

$$\phi^{r,0}(n) \ = \ \mathcal{J}_{\frac{p-2}{2}}(r|X|) \quad , \quad n = \exp(X+A) \in N_p \quad . \tag{8}$$

In Section 5, we shall prove the following result in the case $K = O_p$ (the case $K = SO_p$ is similar and can be found in [5]):

Theorem 3.1. *The bounded spherical functions of (N_p, K), for $K = SO_p$ or $K = O_p$, are the functions $\phi^{r,\Lambda,l,\epsilon}$ given by (6) and (8), where $(r, \Lambda) \in \mathcal{M}$ and $l \in \mathbb{N}^{p_1}$ if $\Lambda \neq 0$, and $\epsilon = \pm 1$ if $K = SO_p$, $\epsilon = \emptyset$ if $K = O_p$.*

In Section 6, we shall also express $\phi^{r,0}$ and $\phi^{r,\Lambda,l}$ in terms of representations of N_p and obtain their eigenvalues for the sub-Laplacian and the radial Plancherel measure.

4. Spherical Function and Representation

In this section, we recall some of the properties of spherical functions, Gelfand pairs and representations, which will be used in the proof of Theorem 3.1.

In this article, we use the following conventions.
The semi-direct product $K \ltimes N$ of two groups K and N such that K acts on N by automorphism, is defined by the law:

$$(k_1, n_1), (k_2, n_2) \in K \ltimes N, \quad (k_1, n_1).(k_2, n_2) \,=\, (k_1 k_2, n_1\, k_1.n_2) \quad .$$

All the groups are supposed locally compact, second countable and separable, and their continuous unitary representations on separable Hilbert spaces.
For such a group G, we denote by $\hat{G}$ the quotient set of the irreducible representations by the equivalence relation $\sim$. We often identify a representation with its equivalence class.
Definitions and properties. Let K be a compact group, which acts continuously on a group N. Let dn be a Haar measure on N, and dk the normalized Haar measure on K. We assume that $dndk$ is a Haar measure on the group $G = K \ltimes N$, and that this group is unimodular.
Let $C^\natural(N)$ be the set of continuous compactly supported K-invariant functions on N.
A K-invariant function ϕ on N is spherical on N if for all $f, g \in C^\natural(N)$ we have:

$$\int_N f * g(n)\phi(n)dn \,=\, \int_N f(n)\phi(n)dn \int_N g(n)\phi(n)dn \ .$$

If ϕ is a spherical function on N for K, the function Φ on $G = K \ltimes N$ given by $\Phi(k, n) = \phi(n)$ is also called a spherical function on G for K.

Remark 4.1. *Suppose K and N are Lie groups and $G = KG^o$, where G^o is the connected component of the neutral element. Then the spherical functions Φ on G are analytic and they are the common eigenfunction of (G-)left and K-invariant differential operators on G, such that $\Phi(0) = 1$. Equivalently, the spherical function ϕ on N are analytic and they are the common eigenfunction of (N-)left and K-invariant differential operators on N, such that $\phi(0) = 1$* [6, ch.X].

As examples of spherical functions, we shall provide their expressions on Heisenberg groups. These will be used during the proof of Theorem 3.1.
If $C^\natural(N)$ is a commutative algebra for the convolution product, then (N, K) is called a Gelfand pair.
We recall the link between bounded spherical functions and representations, which we will use to construct our bounded spherical functions:

Theorem 4.2. *Let (N, K) be a Gelfand pair.*

a) [6, ch.X], [3, ch.IV,I] *The vector space of K-invariant vectors of an irreducible representation on $G = K \ltimes N$ is of dimension at most one.*

   The spherical functions of positive type (on G) are the positive definite functions Φ (on G) which are associated to an irreducible representation with at least one non zero K-invariant vector.

   For the representation associated to a positive definite function, the vector space of K-invariant vectors is $\mathbb{C}\Phi$.

b) [1, Corollary 8.4] *If N is a nilpotent Lie group, then the bounded spherical functions are the spherical functions of positive type.*

It is known [1, Theorem 5.12] that (N_p, SO_p) and consequently (N_p, O_p) are Gelfand pairs. Thus to obtain the bounded spherical functions of (N_p, O_p), we need to describe classes of representations of $G := O_p \ltimes N_p$. In this section, we shall compute those of N_p by the orbit method (see [10] or [8]). We compute $\hat{G}$ using Mackey's Theorem [9, ch.III B Theorem 2], provided that we describe $\hat{N}_p/G$. To describe $\hat{N}_p$, the classes of representations of N, we shall use the orbit method (see [10] or [8]). For a connected simply connected nilpotent Lie group N, we will denote by T_f the classes of representation of N associated to $f \in \mathcal{N}^*$. First, we set the following conventions for elements of $\mathcal{N}^*$.

Conventions regarding elements of $\boldsymbol{\mathcal{N}^*}$. In this section and in the rest of this paper, we write $N = N_p$, its Lie algebra $\mathcal{N}_p = \mathcal{N}$ and the dual $\mathcal{N}_p^* = \mathcal{N}^*$. We denote by $\mathcal{V}^*$ and $\mathcal{Z}^*$ the dual spaces of $\mathcal{V}$ and $\mathcal{Z}$ respectively, and by $X_1^*, \dots, X_p^*$ the dual basis of $X_1, \dots, X_p$.

Let $A^* \in \mathcal{Z}^*$ be identified with an antisymmetric transformation (by the scalar product on $\mathcal{Z}$). We associate to it the bilinear antisymmetric form ω_{A^*} on $\mathcal{V}$, given by: $\omega_{A^*}(X,Y) =< A^*X, Y >$, $X, Y \in \mathcal{V}$. The radical of ω_{A^*} coincides with $\ker A^*$; and its orthogonal complement in $(\mathcal{V}, <,>)$ is $\Im A^*$, the range of A^*. So on $\Im A^*$, ω_{A^*} induces a simplectic form $\omega_{A^*,r}$ and the dimension of $\Im A^*$ is even and will be denoted by $2p_0$.

Suppose we have fixed E_1, a maximal totally isotropic space for $\omega_{A^*,r}$. Then $E_2 = A^*E_1$ is the orthogonal complement of E_1 in $(\Im A^*, <,>)$ and a maximal totally isotropic space for $\omega_{A^*,r}$. The dimension of E_1 and E_2 is p_0. We denote by $q_0 : \mathcal{V} \to \ker A^*$, $q_1 : \mathcal{V} \to E_1$ and $q_2 : \mathcal{V} \to E_2$ the orthogonal projections.

Description of $\boldsymbol{\hat{N}_p}$. Now we describe $\hat{N}_p$, the classes of representations of N (we will only need some of these classes).

