A GENERALIZATION OF THE DRESS CONSTRUCTION FOR A TAMBARA FUNCTOR, AND POLYNOMIAL TAMBARA FUNCTORS

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Abstract. For a finite group G, (semi-)Mackey functors and (semi-)Tambara functors are regarded as G-bivariant analogs of (semi-)groups and (semi-)rings respectively. In fact if G is trivial, they agree with the ordinary (semi-)groups and (semi-)rings, and many naive algebraic properties concerning rings and groups have been extended to these G-bivariant analogous notions.

In this article, we investigate a G-bivariant analog of the semi-group rings with coefficients. Just as a coefficient ring R and a monoid Q yield the semi-group ring $R[Q]$, our constrcution enables us to make a Tambara functor $T[M]$ out of a semi-Mackey functor M, and a coefficient Tambara functor T. This construction is a composant of the Tambarization and the Dress construction.

As expected, this construction is the one uniquely determined by the righteous adjoint property. Besides in analogy with the trivial group case, if M is a Mackey functor, then $T[M]$ is equipped with a natural Hopf structure.

Moreover, as an application of the above construction, we also obtain some G-bivariant analogs of the polynomial rings.

1. Introduction and preliminaries

For a finite group G, a (resp. semi-)Mackey functor is a pair of a contravariant functor and a covariant functor to the category of abelian groups Ab (resp. of commutative monoids Mon), satisfying some conditions (Definition 1.2). Since the category of Mackey functors is a symmetric monoidal abelian category which agrees with Ab when G is trivial, it is regarded as a G-bivariant analog of Ab. Similarly a semi-Mackey functor is regarded as a G-bivariant analog of a commutative monoid.

In this view, a Tambara functor is regarded as a G-bivariant analog of a commutative ring. It consists of an additive Mackey functor structure and a multiplicative semi-Mackey functor structure, satisfying the 'distributive law' (Definition 1.6).

Some naive algebraic properties concerning rings and groups have been extended to these G-bivariant analogous notions. For example, in our previous result ([4]), as a G-bivariant analog of the functor taking semi-group rings

$$\mathbb{Z}[-]\colon Mon \to Ring,$$

we constructed a functor called the *Tambarization functor*

$$\mathcal{T}\colon SMack(G) \to Tam(G),$$

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which is characterized by some natural adjoint property (Fact 2.2). Here $\mathit{SMack}(G)$ denotes the category of semi-Mackey functors, and $\mathit{Tam}(G)$ denotes the category of Tambara functors.

In this article, more generally, we investigate a G-bivariant analog of the semi-group ring *with a coefficient ring*. In the trivial group case, from any commutative ring R and any commutative monoid Q we can make the semi-group ring $R[Q]$, and this gives a functor

$$\mathit{Ring} \times \mathit{Mon} \rightarrow \mathit{Ring}\ ;\ (R, Q) \mapsto R[Q].$$

In section 2, analogously we construct a functor

$$\mathit{Tam}(G) \times \mathit{SMack}(G) \rightarrow \mathit{Tam}(G),$$

which unifies the Tambarization ([4]) and the Dress construction ([5]) as follows:

Theorem 2.5 . For any finite group G, there is a functor

$$\mathcal{F}\colon \mathit{Tam}(G) \times \mathit{SMack}(G) \rightarrow \mathit{Tam}(G)$$

which satisfies the following.

(i) If G is trivial, then $\mathcal{F}$ agrees with the functor taking semi-group rings with coefficients

$$\mathit{Ring} \times \mathit{Mon} \rightarrow \mathit{Ring}\ ;\ (R, Q) \mapsto R[Q].$$

(ii) If $T = \Omega$, we have a natural isomorphism $\mathcal{F}(\Omega, M) \cong \mathcal{T}(M)$ for each semi-Mackey functor M.

(iii) If Q is a finite G-monoid, then we have a natural isomorphism $\mathcal{F}(T, \mathcal{P}_Q) \cong T_Q$ for each Tambara functor T. Here, T_Q is the Tambara functor obtained through the Dress construction ([5]).

As expected from the trivial group case, $\mathcal{F}$ is the unique functor characterized by the following adjoint property:

Theorem 2.5 (iv) . For each Tambara functor T and semi-Mackey functor M, naturally $\mathcal{F}(T, M)$ becomes a T-Tambara functor. Moreover if we fix T, then the induced functor

$$\mathcal{F}(T, -)\colon \mathit{SMack}(G) \rightarrow T\text{-}\mathit{Tam}(G)$$

is left adjoint to the composition of forgetful functors

$$T\text{-}\mathit{Tam}(G) \rightarrow \mathit{Tam}(G) \xrightarrow{(-)^{\mu}} \mathit{SMack}(G)\ ;\ (S, \sigma) \mapsto S^{\mu}.$$

Besides in analogy with the trivial group case, if M is a Mackey functor, then $T[M]$ is equipped with a natural Hopf structure $(T[M], \Delta, \varepsilon, \eta)$ (Corollary 2.9).