We need to describe first the representatives of $\mathcal{N}^*/N$. The co-adjoint representation is given for $n = \exp(X + A) \in N$ by:

$$X^* \in \mathcal{V}^*,\ A^* \in \mathcal{Z}^* \quad \mathrm{Coad}.n(X^* + A^*) \;=\; X^* + A^* - A^*.X \quad .$$

We can thus choose the privileged representative $X^* + A^*$ ($X^* \in \mathcal{V}$ and $A^* \in \mathcal{Z}$), of each orbit $\mathcal{N}^*/N$, such that $X^* \in \ker A^*$. Let $f = X^* + A^*$ have this form. We define the bilinear antisymmetric form on $\mathcal{N}$ associated to f:

$$\forall\, V, V' \in \mathcal{N} \quad : \quad B_f(V, V') \;=\; f([V, V']) \quad .$$

Because of (1), we have:

$$B_f(X + A, X' + A') \;=\; f([X, X']) \;=< A^*, [X, X'] >=\; w_{A^*}(X, X') \quad .$$

Some easy computations show that a polarization $\mathcal{L}_f$ at f and an associated representation $(\mathcal{H}_{X^*,A^*}, U_{X^*,A^*})$ are given by:

- if $A^* = 0$, then $B_f = 0$, $\mathcal{L}_f = \mathcal{N}$, and U_{X^*,A^*} is the one dimensional representation given by: $\exp(X + A) \mapsto \exp(i < X^*, X >)$.
- if $A^* \neq 0$ (with the previous conventions about $A^* \in \mathcal{Z}^*$), we assume that we have chosen a maximal totally isotropic space E_1 for $\omega_{A^*,r}$, and so $\mathcal{L}_f := E_2 \oplus \ker A^* \oplus \mathcal{Z}$ is a polarization at f. Another choice for E_1 gives another polarization at f, but does not change the class of U_{X^*,A^*}. We compute $\mathcal{H}_{X^*,A^*} = L^2(E_1)$, and for $F \in \mathcal{H}_{X^*,A^*}$, $n = \exp(X + A)$,

$X' \in E_1$:

$$U_{X^*,A^*}(n).F(X') \quad = \quad \exp\left(i < A^*, \frac{1}{2}[q_1(X+2X'), q_2(X)] + A >\right)$$

(9) $$e^{i<X^*,X>} F\left(q_1(X) + X'\right) \quad .$$

Kirillov's Theorem gives:

Proposition 4.3. *For $A^* \in \mathcal{Z}^*$, and $X^* \in \ker A^* \subset \mathcal{V}^*$, we have:*

$$U_{X^*,A^*} \in T_{X^*+A^*} \quad .$$

Furthermore, when A^* and X^* ranges over $\mathcal{Z}^*$ and $\ker A^*$ respectively, U_{X^*,A^*} ranges over a set of representatives of each class of $\hat{N}_p$.

Remark 4.4. *The Lie algebra of $\ker U_{X^*,A^*}$ is:*

$$\left(\ker A^* \cap (X^*)^\perp\right) \oplus (A^*)^\perp \quad ,$$

where $(X^*)^\perp$ is the orthogonal space of X^* in $(\mathcal{V}, <,>)$, and $(A^*)^\perp$ is the orthogonal space of A^* in $(\mathcal{Z}, <,>)$.

Remark 4.5. *The restriction of U_{X^*,A^*} on $\mathcal{Z}$ is given by:*

$$\exp A \mapsto \exp(i < A^*, A >) \quad .$$

Consequences of Kirillov's Theorem. Here we give simple consequences of Kirillov's Theorem, which will permit us to describe $\hat{N}_p/G$ (where $G = K \ltimes N_p$).
In this paragraph, N is a connected simply connected nilpotent Lie group, and G a group which acts continuously by automorphisms on N. We denote by $\mathcal{N}$ the Lie algebra of N, and by $\mathcal{N}^*$ the dual of $\mathcal{N}$. Then G acts on $\hat{N}$:

$$g \in G,\ \rho \in \hat{N} \qquad g.\rho := n \mapsto \rho(g^{-1}.n)\ ,$$

and by automorphisms on the vector space $\mathcal{N}^*$:

$$g \in G,\ f \in \mathcal{N}^* \qquad g.f := n \mapsto f(g^{-1}.n)\ .$$

For $g \in G$, we compute: $g.T_f = T_{g.f}$. We deduce:

Corollary 4.6. *The Kirillov map induces a one-to-one map from $(\mathcal{N}^*/N)/G$ onto $\hat{N}/G$, which maps the G-orbit of $f \in \mathcal{N}^*$ to the G-orbit of T_f.*

Under the previous hypothesis, for $\rho \in \hat{N}$, we denote its G-stability group by:

$$G_\rho \;=\; \{g \in G;\ g.\rho = \rho\} \quad .$$

By Kirillov's orbit method, it is easy to see that:

Proposition 4.7. *Let N be a connected simply connected nilpotent Lie group, and K a group which acts continuously by automorphisms on N. Let $G = K \ltimes N$.
We have $(\mathcal{N}^*/N)/G \sim \mathcal{N}^*/G$.
Furthermore, let $\rho \in \hat{N}$ be fixed. We may assume that $\rho = T_f, f \in \mathcal{N}^*$. Then the G-stability group G_ρ is $K_\rho \ltimes N$, where K_ρ is the K-stability group of ρ, or equivalently of the N-orbit $N.f$ of f:*

$$i.e. \qquad K_\rho := \{k \in K :\ k.\rho = \rho\} \;=\; \{k \in K \subset G :\ k.f \in N.f\} \quad .$$

Bounded Spherical Function on the Heisenberg Group. Here, as example of spherical functions, we provide the expressions of the bounded spherical functions on Heisenberg groups for some compact groups. This will be used during the proof of Theorem 3.1.
We use the following law of the Heisenberg group $\mathbb{H}^{p_0}$:

$$\forall\, h = (z_1, \ldots, z_{p_0}, t)\,,\, h' = (z'_1, \ldots, z'_{p_0}, t') \in \mathbb{H}^{p_0} = \mathbb{C}^{p_0} \times \mathbb{R}$$
$$h.h' = (z_1 + z'_1, \ldots, z_{p_0} + z'_{p_0}, t + t' + \frac{1}{2} \sum_{i=1}^{p_0} \Im z_i \bar{z}'_i) \quad .$$

The unitary $p_0 \times p_0$ matrix group U_{p_0} acts by automorphisms on $\mathbb{H}^{p_0}$. Let us describe some subgroups of U_{p_0}. Let $p_0, p_1 \in \mathbb{N}$, and $m = (m_1, \ldots, m_{p_1}) \in \mathbb{N}^{p_1}$ be fixed such that $\sum_{j=1}^{p_1} m_j = p_0$. We define m'_j for $j = 1, \ldots p_1$ by (5). Let $K(m; p_1; p_0)$ be the subgroup of U_{p_0} given by:

$$K(m; p_1; p_0) = U_{m_1} \times \ldots \times U_{m_{p_1}} \quad . \tag{10}$$

The expressions of spherical functions of $(\mathbb{H}^{p_0}, K(m; p_1; p_0))$ can be found in the same way as in the case $m = (p_0), p_1 = 1$ i.e. $K = U_{p_0}$ [4, ch.V,II.6] using Remark 4.1; here, we admit [5]:

Proposition 4.8. $(\mathbb{H}^{p_0}, K(m; p_1; p_0))$ *is a Gelfand pair.*
Its bounded spherical functions on $\mathbb{H}^{p_0}$ *are:*

(1) $\omega = \omega_{\lambda,l}$ *with* $\lambda \in \mathbb{R}^*$ *and* $l = (l_1, \ldots, l_{p_1}) \in \mathbb{N}^{p_1}$ *:*

$$\omega(z_1, \ldots, z_{p_0}, t) \;=\; e^{i\lambda t} \prod_{j=1}^{p_1} \bar{\mathcal{L}}_{l_j, m_j - 1}\Big(\frac{|\lambda|}{2} \sum_{m'_{j-1} < i \leq m'_j} |z_i|^2\Big) \quad ,$$