In the last section, as an application of the construction above, we consider some G-bivariant analogs of the polynomial ring. In the trivial group case, the polynomial ring $R[\mathbf{X}]$ over R with one variable $\mathbf{X}$ was characterized by the natural bijection

$$R\text{-}\mathit{Alg}(R[\mathbf{X}], S) \xrightarrow{\cong} S$$

for each R-algebra S. To any Tambara functor T, we associate two types of 'polynomial' Tambara functors $T[\mathbf{x}]$ and $T[\mathfrak{X}]$. Analogously, we obtain natural bijections for each T-Tambara functor S

$$T\text{-}\mathit{Tam}(G)(T[\mathbf{x}], S) \cong S(G/e)^{G},$$
$$T\text{-}\mathit{Tam}(G)(T[\mathfrak{X}], S) \cong S(G/G),$$

and thus $T[\mathbf{x}]$ and $T[\mathfrak{X}]$ are characterized by these bijections (Theorem 3.2).

Throughout this article, we fix a finite group G, whose unit element is denoted by e. $H \le G$ means H is a subgroup of G. ${}_G set$ denotes the category of finite G-sets and G-equivariant maps. A monoid is always assumed to be unitary and commutative. Similarly a (semi-)ring is assumed to be commutative, and have an additive unit 0 and a multiplicative unit 1. We denote the category of monoids by *Mon*, the category of (resp. semi-)rings by *Ring* (resp. *SRing*), and the category of abelian groups by *Ab*. A monoid homomorphism preserves units, and a (semi-)ring homomorphism preserves 0 and 1.

For any category $\mathcal{K}$ and any pair of objects X and Y in $\mathcal{K}$, the set of morphisms from X to Y in $\mathcal{K}$ is denoted by $\mathcal{K}(X, Y)$. For each $X \in \mathrm{Ob}(\mathcal{K})$, the slice category of $\mathcal{K}$ over X is denoted by $\mathcal{K}/X$.

Definition 1.1. An *additive contravariant functor F on G* means a contravariant functor

$$F\colon {}_G set \to Mon,$$

which sends coproducts in ${}_G set$ to products in *Mon*. A *morphism* from one additive contravariant functor to another merely means a natural transformation. The category of additive contravariant functors is denoted by $Madd(G)$.

Definition 1.2. A *semi-Mackey functor M on G* is a pair $M = (M^*, M_*)$ of a covariant functor

$$M_*\colon {}_G set \to Mon$$

and an additive contravariant functor

$$M^*\colon {}_G set \to Mon,$$

satisfying $M^*(X) = M_*(X)$ for any $X \in \mathrm{Ob}({}_G set)$, and the following Mackey condition:

- (Mackey condition)
  If we are given a pull-back diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle x}\downarrow & \square & \downarrow{\scriptstyle y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

  in ${}_G set$, then

$$\begin{array}{ccc} M(X') & \xleftarrow{M^*(f')} & M(Y') \\ {\scriptstyle M_*(x)}\downarrow & \circlearrowright & \downarrow{\scriptstyle M_*(y)} \\ M(X) & \xleftarrow[M^*(f)]{} & M(Y) \end{array}$$

  is commutative.

Here we put $M(X)=M^*(X)=M_*(X)$ for each $X\in \mathrm{Ob}({}_G\mathit{set})$. Those $M_*(f)$ and $M^*(f)$ for morphisms f in ${}_G\mathit{set}$ are called *structure morphisms* of M. For each $f\in {}_G\mathit{set}(X,Y)$, $M^*(f)$ is called the *restriction*, and $M_*(f)$ is called the *transfer* along f.

For semi-Mackey functors M and N, a *morphism* from M to N is a family of monoid homomorphisms

$$\varphi=\{\varphi_X\colon M(X)\to N(X)\}_{X\in \mathrm{Ob}({}_G\mathit{set})},$$

natural with respect to both of the contravariant and the covariant parts. The category of semi-Mackey functors is denoted by $\mathit{SMack}(G)$.

If $M(X)$ is an abelian group for each $X\in\mathrm{Ob}({}_G\mathit{set})$, namely if M^* and M_* are functors to Ab, then a semi-Mackey functor $M=(M^*,M_*)$ is called a *Mackey functor*. The full subcategory of Mackey functors in $\mathit{SMack}(G)$ is denoted by $\mathit{Mack}(G)$.

Example 1.3.

(1) The Burnside ring functor $\Omega\in\mathrm{Ob}(\mathit{Mack}(G))$ (see for example [1], [5] or [4]).

(2) If Q is a (not necessarily finite) G-monoid, then the correspondence

$$\mathcal{P}_Q(X)=\{G\text{-equivariant maps from } X \text{ to } Q\}$$

forms a semi-Mackey functor $\mathcal{P}_Q\in\mathrm{Ob}(\mathit{SMack}(G))$ with structure morphisms defined by

$$\mathcal{P}_Q^*(f)\colon \mathcal{P}_Q(Y)\to\mathcal{P}_Q(X)\ ;\ \beta\mapsto\beta\circ f$$

$$(\mathcal{P}_Q)_*(f)\colon \mathcal{P}_Q(X)\to\mathcal{P}_Q(Y)\ ;\ \alpha\mapsto (Y\ni y\mapsto \prod_{x\in f^{-1}(y)}\alpha(x)\in Q)$$

for each $f\in {}_G\mathit{set}(X,Y)$, where $\prod$ denotes the multiplication of elements in Q. This $\mathcal{P}_Q$ is called the *fixed point functor* associated to Q (see for example [6] or [4]). If we denote the category of G-monoids by G-Mon, this construction gives a fully faithful functor

$$\mathcal{P}\colon G\text{-}\mathit{Mon}\to\mathit{SMack}(G)\ ;\ Q\mapsto\mathcal{P}_Q.$$

Thus G-Mon can be regarded as a full subcategory of $\mathit{SMack}(G)$ through $\mathcal{P}$.