(2) $\omega = \omega_\mu$ *with* $\mu = (\mu_1, \ldots, \mu_{p_1})$ *and* $\mu_i > 0$ *:*

$$\omega(z, t) \;=\; \prod_{j=1}^{p_1} \mathcal{J}_{m_j - 1}\Big(\mu_j \sqrt{\sum_{m'_{j-1} < i \leq m'_j} |z_i|^2}\Big) \quad .$$

During the proof of Theorem 3.1, we will use the following notations. To a spherical function ω for the Gelfand pair $(\mathbb{H}^{p_0}, K(m; p_0; p_1))$, we associate the corresponding spherical function Ω^ω on $H_{heis} = K(m; p_0; p_1) \ltimes \mathbb{H}^{p_0}$, and the irreducible representation $(\mathcal{H}_\omega, \Pi_\omega)$ on H_{heis} associated with Ω^ω. We compute easily:

$$\text{if}\, \omega = \omega_{\lambda,l} \qquad \Pi_\omega(0, t) \;=\; \exp(i\lambda t) \tag{11}$$
$$\text{if}\, \omega = \omega_\mu \qquad \Pi_\omega(0, t) \;=\; 1 \quad . \tag{12}$$

5. Expression of the Bounded Spherical Functions

This section is devoted to the proof of Theorem 3.1 for $K = O_p$. Let G be the group $K \ltimes N$, where $N = N_p$ and $K = O_p$. We fix the Haar measure $dkdn$ on G.
Overview of the proof. For $\rho \in \hat{N}$, we denote by:

- G_ρ the G-stability group of ρ,
- $\check{G}_\rho$ the set of $\nu \in \hat{G}_\rho$ such that $\nu_{|N}$ is a multiple of ρ,
- $\tilde{G}_\rho$ the set of $\nu \in \check{G}_\rho$ such that the dimension of the space of K_ρ-invariant vectors is one.

By Mackey's Theorem [9, ch.III B Theorem 2], when ρ and ν range over a representative of each class of $\hat{N}$ and $\check{G}_\rho$ respectively, the representation induced by ν on G gives a representative of each class of $\hat{G}$.
Because of the subgroup and intertwining number Theorems [9, ch.II A, Theorem 1 and Lemma 5 respectively], we easily get for $\nu \in \hat{G}_\rho$:

$$
\begin{aligned}
\nu \in \tilde{G}_\rho \quad &\Longleftrightarrow \quad \nu_{|K_\rho} \text{ contains exactly one times } 1_{K_\rho} \\
&\Longleftrightarrow \quad \text{the space of } K\text{-invariant vectors of } \mathrm{Ind}_{G_\rho}^G \nu \text{ is a line.}
\end{aligned} \tag{13}
$$

The proof of Theorem 3.1 is based on the two theorems and proposition which follow. We will explain after their statements how we deduce from them the expression of all bounded spherical functions.
First, we express the bounded spherical functions in terms of representations $\rho \in \hat{N}/G$ and $\nu \in \tilde{G}_\rho$:

Theorem 5.1. *Let $\rho \in \hat{N}$ and $(\mathcal{H}^\nu, \nu) \in \tilde{G}_\rho$. Then because of (13), $\mathrm{Ind}_{G_\rho}^G \nu \in \hat{G}$ has also a (non-zero) K-invariant line and the associated bounded spherical function is the function ϕ^ν given by :*

$$
\phi^\nu(n) = \int_K \left\langle \nu(I, k.n).\vec{u}_\nu \, , \, \vec{u}_\nu \right\rangle_{\mathcal{H}^\nu} dk, \quad n \in N \quad , \tag{14}
$$

where $\vec{u}_\nu \in \mathcal{H}^\nu$ is any unit K_ρ-invariant vector.
Furthermore, we obtain all the bounded spherical functions as ϕ^ν when ρ and ν range over a set of representatives of $\hat{N}/G$, and $\tilde{G}_\rho$ respectively.

Next, to obtain all representations $\rho \in \hat{N}/G$, we describe $\mathcal{N}^*/G$ (see Corollary 4.6):

Proposition 5.2. *Let $O(r, \Lambda) = G.(rX_p^* + D_2(\Lambda)) \subset \mathcal{N}^*$.*
Then the mapping

$$
\begin{array}{ccc}
\mathcal{M} & \to & \mathcal{N}^*/G \\
(r, \Lambda) & \mapsto & O(r, \Lambda)
\end{array}
$$

is a bijection.

Now, we describe $\tilde{G}_\rho$, where ρ is a representation associated (by Kirillov) to a linear form on $\mathcal{N}$, which is a privileged representative of a G-co-adjoint orbit (just given in Proposition 5.2):

Theorem 5.3. *Let $\rho \in T_f$ where $f = rX_p^* + D_2(\Lambda)$ and $(r, \Lambda) \in \mathcal{M}$.*

a) *If $\Lambda = 0$, $\tilde{G}_\rho = \{\nu^{r,0}\}$. The spherical function ϕ^ν which is associated (by (14)) to $\nu = \nu^{r,0}$ is $\phi^{r,0}$ (given by (8)).*
b) *If $\Lambda \neq 0$, $\tilde{G}_\rho \subset \{\nu^{r,\Lambda,l} \, , \, l \in \mathbb{N}^{p_1}\}$. Each representation $\nu = \nu^{r,\Lambda,l} \in \hat{G}_\rho$ has a K_ρ-invariant line, and the spherical function ϕ^ν associated (by (14)) is $\phi^{r,\Lambda,l}$ (given by (6)).*

The representations $\nu^{r,0}$ and $\nu^{r,\Lambda,l}$ will be described during the proof (see (17) and (18)).
For the moment, we will admit these two theorems and the proposition, and keep their notations. From Corollary 4.6 and Proposition 5.2, we deduce that:

$$
\hat{N}/G = \{T_{rX_p^* + D_2(\Lambda)} \, , \, (r, \Lambda) \in \mathcal{M}\} \quad .
$$

Under Theorems 5.1 and 5.3, the spherical bounded functions are the functions $\phi^{r,0}$, when $r \in \mathbb{R}^+$, and $\phi^{r,\Lambda,l}$ when $(r,\Lambda) \in \mathcal{M}$ and $l \in \mathbb{N}^{p_1}$.
If we prove Theorems 5.1 and 5.3, and Proposition 5.2, Theorem 3.1 will follow. The rest of this section will be devoted to this. We start with the proofs of Theorem 5.1 and Proposition 5.2. Then for a representation $\rho \in T_{rX_p^*+D_2(\Lambda)}$, we describe its G-stability group and the quotient group $\overline{N} = N/\ker\rho$. We finish with the proof of Theorem 5.3.
Set $\tilde{\boldsymbol{G}}_{\boldsymbol{\rho}}$. The aim of this paragraph is to prove Theorem 5.1.
Let $\rho \in \hat{N}$ be fixed. Under Proposition 4.7, the G-stability group of ρ is $G_\rho = K_\rho \ltimes N$, where K_ρ is the K-stability group (which is a compact subgroup of K). We fix the normalized Haar measure dk_ρ on K_ρ, and the Haar measure $dk_\rho dn$ on G_ρ.
We fix $(\mathcal{H}^\nu, \nu) \in \tilde{G}_\rho$ and a unit K_ρ-invariant vector $\vec{u} = \vec{u}_\nu \in \mathcal{H}^\nu$. We denote by $(\mathcal{H}^\Pi, \Pi)$ the induced representation $\mathrm{Ind}_{G_\rho}^G \nu$ of ν:

$$\forall\, g, g' \in G \quad f \in \mathcal{H}^\Pi \quad : \quad \Pi(g).f(g') \,=\, f(g'g) \quad ,$$

and by f the function on G given by :

$$f(k,n) \,=\, \nu(I,n).\vec{u}\,, \quad (k,n) \in G \quad ;$$

the vector $f \in \mathcal{H}^\Pi$ is K-invariant and of norm one. We can then associate to Π and f the bounded spherical function ϕ^ν:

$$\phi^\nu(g) \,=\, \left\langle \Pi(g).f, f \right\rangle_{\mathcal{H}^\Pi}\,, \quad g \in G \quad .$$