Remark 1.4. For each pair of Mackey functors M and N, its tensor product $M\underset{\Omega}{\otimes}N$ is defined (also denoted by $M\widehat{\otimes}N$ in [1]), and $\mathit{Mack}(G)$ becomes a symmetric monoidal category with this tensor product and the unit Ω.

The category of monoids in $\mathit{Mack}(G)$ is denoted by $\mathit{Green}(G)$, and a monoid A in $\mathit{Mack}(G)$ is called a *Green functor on* G.

By definition of the tensor product ([1]), for each $X\in\mathrm{Ob}({}_G\mathit{set})$

$$(M\underset{\Omega}{\otimes}N)(X)=(\bigoplus_{A\overset{p}{\to}X}M(A)\underset{\mathbb{Z}}{\otimes}N(A))/\mathcal{I},$$

where $A\overset{p}{\to}X$ runs over the objects in ${}_G\mathit{set}/X$, and $\mathcal{I}$ is the submodule generated by the elements

$$M^*(a)(s')\otimes t-s'\otimes N_*(a)(t),\quad M_*(a)(s)\otimes t'-s\otimes N^*(a)(t')$$

$$(a\in{}_G\mathit{set}/X((A\overset{p}{\to}X),(A'\overset{p'}{\to}X)), s\in M(A), t\in N(A), s'\in M(A'), t'\in N(A')).$$

In the component of $A \overset{p}{\to} X$, the image of $s \otimes t$ in $(M \underset{\Omega}{\otimes} N)(X)$ is denoted by $[s \otimes t]_{(A,p)}$ for each $s \in M(A)$ and $t \in N(A)$.

A priori an element ω in $(M \underset{\Omega}{\otimes} N)(X)$ is a finite sum of the elements of the above form

$$\omega = \sum_{1 \le i \le n} [s_i \otimes t_i]_{(A_i, p_i)},$$

however ω can be written by one such an element. In fact, if we put

$$A = \coprod_{1 \le i \le n} A_i, \quad p = \bigcup_{1 \le i \le n} p_i, \quad \iota_i \colon A_i \hookrightarrow A \ \ \text{(inclusion)}$$

and put

$$s = \sum_{1 \le i \le n} (\iota_i)_*(s_i), \quad t = \sum_{1 \le i \le n} (\iota_i)_*(t_i),$$

then we have

$$[s \otimes t]_{(A,p)} = \sum_{1 \le i \le n} [s_i \otimes t_i]_{(A_i, p_i)}.$$

Definition 1.5. For each $f \in {}_G set(X, Y)$ and $p \in {}_G set(A, X)$, the *canonical exponential diagram* generated by f and p is the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{e} & X \underset{Y}{\times} \Pi_f(A) \\ {\scriptstyle f}\downarrow & & exp & & \downarrow{\scriptstyle f'} \\ Y & & \xleftarrow[\pi]{} & & \Pi_f(A) \end{array}$$

where

$$\Pi_f(A) = \left\{ (y, \sigma) \;\middle|\; \begin{array}{l} y \in Y, \\ \sigma \colon f^{-1}(y) \to A \ \text{ is a map of sets,} \\ p \circ \sigma \ \text{ is identity on } f^{-1}(y) \end{array} \right\},$$

$$\pi(y, \sigma) = y, \qquad e(x, (y, \sigma)) = \sigma(x),$$

and f' is the pull-back of f by π. A diagram in ${}_G set$ isomorphic to one of the canonical exponential diagrams is called an *exponential diagram*. For the properties of exponential diagrams, see [6].

Definition 1.6. A *semi-Tambara functor* T *on* G is a triplet $T = (T^*, T_+, T_\bullet)$ of two covariant functors

$$T_+ \colon {}_G set \to Mon, \quad T_\bullet \colon {}_G set \to Mon$$

and one additive contravariant functor

$$T^* \colon {}_G set \to Mon$$

which satisfies the following.

(1) $T^\alpha = (T^*, T_+)$ and $T^\mu = (T^*, T_\bullet)$ are objects in $SMack(G)$. T^α is called the *additive part* of T, and T^μ is called the *multiplicative part* of T.

(2) (Distributive law) If we are given an exponential diagram

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{\lambda} & Z \\ {\scriptstyle f}\downarrow & & exp & & \downarrow{\scriptstyle \rho} \\ Y & & \xleftarrow[q]{} & & B \end{array}$$

in ${}_G set$, then

$$\begin{array}{ccccc} T(X) & \xleftarrow{T_+(p)} & T(A) & \xrightarrow{T^*(\lambda)} & T(Z) \\ {\scriptstyle T_\bullet(f)}\downarrow & & \circlearrowright & & \downarrow{\scriptstyle T_\bullet(\rho)} \\ T(Y) & & \xleftarrow[T_+(q)]{} & & T(B) \end{array}$$

is commutative.

If $T = (T^*, T_+, T_\bullet)$ is a semi-Tambara functor, then $T(X)$ becomes a semi-ring for each $X \in \mathrm{Ob}({}_G set)$, whose additive (resp. multiplicative) monoid structure is induced from that on $T^\alpha(X)$ (resp. $T^\mu(X)$). Those $T^*(f), T_+(f), T_\bullet(f)$ for morphisms f in ${}_G set$ are called *structure morphisms* of T. For each $f \in {}_G set(X, Y)$,

- $T^*\colon T(Y) \to T(X)$ is a semi-ring homomorphism, called the *restriction* along f.
- $T_+(f)\colon T(X) \to T(Y)$ is an additive homomorphism, called the *additive transfer* along f.
- $T_\bullet(f)\colon T(X) \to T(Y)$ is a multiplicative homomorphism, called the *multiplicative transfer* along f.

$T^*(f), T_+(f), T_\bullet(f)$ are often abbreviated to $f^*, f_+, f_\bullet$.

A *morphism* of semi-Tambara functors $\varphi\colon T \to S$ is a family of semi-ring homomorphisms

$$\varphi = \{\varphi_X\colon T(X) \to S(X)\}_{X \in \mathrm{Ob}({}_G set)},$$

natural with respect to all of the contravariant and the covariant parts. We denote the category of semi-Tambara functors by $STam(G)$.

If $T(X)$ is a ring for each $X \in \mathrm{Ob}({}_G set)$, then a semi-Tambara functor T is called a *Tambara functor*. The full subcategory of Tambara functors in $STam(G)$ is denoted by $Tam(G)$.

Remark 1.7. A semi-Tambara functor T is a Tambara functor if and only if T^α is a Mackey functor. Taking the additive parts and the multiplicative parts, we obtain functors

$$(-)^\alpha\colon Tam(G) \to Mack(G),$$
$$(-)^\mu\colon Tam(G) \to SMack(G).$$

(In fact, $(-)^\alpha$ factors through the category $Green(G)$. For this, see [4] or [7].)

If G is trivial, these functors are nothing other than the forgetful functors

$$(-)^\alpha\colon Ring \to Ab,$$
$$(-)^\mu\colon Ring \to Mon,$$

where, for each ring R, its image R^α (resp. R^μ) is the underlying additive (resp. multiplicative) monoid of R.

Example 1.8. The Burnside ring functor Ω is the initial object in $Tam(G)$. Ω can be regarded as the G-bivariant analog of $\mathbb{Z}$.

The following is shown in [7].

Remark 1.9 (§12 in [7]). If T and S are Tambara functors, then so is $T \underset{\Omega}{\otimes} S$. Besides, there exist morphisms $\iota_T \in Tam(G)(T, T \underset{\Omega}{\otimes} S)$ and $\iota_S \in Tam(G)(S, T \underset{\Omega}{\otimes} S)$, which make $T \underset{\Omega}{\otimes} S$ the coproduct of T and S in $\mathit{Tam}(G)$. Ω is the unit for this tensor product.

Proof. A simple proof using a functor category will be found in [7]. For the later use, we briefly introduce an explicit construction of the structure morphisms of $T \underset{\Omega}{\otimes} S$.

Let $f \in {}_G\mathit{set}(X, Y)$ be any morphism. For any $[v \otimes u]_{(B,q)} \in (T \underset{\Omega}{\otimes} S)(Y)$, we define $f^*([v \otimes u]_{(B,q)})$ by

$$f^*([v \otimes u]_{(B,q)}) = [T^*(p_B)(v) \otimes S^*(p_B)(u)]_{(X \underset{Y}{\times} B, p_X)},$$

where

$$\begin{array}{ccc} X \times_Y B & \xrightarrow{p_B} & B \\ {\scriptstyle p_X}\downarrow & \square & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{1.1}$$

is the canonical pull-back.

For any $[t \otimes s]_{(A,p)} \in (T \underset{\Omega}{\otimes} S)(X)$, we define $f_+([t \otimes s]_{(A,p)})$ and $f_\bullet([t \otimes s]_{(A,p)})$ by

$$f_+([t \otimes s]_{(A,p)}) = [t \otimes s]_{(A, f \circ p)},$$

$$f_\bullet([t \otimes s]_{(A,p)}) = [T_\bullet(f')T^*(e)(t) \otimes S_\bullet(f')S^*(e)(s)]_{(\Pi_f(A), \pi)},$$

where

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{e} & X \underset{Y}{\times} \Pi_f(A) \\ {\scriptstyle f}\downarrow & & \circlearrowleft & & \downarrow{\scriptstyle f'} \\ Y & & \xleftarrow[\pi]{} & & \Pi_f(A) \end{array} \tag{1.2}$$

is the canonical exponential diagram. With these structure morphisms, $T \underset{\Omega}{\otimes} S$ becomes a Green functor as shown in [1]. Moreover for these (to-be-)structure morphisms, the functoriality of $(T \underset{\Omega}{\otimes} S)_\bullet$ follows from (1.3) in [6], the Mackey condition for $(T \underset{\Omega}{\otimes} S)^\mu$ follows from (1.1) in [6], the distributive law for $T \underset{\Omega}{\otimes} S$ follows from (1.2) in [6], and thus $T \underset{\Omega}{\otimes} S$ becomes a Tambara functor. $\square$