We can easily obtain for $g = (k,n), g' = (k',n') \in G$:

$$\begin{aligned} \Pi(g).f(g') &= \nu(I,n')\nu(I,k'.n).\vec{u} \quad , \\ \left\langle \Pi(g).f(g'), f(g') \right\rangle_{\mathcal{H}^\nu} &= \left\langle \nu(I,k'.n).\vec{u}\,,\, \vec{u} \right\rangle_{\mathcal{H}^\nu} \quad . \end{aligned}$$

We thus obtain the formula (14).
We can now complete our proof of Theorem 5.1. Under Mackey's Theorem and property (13), when ρ and ν range over a set of representatives of $\hat{N}/G$ and $\tilde{G}_\rho$ respectively, we get all the irreducible representations $\Pi = \mathrm{Ind}_{G_\rho}^G \nu$ having a K-invariant line. Under Theorem 4.2, the positive definite functions ϕ^ν associated to Π give all the bounded spherical functions.
Description of $\boldsymbol{\mathcal{N}^*/G}$. Here we prove Proposition 5.2. We easily compute for $g = (k,n) \in G$ with $n = \exp(X+A) \in N$ and $X^* \in \mathcal{V}^*, A^* \in \mathcal{Z}^*$:

$$\text{Coad}.g(X^* + A^*) \,=\, k.X^* + k.A^* - (k.A^*).X \quad . \tag{15}$$

Let $O \in \mathcal{N}^*/G$ be a fixed orbit. We associate to it $\Lambda \in \overline{\mathcal{L}}$ such that all the antisymmetric matrices A_f^*, where $f = X_f^* + A_f^* \in O$, are K-conjugate, and K-conjugated to $D_2(\Lambda)$.
Let $f = X_f^* + A_f^* \in O$ be fixed. We make the following choices:

(1) let $k_0 \in K$ be such that $k_0.A_f^* = D_2(\Lambda)$;
(2) let $X_0 \in \mathcal{V}$ be such that $(k_0.A_f^*).X_0 \in \mathcal{V}^*$ is the orthogonal projection X_0^* of $k_0.X_f^* \in \mathcal{V}^*$ on the kernel $\ker k_0.A_f^* = \ker D_2(\Lambda)$; in particular, $X_0^* = 0$ if $\Im D_2(\Lambda) = \mathcal{V}$;
(3) let $k_0' \in K$ be such that $k_0'.X \in \Im D_2(\Lambda)$ for all $X \in \Im D_2(\Lambda)$ and $k_0' X_0^* = rX_p^*$, $r \in \mathbb{R}^+$.

We get $(k_0'k_0, \exp X_0).f = rX_p^* + D_2(\Lambda)$.
We remark that $\Im D_2(\Lambda) = \mathcal{V}$ is equivalent to $p = 2p_0$ and $\lambda_i \neq 0$, $i = 1, \ldots, p'$.
Proposition 5.2 is thus proved.
Stability Group $\boldsymbol{K_\rho}$. The aim of this paragraph is to describe the stability group K_ρ of $\rho \in T_{rX_p^*+D_2(\Lambda)}$.
Before this, let us recall that the orthogonal $2n \times 2n$ matrices which commutes with $D_2(1, \ldots, 1)$ (see (2) for this notation) have determinant one and form the group $Sp_n \cap O_n$. This group is isomorphic to U_n; the isomorphism is denoted $\psi_1^{(n)}$, and satisfies:

$$\forall\, k, X \quad : \quad \psi_c^{(n)}(k.X) \;=\; \psi_1^{(n)}(k).\psi_c^{(n)}(X) \quad ,$$

where $\psi_c^{(n)}$ is the complexification:

$$\psi_c^{(n)}(x_1, y_1; \ldots; x_n, y_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \quad .$$

Now, we can describe K_ρ:

Proposition 5.4. *Let $(r, \Lambda) \in \mathcal{M}$. Let p_0 be the number of $\lambda_i \neq 0$, where $\Lambda = (\lambda_1, \ldots, \lambda_{p'})$, and p_1 the number of distinct $\lambda_i \neq 0$. We set $\tilde{\Lambda} = (\lambda_1, \ldots, \lambda_{p_0}) \in \mathbb{R}^{p_0}$.*
Let $\rho \in T_f$ where $f = rX_p^* + D_2(\Lambda)$.

- *If $\Lambda = 0$, then K_ρ is the subgroup of K such that $k.rX_p^* = rX_p^*$ for all $k \in K_\rho$.*
- *If $\Lambda \neq 0$, then K_ρ is the direct product $K_1 \times K_2$, where:*

$$K_1 \;=\; \left\{ k_1 \;=\; \begin{bmatrix} \tilde{k}_1 & 0 \\ 0 & \mathrm{Id} \end{bmatrix} \;\Big/\; \begin{array}{l} \tilde{k}_1 \in SO(2p_0) \\ D_2(\tilde{\Lambda})\tilde{k}_1 = \tilde{k}_1 D_2(\tilde{\Lambda}) \end{array} \right\} \quad ,$$

$$K_2 \;=\; \left\{ k_2 \;=\; \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \tilde{k}_2 \end{bmatrix} \in K \;\Big/\; \tilde{k}_2.rX_p^* = rX_p^* \right\} \quad .$$

Furthermore, K_1 is isomorphic to the group $K(m; p_0; p_1)$ given by (10).

Proof. We keep the notations of this proposition, and we set $A^* = D_2(\Lambda)$ and $X^* = rX_p^*$. With Propositions 4.7 and the expression (15) of the co-adjoint representation, it is easy to prove:

$$K_\rho \;=\; \{k \in K \;:\; kA^* = A^*k \quad \text{and} \quad kX^* = X^*\} \quad . \tag{16}$$

If $\Lambda = 0$, because of (16), K_ρ is the stability group in K of $X^* \in \mathcal{V}^* \sim \mathbb{R}^p$. So the first part of Proposition 5.4 is proved.