Definition 1.10. Fix a Tambara functor $T \in \mathrm{Ob}(\mathit{Tam}(G))$. A *$T$-Tambara functor* is a pair (S, σ) of a Tambara functor S and $\sigma \in \mathit{Tam}(G)(T, S)$. We often represent (S, σ) merely by S.

If (S, σ) and (S', σ') are T-Tambara functors, then a *morphism* φ from (S, σ) to (S', σ') is a morphism $\varphi \in \mathit{Tam}(G)(S, S')$ satisfying $\sigma' = \varphi \circ \sigma$. The category of T-Tambara functors is denoted by T-$\mathit{Tam}(G)$. Remark that Ω-$\mathit{Tam}(G)$ is nothing other than $\mathit{Tam}(G)$.

By Remark 1.9, for any $T, S \in \mathrm{Ob}(\mathit{Tam}(G))$, their tensor product $T \underset{\Omega}{\otimes} S$ can be naturally regarded as a T-Tambara functor $(T \underset{\Omega}{\otimes} S, \iota_T)$, or a S-Tambara functor $(T \underset{\Omega}{\otimes} S, \iota_S)$.

2. Generalization of the Dress construction

The *Dress construction* of a Tambara functor is a process making a Tambara functor T_Q out of a Tambara functor T and a finite G-monoid Q. This is realized as a functor

$$\mathit{Tam}(G) \times G\text{-}\mathit{mon} \to \mathit{Tam}(G)$$

where G-*mon* is the category of *finite* G-monoids. Remark here G-*mon* is a full subcategory of G-*Mon*, and thus can be regarded as a full subcategory of $\mathit{SMack}(G)$ through $\mathcal{P}$ (Example 1.3).

In this section, we extend this functor to

$$\mathit{Tam}(G) \times \mathit{SMack}(G) \to \mathit{Tam}(G),$$

through a G-bivariant analogical construction of a semi-group ring with a coefficient, by means of the *Tambarization functor*. First, we recall the Dress construction for a Tambara functor:

Definition 2.1 (Theorem 2.9 in [5])**.** Let Q be a finite G-monoid, and T be a Tambara functor on G. In [5], $T_Q \in \mathrm{Ob}(\mathit{Tam}(G))$ is defined by $T_Q(X) = T(X \times Q)$ for each $X \in \mathrm{Ob}({}_G\mathit{set})$, whose structure morphisms are given by

$$\begin{aligned} (T_Q)^*(f) &= T^*(f \times Q) \\ (T_Q)_+(f) &= T_+(f \times Q) \\ (T_Q)_\bullet(f) &= T_+(\mu_f) \circ T_\bullet(f') \circ T^*(e) \end{aligned}$$

for each morphism $f \in {}_G\mathit{set}(X,Y)$. Here,

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & X \times Q & \xleftarrow{e} & X \times_Y \Pi_f(X \times Q) \\ {\scriptstyle f}\downarrow & & \mathit{exp} & & \downarrow{\scriptstyle f'} \\ Y & & \xleftarrow[\pi]{\qquad\qquad} & & \Pi_f(X \times Q) \end{array}$$

is the canonical exponential diagram with $p_X \colon X \times Q \to X$ the projection, and $\mu_f \colon \Pi_f(X \times Q) \to Y \times Q$ is the morphism defined by

$$\mu_f(y, \sigma) = (y, \prod_{x \in f^{-1}(y)} p_Q \circ \sigma(x)),$$

where $p_Q \colon X \times Q \to Q$ is the projection.

Especially if $T = \Omega$, then Ω_Q is called the *crossed Burnside ring functor*.

The crossed Burnside ring functors were generalized by the following Theorems.

Fact 2.2 (Theorem 2.15 in [4])**.** $(-)^\mu \colon \mathit{Tam}(G) \to \mathit{SMack}(G)$ has a left adjoint functor

$$\mathcal{T} \colon \mathit{SMack}(G) \to \mathit{Tam}(G),$$

which we call the *Tambarization functor*.

Proof. We briefly review the structure of $\mathcal{T}(M)$. For the entire proof, see [4].

For each $X \in \mathrm{Ob}({}_G\mathit{set})$, we define $\mathcal{T}(M)$ to be the Grothendieck ring of the category of pairs $(A \xrightarrow{p} X, m_A)$ of $(A \xrightarrow{p} X) \in \mathrm{Ob}({}_G\mathit{set}/X)$ and $m_A \in M(A)$.