Let us show the second part. $\Lambda \neq 0$ so we have

$$A^* \;=\; \left[\begin{array}{c|c} D_2(\tilde{\Lambda}) & 0 \\ \hline 0 & 0 \end{array}\right] \quad \text{with} \quad D_2(\tilde{\Lambda}) \;=\; \begin{bmatrix} \mu_1 J_{m_1} & 0 & 0 \\ & \ddots & \\ 0 & 0 & \mu_{p_1} J_{m_{p_1}} \end{bmatrix} \quad .$$

where $\mu_1, \ldots, \mu_{p_1}$ are defined by (4), and m_j is the number of $\lambda_i = \mu_j$. We define m_j' for $j = 1, \ldots p_1$ by (5).
Let $k \in K_\rho$. Because of (16), the matrices k and A^* commute and we have:

$$k \;=\; \begin{bmatrix} \tilde{k}_1 & 0 \\ 0 & \tilde{k}_2 \end{bmatrix} \quad \text{with } \tilde{k}_1 \in O(2p_0) \text{ and } \tilde{k}_2 \in O(p - 2p_0) \quad ;$$

furthermore, by (16), $\tilde{k}_2.X^* = X^*$, and the matrices $\tilde{k}_1$ and $D_2(\tilde{\Lambda}^*)$ commute. So $\tilde{k}_1$ is a diagonal block matrix, with blocks $[\tilde{k}_1]_j \in O(m_j)$ for $i = 1, \ldots, p_1$. Each block $[\tilde{k}_1]_j \in O(m_j)$ commutes with J_{m_j}. So on one hand we have $\det [\tilde{k}_1]_j = 1$, $\det \tilde{k}_1 = 1$, and on the other hand, $[\tilde{k}_1]_j \in O(m_j)$ corresponds to a unitary matrix $\psi_1^{(m_j)}([\tilde{k}_1]_j)$. Now, we set for $k_1 \in K_1$:

$$\Psi_1(k_1) \;=\; \left(\psi_1^{(m_1)}([\tilde{k}_1]_1), \ldots, \psi_1^{(m_{p_1}}([\tilde{k}_1]_{p_1})\right) \quad .$$

$\Psi_1 : K_1 \to K(m; p_0; p_1)$ is a group isomorphism. □

As $\psi_1^{(n)}$ is an isomorphism which respects complexification, we have:

Corollary 5.5. *The isomorphism Ψ_1 given during the previous proof respects complexification:*

$$\psi_c^{(p_0)}\{\tilde{k}_1.(x_1, y_1, \ldots, x_{p_0}, y_{p_0})\} \;=\; \Psi_1(k_1).\psi_c^{(p_0)}(x_1, y_1, \ldots, x_{p_0}, y_{p_0}) \quad .$$

Quotient Group $\overline{N} = N/\ker\rho$. In this paragraph, we describe the quotient groups $N/\ker\rho$ and $G/\ker\rho$, for some $\rho \in \hat{N}$. This will permit in the next paragraph to reduce the construction of the bounded spherical functions on N_p to known questions on Euclidean and Heisenberg groups. For a representation $\rho \in \hat{N}$, we will denote by:

- $\ker\rho$ the kernel of ρ,
- $\overline{N} = N/\ker\rho$ its quotient group and $\overline{\mathcal{N}}$ its Lie algebra,
- $(\mathcal{H}, \overline{\rho})$ the induced representation on $\overline{N}$,
- $\overline{n} \in \overline{N}$ and $\overline{Y} \in \overline{\mathcal{N}}$ the image of $n \in N$ and $Y \in \mathcal{N}$ respectively by the canonical projections $N \to \overline{N}$ and $\mathcal{N} \to \overline{\mathcal{N}}$.

Now, with the help of the canonical basis, we choose the privileged representative ρ of $T_{rX_p^* + D_2(\Lambda)}$ as $\rho = U_{rX_p^*, D_2(\Lambda)}$ given by (9), with $E_1 = \mathbb{R}X_1 \oplus \ldots \oplus \mathbb{R}X_{2p_0-1}$ as maximal totally isotropic space for $\omega_{D_2(\Lambda), r}$. Because of Remark 4.4, the quotient Lie algebra $\overline{\mathcal{N}}$ has the natural basis:

$$\overline{X_1}, \ldots, \overline{X_{2p_0}}, \overline{B} = |\Lambda|^{-1}\overline{D_2(\Lambda)} \quad \text{with } X_p \text{ if } r \neq 0 \quad ;$$

here, we have denoted $|\Lambda| = \left(\sum_{j=1}^{p'} {\lambda_j}^2\right)^{\frac{1}{2}} = |D_2(\Lambda)|$ (for the Euclidean norm on $\mathcal{Z}$). We compute that each Lie bracket of two vectors of this basis equals zero, except:

$$[\overline{X_{2i-1}}, \overline{X_{2i}}] \;=\; \frac{\lambda_i}{|\Lambda|}\overline{B}, \quad i = 1, \ldots, p_0 \quad .$$

Let $\overline{\mathcal{N}_1}$ be the Lie sub-algebra of $\overline{\mathcal{N}}$, with basis $\overline{X_1}, \ldots, \overline{X_{2p_0}}, \overline{B}$, and let $\overline{N_1}$ be its corresponding connected simply connected nilpotent Lie group. We define the mapping $\Psi_2 : \mathbb{H}^{p_0} \to \overline{N_1}$ for $h = (x_1 + iy_1, \ldots, x_{p_0} + iy_{p_0}, t) \in \mathbb{H}^{p_0}$ by:

$$\Psi_2(h) \;=\; \exp\left(\sum_{j=1}^{p_0} \sqrt{\frac{|\Lambda|}{\lambda_j}}\left(x_j\overline{X_{2j-1}} + y_i\overline{X_{2j}}\right) \;+ t\overline{B}\right) \quad .$$

Because of the values of the Lie brackets in $\overline{N_1}$, it is easy to see:

Lemma 5.6. *Ψ_2 is a group isomorphism between N_1 and $\mathbb{H}^{p_0}$.*

With our choice for representations and notations, we describe the induced representation and action:

Proposition 5.7. *With the notations set just above,*

a) *If $\Lambda \neq 0$, the two groups $\overline{N}$ and $\overline{N_1} \times \overline{N_2}$ are isomorphic and the representations $\overline{\rho}$ and $\overline{\rho_1} \otimes \overline{\rho_2}$ are equivalent, where*
   (1) *$\overline{\rho}_1$ is a representation on $\overline{N_1}$ (whose expression may be computed);*
   (2) *$\overline{N_2}$ and $\overline{\rho}_2$ are described by:*
      – *either $r = 0$, then $\overline{N_2}$ and $\overline{\rho}_2$ are trivial;*
      – *or $r \neq 0$, then $\overline{N_2} \sim \mathbb{R}\overline{X}_p$, and $\overline{\rho}_2 \sim \exp(x\overline{X}_p) \to \exp(ix)$.*

b) *If $\Lambda = 0$, then $\overline{N}$ and $\overline{\rho}$ are the same as $\overline{N_2}$ and $\overline{\rho}_2$ above.*

Remark 5.8. *Because of Remark 4.5, the restriction of $\overline{\rho}_1$ on the center $\exp \mathbb{R}\overline{B}$ of $\overline{N_1}$ is given by: $\exp(a\overline{B}) \mapsto \exp(ia|\Lambda|)$.*

As K_ρ is the K-stability group of $\rho \in \hat{N}$, it acts by automorphisms on $\overline{N}$. Simple computations show that:

Proposition 5.9. *We keep the notations of Propositions 5.4 and 5.7.*

a) *If $\Lambda = 0$, K_ρ acts trivially on $\overline{N}$.*

b) *If $\Lambda \neq 0$,*
   – *K_1 acts by automorphisms on $\overline{N_1}$ and trivially on the center $\exp \mathbb{R}\overline{B}$ of $\overline{N_1}$,*
   – *K_2 acts trivially on $\overline{N_2}$.*

   So the groups $K_\rho \ltimes \overline{N}$ and $(K_1 \ltimes \overline{N_1}) \times K_2 \times \overline{N_2}$ are isomorphic.