For each $f \in {}_G\mathit{set}(X,Y)$, the structure morphisms induced from f are those determined by

$$\mathcal{T}(M)^*(f)(B \xrightarrow{q} Y, m_B) = (X \times_Y B \xrightarrow{p_X} X, M^*(p_B)(m_B))$$

$$\mathcal{T}(M)_+(A \stackrel{p}{\to} X, m_A) = (A \stackrel{f\circ p}{\longrightarrow} Y, m_A)$$
$$\mathcal{T}(M)_\bullet(A \stackrel{p}{\to} X, m_A) = (\Pi_f(A) \stackrel{\pi}{\to} Y, M_*(f')M^*(e)(m_A))$$

for each $(A \stackrel{p}{\to} X, m_A)$ and $(B \stackrel{q}{\to} Y, m_B)$, where (1.1) is the canonical pull-back, and (1.2) is the canonical exponential diagram. □

Fact 2.3 (Proposition 3.2 in [4])**.** If Q is a finite G-monoid, then we have an isomorphism of Tambara functors $\Omega_Q \cong \mathcal{T}(\mathcal{P}_Q)$.

Thus $\mathcal{T}(M)$, where M can be taken as an arbitrary semi-Mackey functor on G, is regarded as a generalization of the crossed Burnside ring functors.

Remark 2.4. By the adjoint property in Fact 2.2, $\mathcal{T}(M)$ can be also regarded as a G-bivariant analog of the semi-ring. In fact if G is trivial, then $\mathcal{T}$ is nothing other than the functor taking semi-group rings

$$\mathbb{Z}[-]\colon \mathit{Mon} \to \mathit{Ring}.$$

In this view, from here we denote $\mathcal{T}(M)$ by $\Omega[M]$ instead, for each semi-Mackey functor M on G.

Our aim is to show the following, which unifies the Tambarization and the Dress construction.

Theorem 2.5. *For any finite group G, there is a functor*

$$\mathcal{F}\colon \mathit{Tam}(G) \times \mathit{SMack}(G) \to \mathit{Tam}(G)$$

which satisfies the following.

(i) *If G is trivial, then $\mathcal{F}$ agrees with the functor taking semi-group rings with coefficients*

$$\mathit{Ring} \times \mathit{Mon} \to \mathit{Ring}\ ;\ (R, Q) \mapsto R[Q].$$

(ii) *If $T = \Omega$, we have a natural isomorphism $\mathcal{F}(\Omega, M) \cong \Omega[M]$ for each semi-Mackey functor M.*

(iii) *If Q is a finite G-monoid, then we have a natural isomorphism $\mathcal{F}(T, \mathcal{P}_Q) \cong T_Q$ for each Tambara functor T.*

(iv) *For each T and M, naturally $\mathcal{F}(T, M)$ becomes a T-Tambara functor. Moreover if we fix a Tambara functor T, then the induced functor*

$$\mathcal{F}(T, -)\colon \mathit{SMack}(G) \to T\text{-}\mathit{Tam}(G)$$

is left adjoint to the composition of forgetful functors

$$T\text{-}\mathit{Tam}(G) \to \mathit{Tam}(G) \stackrel{(-)^\mu}{\longrightarrow} \mathit{SMack}(G)\ ;\ (S, \sigma) \mapsto S^\mu.$$

Proof. As a consequence of Remark 1.9, we obtain a functor

$$\mathit{Tam}(G) \times \mathit{Tam}(G) \to \mathit{Tam}(G)\ ;\ (T, S) \mapsto T \underset{\Omega}{\otimes} S.$$

Combining this with the Tambarization functor, we define $\mathcal{F}$ by

$$\mathcal{F}\colon \mathit{Tam}(G) \times \mathit{SMack}(G) \to \mathit{Tam}(G)\ ;\ (T, M) \mapsto T \underset{\Omega}{\otimes} \Omega[M].$$

Then properties (i) and (iv) follows immediately from the adjoint property of the Tambarization functor, and the universality of the coproduct. Also, (ii) follows from the unit isomorphism $\Omega \underset{\Omega}{\otimes} \Omega[M] \cong \Omega[M]$. Thus it remains to show (iii).

To show (iii), let T be a Tambara functor, and let Q be a finite G-monoid. We construct a natural isomorphism $T_Q \cong T \underset{\Omega}{\otimes} \Omega[\mathcal{P}_Q]$ of Tambara functors. Remark that for each $X \in \mathrm{Ob}({}_G set)$, any element ω in $(T \underset{\Omega}{\otimes} \Omega[\mathcal{P}_Q])(X)$ can be written in the form of

$$\omega = [s \otimes (R \overset{r}{\to} A, m_R)]_{(A,p)}, \tag{2.1}$$

where $s \in T(A)$, $(A \overset{p}{\to} X) \in \mathrm{Ob}({}_G set/X)$, $r \in {}_G set(R, A)$ and $m_R \in {}_G set(R, Q)$.