Recall $H_{heis} = K(m; p_0; p_1) \ltimes \mathbb{H}^{p_0}$; let us also define the group $H = K_1 \ltimes \overline{N_1}$ and the map:

$$\Psi_0 \ : \ \begin{cases} \quad H_{heis} & \longrightarrow \quad H \\ (k_1\, ,\, h) & \longmapsto \quad (\Psi_1(k_1)\, ,\, \Psi_2(h)) \end{cases} \quad .$$

Corollary 5.5, Proposition 5.9, and Lemma 5.6 imply:

Proposition 5.10. *Ψ_0 is a group isomorphism between H_{heis} and H.*

Expression of $\boldsymbol{\phi^\nu}$. Here we prove Theorem 5.3. Let $\rho \in \hat{N}$ and $\nu \in \tilde{G}_\rho$ be fixed. We have $\nu_{|N} = c.\rho$, $1 \leq c \leq \infty$, and we denote by $\bar{\nu}$ the induced representation on $K_\rho \ltimes \overline{N}$.

a) Case of the orbit $\boldsymbol{O(r,0)}$. We assumed that $\rho = \rho_{r,0}$. By Proposition 5.9, we have $K_\rho \ltimes \overline{N} = K_\rho \times \overline{N}$. So $\overline{\nu}$ is the tensor product of an irreducible representation over $\overline{N}$, which coincides with $c.\overline{\rho}$ (and so $c = 1$), with an irreducible representation over K_ρ with a K_ρ-invariant vector, which is thus the trivial representation. We obtain that $\overline{\nu}$ coincides with $(k,n) \in K_\rho \times \overline{N} \mapsto \overline{\rho}(n)$. Now because of Proposition 5.7.b), $\nu = \nu^{r,0}$ is given by:

$$(k, \exp(X + A)) \longmapsto e^{i<rX_p^*, X>} \quad . \tag{17}$$

So $\tilde{G}_\rho$ is the set of the classes of $\nu^{r,0}$, where r ranges over $\mathbb{R}^+$, and we compute that the function ϕ^ν for $\nu = \nu^{r,0}$ is given by (14):

$$\phi^\nu(n) \ = \int_{k \in K} e^{i<rX_p^*, k.X>} dk, \quad n = \exp(X + A) \in N_p \quad .$$

By (3), we have $\phi^\nu = \phi^{r,0}$, and Theorem 5.3.a) is proved.

b) Case of the orbit $\boldsymbol{O(r,\Lambda)}$. We assume $\Lambda \neq 0$ and $\rho = \rho_{r,\Lambda}$. For each bounded spherical function ω of the Gelfand pair $(\mathbb{H}^{p_0}, K(m;p_0;p_1))$, we define the representation $(\mathcal{H}^\omega, \Pi^\omega)$ of H such that :

$$\mathcal{H}^\omega = \{F \circ \Psi_0^{-1}\ ,\quad F \in \mathcal{H}_\omega\} \quad \text{and} \quad \Pi^\omega = \Pi_\omega \circ \Psi_0^{-1} \quad .$$

By Proposition 5.9.b), $\overline{\nu}$ is the tensor product of three irreducible representations, of $\overline{N_2}$, H and K_2, such that the vector space of K_1 (respectively K_2)-invariant vectors of the representations of H (respectively K_2) is a line. So the representation of K_2 is trivial, and $\overline{\nu}$ induces a unitary irreducible representation $\overline{\overline{\nu}}$ of $H \times \overline{N_2}$ which coincides with $c.\overline{\rho}$ on $\overline{N}$, and such that the vector space of K_1-invariant vectors is a line.

We have $\overline{\overline{\nu}} = \gamma_1 \otimes \gamma_2$ where

(a) γ_1 is an irreducible representation of H; the space of its K_1-invariant vectors is a line; it coincides with $c.\overline{\rho}$ over $\overline{N_1}$;

(b) γ_2 is an irreducible representation over $\overline{N_2}$ and coincides with $c.\overline{\rho}$ on $\overline{N_2}$.

Because of irreducibility, (a) implies $c = 1$ and so (b) implies $\gamma_2 \sim \rho_2$. Furthermore, by Proposition 5.9 and Theorem 4.2, the irreducible representations on H such that the vector space of K_1-invariant vectors is a line, are all the representations $(\mathcal{H}_\omega, \Pi_\omega)$, where ω ranges over the set of bounded spherical functions of H_{heis}; the K_1-invariant line of $\mathcal{H}^\omega$ is $\mathbb{C}\Omega^\omega \circ \Psi_0^{-1}$. Thus, the representations γ_1 satisfying (a) are the representations such that $\gamma_1 \sim \Pi^\omega$ and $\Pi^\omega_{|\overline{N_1}} \sim \overline{\rho}_1$. Because of the expressions of γ_1 and Π^ω on the center of $\overline{N_1}$ and H (see Remark 5.8 and equalities (11), (12)) the case $\omega = \omega_\mu$ is impossible if $\Pi^\omega_{|\overline{N_1}} \sim \overline{\rho}_1$.

We have shown that $\overline{\overline{\nu}} = \gamma_1 \otimes \gamma_2$, where $\gamma_2 \sim \rho_2$ is given in Proposition 5.7, and γ_1 is among the representations equivalent to $(\mathcal{H}_\omega, \Pi_\omega)$ with $\omega = \omega_{\lambda,l}$, $l \in \mathbb{N}^{p_1}$.

Then ν is among the representations equivalent to $(\mathcal{H}^\omega, \nu^{r,\Lambda,l})$ with $\omega = \omega_{|\Lambda|,l}$, defined for $n = \exp(X + A) \in N$, and $k = k_1 k_2 \in K_\rho$ where $k_1 \in K_1$, and $k_2 \in K_2$ by:

$$\nu^{r,\Lambda,l}(k,n) \;=\; e^{ir<X_p^*,X>}\, \Pi^\omega(k_1, \bar{q_1}(n)) \quad , \tag{18}$$

where $\bar{q_1} : N \to \overline{N_1}$ is the canonical projection.

We denote by $\tilde{G}'_\rho$ the set of classes of the representations $\nu^{r,\Lambda,l}$, $l \in \mathbb{N}^{p_1}$. A representation $\nu^{r,\Lambda,l}$, has a unitary K_ρ-invariant vector $\vec{u} = \Omega^\omega \circ \Psi_0^{-1}$.