For each $X \in \mathrm{Ob}({}_G set)$, define

$$\varphi_X : T_Q(X) \to (T \underset{\Omega}{\otimes} \Omega[\mathcal{P}_Q])(X)$$
$$\psi_X : (T \underset{\Omega}{\otimes} \Omega[\mathcal{P}_Q])(X) \to T_Q(X)$$

by

$$\varphi_X(t) = [t \otimes (X \times Q \overset{\mathrm{id}}{\to} X \times Q, p_Q)]_{(X \times Q, p_X)} \quad ({}^\forall t \in T_Q(X)),$$
$$\psi_X(\omega) = T_+((p \circ r, m_R))T^*(r)(s) \quad ({}^\forall \omega \text{ as in } (2.1)).$$

Then we have

$$\psi_X \circ \varphi_X(t) = T_+((p_X \circ \mathrm{id}_{X \times Q}, p_Q))T^*(\mathrm{id})(t) = t$$

for any t, and

$$\begin{aligned} \varphi_X \circ \psi_X(\omega) &= [T_+((p \circ r, m_R))T^*(r)(s) \otimes (X \times Q \overset{\mathrm{id}}{\to} X \times Q, p_Q)]_{(X \times Q, p_X)} \\ &= [s \otimes \Omega_{Q+}(r)\Omega_Q^*((p \circ r, m_R))(X \times Q \overset{\mathrm{id}}{\to} X \times Q, p_Q)]_{(A,p)} \\ &= [s \otimes (R \overset{r}{\to} A, m_R)]_{(A,p)} = \omega \end{aligned}$$

for any ω, namely φ_X and ψ_X are mutually inverses. Thus it remains to show the following:

Claim 2.6. *$\varphi = \{\varphi_X\}_{X \in \mathrm{Ob}({}_G set)}$ gives a morphism of Tambara functors*

$$\varphi \colon T_Q \to T \underset{\Omega}{\otimes} \Omega[\mathcal{P}_Q].$$

Proof. Let $f \in {}_G set(X, Y)$ be any morphism. It suffices to show φ is compatible with f^*, f_+ and $f_\bullet$. Denote the projections onto Q by $p_Q \colon X \times Q \to Q$ and $p'_Q \colon Y \times Q \to Q$.

(a) (compatibility with f^*)

Let u be any element in $T_Q(Y)$. Then we have

$$\begin{aligned} f^*\varphi_Y(u) &= f^*([u \otimes (Y \times Q \overset{\mathrm{id}}{\to} Y \times Q, p'_Q)]_{(Y \times Q, p_Y)}) \\ &= [T^*(f \times Q)(u) \otimes (X \times Q \overset{\mathrm{id}}{\to} X \times Q, p_Q)]_{(X \times Q, p_X)} \\ &= \varphi_X(T^*(f \times Q)(u)) \\ &= \varphi_X T_Q^*(f)(u). \end{aligned}$$

(b) (compatibility with f_+)

Let t be any element in $T_Q(X)$. Then we have

$$\begin{aligned}
f_+\varphi_X(t) &= f_+([t\otimes(X\times Q\overset{\mathrm{id}}{\to}X\times Q)]_{(X\times Q,p_X)})\\
&= [t\otimes(X\times Q\overset{\mathrm{id}}{\to}X\times Q)]_{(X\times Q,f\circ p_X)}\\
&= [t\otimes(\Omega[\mathcal{P}_Q]^*(f\times Q))(Y\times Q\overset{\mathrm{id}}{\to}Y\times Q,p'_Q)]_{(X\times Q,f\circ p_X)}\\
&= [T_+(f\times Q)(t)\otimes(Y\times Q\overset{\mathrm{id}}{\to}Y\times Q,p'_Q)]_{(X\times Q,f\circ p_X)}\\
&= \varphi_Y(T_+(f\times Q)(t))\\
&= \varphi_Y T_{Q+}(f)(t).
\end{aligned}$$

(c) (compatibility with $f_\bullet$)

We use the notation in Definition 2.1. For any element t in $T_Q(X)$, we have

$$\begin{aligned}
\varphi_Y T_{Q\bullet}(t) &= \varphi_Y(T_+(\mu_f)T_\bullet(f')T^*(e)(t))\\
&= [T_+(\mu_f)T_\bullet(f')T^*(e)(t)\otimes(Y\times Q\overset{\mathrm{id}}{\to}Y\times Q,p'_Q)]_{(Y\times Q,p_Y)}\\
&= [T_\bullet(f')T^*(e)(t)\otimes\Omega[\mathcal{P}_Q]^*(\mu_f)(Y\times Q\overset{\mathrm{id}}{\to}Y\times Q,p'_Q)]_{(\Pi_f(X\times Q),\pi)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&f_\bullet\varphi_X(t)=f_\bullet([t\otimes(X\times Q\overset{\mathrm{id}}{\to}X\times Q,p_Q)]_{(X\times Q,p_X)})\\
&=[T_\bullet(f')T^*(e)(t)\otimes\Omega[\mathcal{P}_Q]_\bullet(f')\Omega[\mathcal{P}_Q]^*(e)(X\times Q\overset{\mathrm{id}}{\to}X\times Q,p_Q)]_{(\Pi_f(X\times Q),\pi)}.
\end{aligned}$$

Thus it suffices to show

$$\begin{aligned}
(2.2)\qquad &\Omega[\mathcal{P}_Q]^*(\mu_f)(Y\times Q\overset{\mathrm{id}}{\to}Y\times Q,p'_Q)\\
&\qquad= \Omega[\mathcal{P}_Q]_\bullet(f')\Omega[\mathcal{P}_Q]^*(e)(X\times Q\overset{\mathrm{id}}{\to}X\times Q,p_Q).
\end{aligned}$$