We still have to show that under formula (14), the function ϕ^ν for $\nu = \nu^{r,\Lambda,l}$ satisfies $\phi^\nu = \phi^{r,\Lambda,l}$. For $n = \exp(X + A)$, it is given by:

$$\phi^\nu(n) \;=\; \int_K e^{ir<X_p^*,k.X>} \omega \circ \Psi_2^{-1} \circ \bar{q_1}(k.n) dk \quad , \tag{19}$$

where $\omega = \omega_{|\Lambda|,l}$. We compute

$$\begin{aligned} \nu(I,n)\vec{u} &= e^{ir<X_p^*,X>}\, \Pi^\omega(I, \bar{q_1}(n)) \Omega^\omega \circ \Psi_0 \quad , \\ \langle \nu(I,n).\vec{u}\,,\, \vec{u} \rangle_{\mathcal{H}^\nu} &= e^{ir<X_p^*,X>}\, \langle \Pi_\omega(I, \Psi_2^{-1} \circ \bar{q_1}(n)) \Omega^\omega,\, \Omega^\omega \rangle_{H_\omega} \quad . \end{aligned}$$

As Ω^ω is the positive definite function associated to Π_ω, we have:

$$\begin{aligned} \langle \Pi_\omega(I, \Psi_2^{-1} \circ \bar{q_1}(n)) \Omega^\omega,\, \Omega^\omega \rangle_{H_\omega} &= \Omega^\omega \left(I, \Psi_2^{-1} \circ \bar{q_1}(n)\right) \\ &= \omega \circ \Psi_2^{-1} \circ \bar{q_1}(n) \quad . \end{aligned}$$

We know the expression $\omega = \omega_{|\Lambda|,l}$ (Proposition 4.8), and we compute those of Ψ_2^{-1} and $\bar{q_1}$. Assuming (4), we obtain:

$$\Theta^{r,\Lambda,l}(\exp(X+A)) = e^{ir<X_p^*,X>}\omega \circ \Psi_2^{-1} \circ \bar{q_1}(\exp(X+A)) \quad ,$$

where $\Theta^{r,\Lambda,l}$ is given by (7). Because of (19), $\phi^\nu = \phi^{r,\Lambda,l}$, and Theorem 5.3.b) is consequently proved.

The proof of Theorem 5.3 is now over. Theorem 3.1 is thus proved.

6. Representation over N_p

We give here the bounded spherical functions in terms of representations over N_p [1, Theorem G], which is an equivalent way for constructing them.
We deduce then the eigenvalues of the sub-Laplacian for the bounded spherical functions and the expression of the radial Plancherel measure.
Another expression of the bounded spherical functions $\boldsymbol{\phi^{r,\Lambda,l}}$. Let us recall a few facts contained in [1]. Let (N,K) be a Gelfand pair. Let $(\mathcal{H},\Pi) \in \hat{N}$, and K_Π its K-stability group. There exists a projective representation W_Π of K_Π on $\mathcal{H}$, and an orthogonal decomposition of $\mathcal{H} = \sum V_l$ into irreducible subspaces W_Π-invariant. For $\zeta \in \mathcal{H}$, let us define the function $\phi_{\Pi,\zeta}$ by:

$$\phi_{\Pi,\zeta}(n) = \int_K < \Pi(k.n)\zeta, \zeta > dk \,, \quad n \in N \quad .$$

The spherical functions are the $\phi_{\Pi,\zeta}$ for $\zeta \in V_l$, $|\zeta| = 1$ and $\Pi \in \hat{N}$; the spherical function $\phi_{\Pi,\zeta}$ is independent of $\zeta \in V_l$, $|\zeta| = 1$ and of the choice of the representative of Π.

Remark 6.1. *Furthermore, for a K-invariant function f on N and a K left-invariant differential operator D on N, $\Pi(f)$ and $d\Pi(D)$ are W_Π-invariant; they equal the identity on each V_l up to a constant; and for V_l, and the corresponding spherical function ϕ, the constants are respectively $< f, \phi >$ and the eigenvalue of D for ϕ.*

Now, we give the link between the given spherical functions and the above description for each class of $\hat{N}_p$, which corresponds to a class $O(X^*,A^*)$.
Before this, we recall definitions and properties of Hermite functions. We denote by H_k the Hermite polynomial:

$$H_k(s) = (-1)^k e^{s^2}(d/ds)^k e^{-s^2} \quad ,$$

by $h_k, k \in \mathbb{N}$ the Hermite functions on $\mathbb{R}$ [12, §5.5]:

$$h_k(x) = (2^k k! \sqrt{\pi})^{-\frac{k}{2}} e^{-\frac{x^2}{2}} H_k(x) \quad ,$$

and by $h_\alpha, \alpha \in \mathbb{N}^n$ the Hermite functions on $\mathbb{R}^n$: $h_\alpha = \Pi_{i=1}^n h_{\alpha_i}$. We recall that the Hermite functions $h_k, k \in \mathbb{N}$ on $\mathbb{R}$ form an orthonormal basis $L^2(\mathbb{R})$.
Case $A^* = 0$. $\rho = \rho^{r^*,0}$ is the one-dimensional representation given by

$$\exp(X+A) \mapsto \exp(i < rX_p^*, X >) \quad .$$

Case $A^* \neq 0$. Let $(r,\Lambda) \in \mathcal{M}$ with $\Lambda \neq 0$ et $l \in \mathbb{N}^{p_1}$. Let E_l be the set of $\alpha = (\alpha^1, \dots, \alpha^{p_1})$ where $\alpha^j = (\alpha_i^j)_{m'_{j-1} < i \leq m'_j} \in \mathbb{N}^{m_j}$ such that

$$|\alpha^j| = \sum_{m'_{j-1} < i \leq m'_j} \alpha_i^j = l_j \quad \text{for } j = 1, \dots, p_1 \quad .$$

Let us define the representation $(\mathcal{H}, \Pi) = (L^2(\mathbb{R}^{p_0}), \Pi_{r,\Lambda}) \in \hat{N}_p$ for $f \in \mathcal{H}$, $(y_1, \dots, y_{p_0}) \in \mathbb{R}^{p_0}$, $n = \exp(X + A) \in N_p$ by:

$$\begin{aligned} \Pi(n).f(y) \;=\; & e^{i<D_2(\Lambda),A>+i<rX_p^*,X>+i\sum_{j=1}^{p'} \frac{\lambda_j}{2} x_{2j}x_{2j-1} + \sqrt{\lambda_j} x_{2j} y_j} \\ & f(y_1 + \sqrt{\lambda_1} x_1, \dots, y_{p_0} + \sqrt{\lambda_{p_0}} x_{2p_0 - 1}) \quad , \end{aligned}$$

where $X = \sum_{j=1}^{p} x_j X_j$.
It is easy to see that the representation Π is equivalent to $\rho_{r,\Lambda}$. So $\Pi \in T_{rX^* + D_2(\Lambda)}$, Π is irreducible, and its K-stability group denoted K_Π equals K_ρ.
Let $\zeta_\alpha \in \mathcal{H}, \alpha \in E_l$ be given by:

$$\zeta_\alpha : \left\{ \begin{array}{rcl} \mathbb{R}^{p_0} & \longrightarrow & \mathbb{R} \\ y_1, \dots, y_{p_0} & \longmapsto & \prod_{j=1}^{p_1} h_{\alpha^j}(y_{m'_{j-1}+1}, \dots, y_{m'_j}) \end{array} \right. \quad ,$$

The vectors $\zeta_\alpha, \alpha \in E_l, l \in \mathbb{N}^{p_1}$ form an orthonormal basis of $\mathcal{H}$.
It can be proved that each vector space V_l generated by $\zeta_\alpha, \alpha \in E_l$ is K_Π-invariant.
Using relation between Laguerre and Hermite functions, we obtain:

Lemma 6.2. *The spherical function associated to $\Pi_{r,\Lambda}$ and V_l is $\phi^{r,\Lambda,l}$.*

Consequences for Sub-Laplacian. The (Kohn) sub-Laplacian is

$$L := -\sum_{i=1}^{p} X_i^2 \quad .$$

It is a sub-elliptic K-invariant operator (with analytic coefficients). Consequently the spherical functions are eigenfunctions for this operator (see Remark 4.1).
Each representation $\Pi = \Pi_{r,\Lambda}$ induces the representation $d\Pi$ on the algebra of differential left invariant operators on N_p, over the space of Schwartz functions $\mathcal{S}(\mathbb{R}^{p_0})$:

$$\begin{array}{ccclr} j = 1, \dots, p_0 & d\Pi(X_{2j-1}) & = & \sqrt{\lambda_j} \partial_{y_j} & , \\ j = 1, \dots, p_0 & d\Pi(X_{2j}) & = & i\sqrt{\lambda_j} y_j & , \\ 2p_0 < j < p & d\Pi(X_j) & = & 0 & , \\ j = p & d\Pi(X_p) & = & ir\mathrm{Id} & , \\ \forall i < j & d\Pi(X_{i,j}) & = & \left\{ \begin{array}{ll} i\lambda_{j'}\mathrm{Id} & \text{if } (i,j) = (2j'-1, 2j') \\ 0 & \text{otherwise} \end{array} \right. & . \end{array}$$

For L, we get:

$$d\Pi(L) \;=\; r^2 \mathrm{Id} - \sum_{i=1}^{p_0} \lambda_i \left(\partial_{y_i}^2 - y_i^2 \right) \quad .$$

We recall the Hermite function $y = h_k$ satisfies the differential equation $y'' + (2k + 1 - x^2)y = 0$ [12, formula (5.5.2)]. So, we get:

$$d\Pi(L).\zeta_\alpha \;=\; \left(\sum_{j=1}^{p_1} \lambda_j(2l_j + m_j) + r^2\right) \zeta_\alpha \quad , \quad \alpha \in E_l,\, l \in \mathbb{N}^{p_1} \quad .$$

We deduce (see Remark 6.1):

$$L.\phi^{r,\Lambda,l} \;=\; \left(\sum_{j=1}^{p_1} \lambda_j(2l_j + m_j) + r^2\right) \phi^{r,\Lambda,l} \quad ;$$

this equality may also be computed directly using properties of the Laguerre functions.
Radial Plancherel measure. Here, we give the radial Plancherel measure.

Let $\mathcal{L}$ be the set of $\Lambda = (\lambda_1, \ldots, \lambda_{p'}) \in \mathbb{R}^{p'}$ such that $\lambda_1 > \ldots > \lambda_{p'} > 0$. We define the following measure on $\mathcal{L}$:

- $d\Lambda = d\lambda_1 \ldots d\lambda_{p'}$ is the restricted Lebesgue measure on $\mathcal{L}$,
- η is the measure on $\mathcal{L}$ such that:

$$d\eta(\Lambda) = \begin{cases} c\Pi_{j<k}(\lambda_j^2 - \lambda_k^2)^2 d\Lambda & \text{if } p = 2p' \\ c\Pi_i \lambda_i^2 \Pi_{j<k}(\lambda_j^2 - \lambda_k^2)^2 d\Lambda & \text{if } p = 2p'+1 \end{cases} \quad ,$$

  where the constant c is chosen in order to yield the polar change of variables over the space of antisymmetric matrices $\mathcal{A}_p$:

$$\int_{\mathcal{A}_p} g(A)dA \;=\; \int_{O_p}\int_{\mathcal{L}} g(k.D_2(\Lambda))d\eta(\Lambda)dk \quad .$$

- η' is the measure on $\mathcal{L}$ given by: $d\eta'(\Lambda) = \Pi_{i=1}^{p'} \lambda_i d\eta(\Lambda)$,

Over $\mathbb{R}^+$, we define the measure τ given as the Lebesgue measure if $p = 2p'+1$, and the Dirac measure in 0 if $p = 2p'$.

The (non radial) Plancherel measure is already known [11, Section 6]. With our notations, it is the measure m given as the tensor product of the Haar probability measure dk on $K = O(p)$, the measure η' on $\mathcal{L}$, and the measure τ on $\mathbb{R}^+$, up to the constant $c(p)$ given by:

$$c(p) \;=\; \begin{cases} (2\pi)^{-\frac{p(p-1)}{2}+p'} & \text{if } p = 2p' \;, \\ 2(2\pi)^{-\frac{p(p-1)}{2}+p'-1} & \text{if } p = 2p'+1 \;. \end{cases}$$

Theorem 6.3. *m is the (non radial) Plancherel measure; i.e. for $\psi \in L^2(N)$, we have:*

$$\|\psi\|^2_{L^2(N)} \;=\; \int \|k.\Pi_{r,\Lambda}(\psi)\|^2_{HS}\, dm(r,k,\Lambda) \quad .$$

where $\|.\|_{HS}$ denotes the Hilbert-Schmidt norm.

If we compute the Hilbert-Schmidt square-norm of $k.\Pi_{r,\Lambda}(\psi)$ with the orthonormal basis $\{\zeta_\alpha, \alpha \in E_l, l \in \mathbb{N}^{p_1}\}$, we deduce the radial Plancherel measure (see Lemma 6.2). This is the measure, which we denote by $m^\natural$, given as the tensor product of η' on $\mathcal{L}$, and the counting measure $\sum$ on $\mathbb{N}^{p'}$, and the measure τ on $\mathbb{R}^+$, up to the normalizing constant $c(p)$:

Theorem 6.4. *$m^\natural$ is the radial Plancherel measure for (N_p, O_p), i.e. for a K-invariant function $\psi \in L^2(N)$, we have:*

$$\|\psi\|^2_{L^2(N)} = \int |<\psi, \phi^{r,\Lambda,l}>|^2 dm^\natural(r,\Lambda,l) \quad .$$

We can also compute directly the radial Plancherel measure $m^\natural$, using the properties of Laguerre functions and Euclidean Fourier transform (see [5]).

References

[1] Chal, B., and J. Jenkins, and G. Ratcliff *On Gelfand pairs associated with solvable Lie groups* Trans. Amer. Math. Soc. **321(1)** (1990), 85–116

[2] Chal, B., and J. Jenkins, and G. Ratcliff Bounded K-spherical functions on Heisenberg groups J. Funct. Anal. **105(2)** (1992) 409–443

[3] Faraut J. “Analyse harmonique” Les cours du CIMPA (1982) chapter IV

[4] Faraut J. and K. Harzallah “Deux cours d’analyse harmonique” Progress in Mathematics volume 69, Birkhäuser Boston Inc., Boston, MA, 1987

[5] V. Fischer “Études de deux classes de groupes nilpotents de pas deux” PhD thesis, Université Paris-Sud, 2004

[6] S. Helgason “Differential geometry and symmetric spaces” Pure and Applied Mathematics **XII** (1962) Academic Press, New York

[7] N. Jacobson “Lie algebras” Interscience Tracts in Pure and Applied Mathematics, No. 10. 1962.

[8] A. Kirillov “Éléments de la théorie des représentations” Éditions Mir, Moscow, 1974

[9] R.-L. Lipsman “Group representations” Lecture Notes in Mathematics, Vol. 388, Springer-Verlag, Berlin, 1974

[10] L. Pukanszky “Leçons sur les représentations des groupes” Monographies SMF, No. 2. Dunod, Paris, 1967

[11] R.. Strichartz *L^p harmonic analysis and Radon transforms on the Heisenberg group* J. Funct. Anal. **96(2)**, 350–406, 1991

[12] G. Szegő “Orthogonal polynomials” American Mathematical Society, 1975

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