The left hand side of (2.2) is equal to

$$(\Pi_f(X\times Q)\overset{\mathrm{id}}{\to}\Pi_f(X\times Q),p'_Q\circ\mu_f). \tag{2.3}$$

Remark that $p'_Q\circ\mu_f\colon\Pi_f(X\times Q)\to Q$ is the morphism which satisfies

$$p'_Q\circ\mu_f(y,\sigma)=p'_Q(y,\prod_{x\in f^{-1}(y)}p_Q\circ\sigma(x))=\prod_{x\in f^{-1}(y)}p_Q\circ\sigma(x)$$

for any $(y,\sigma)\in\Pi_f(X\times Q)$.

On the other hand, since there is an exponential diagram

$$\begin{array}{ccccc}
X\underset{Y}{\times}\Pi_f(X\times Q) & \xleftarrow{\mathrm{id}} & X\underset{Y}{\times}\Pi_f(X\times Q) & \xleftarrow{\mathrm{id}} & X\underset{Y}{\times}\Pi_f(X\times Q)\\
{\scriptstyle f'}\big\downarrow & & exp & & \big\downarrow{\scriptstyle f'}\\
\Pi_f(X\times Q) & & \xleftarrow[\mathrm{id}]{\qquad\qquad} & & \Pi_f(X\times Q)\,,
\end{array}$$

the right hand side of (2.2) is equal to

$$(\Pi_f(X\times Q)\overset{\mathrm{id}}{\to}\Pi_f(X\times Q),(\mathcal{P}_Q)_*(f')(p_Q\circ e)),$$

where $(\mathcal{P}_Q)_*(f')(p_Q\circ e)\colon\Pi_f(X\times Q)\to Q$ is the morphism satisfying

$$((\mathcal{P}_Q)_*(f')(p_Q\circ e))(y,\sigma)=\prod_{(x,(y,\sigma))\in f'^{-1}(y,\sigma)}p_Q\circ e(x,(y,\sigma)) \tag{2.4}$$

for each $(y,\sigma)\in\Pi_f(X\times Q)$.

Since $f'^{-1}(y,\sigma) = \{(x,(y,\sigma)) \mid x \in f^{-1}(y)\}$, we have

$$\prod_{(x,(y,\sigma))\in f'^{-1}(y,\sigma)} p_Q \circ e(x,(y,\sigma)) = \prod_{(x,(y,\sigma))\in f'^{-1}(y,\sigma)} p_Q \circ \sigma(x) = \prod_{x\in f^{-1}(y)} p_Q \circ \sigma(x) \tag{2.5}$$

By (2.3), (2.4) and (2.5), we obtain

$$p'_Q \circ \mu_f = (\mathcal{P}_Q)_*(f')(p_Q \circ e),$$

and the equality of (2.2) follows. □

□

Remark 2.7. By (i) and (iv) in Theorem 2.5, $\mathcal{F}$ can be regarded as a G-bivariant analog of the functor taking semi-group rings with coefficients. In this view, from here we denote $\mathcal{F}(T,M)$ by $T[M]$ instead.

As in the trivial group case, $T[M]$ is equipped with a natural Hopf structure if M is a Mackey functor. To state this, first we remark the following.

Remark 2.8. For each pair $M,N \in \mathrm{Ob}(\mathit{Mack}(G))$, the coproduct of $T[M]$ and $T[N]$ in $\mathit{Tam}(G)$ is nothing other than $T[M \oplus N]$, where $M \oplus N$ is the coproduct of M and N in $\mathit{Mack}(G)$ defined in an obvious way. $T[M \oplus N]$ is denoted by $T[M] \underset{T}{\otimes} T[N]$.

Proof. This immediately follows from the adjointness in Theorem 2.5. □

Corollary 2.9. *For any $M \in \mathrm{Ob}(\mathit{Mack}(G))$, there exist morphisms of T-Tambara functors*

$$\begin{aligned} \Delta &: T[M] \to T[M] \underset{T}{\otimes} T[M] \\ \varepsilon &: T[M] \to T \\ \eta &: T[M] \to T[M] \end{aligned}$$

satisfying

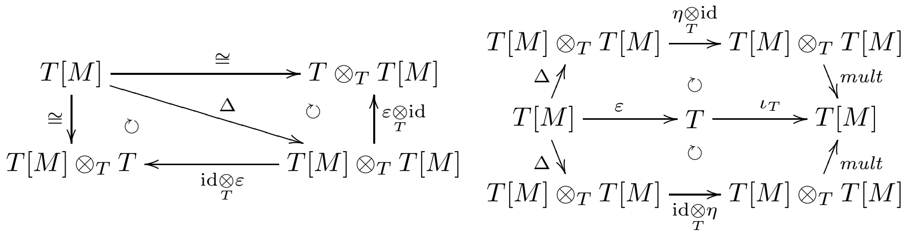
$$(\Delta \underset{T}{\otimes} \mathrm{id}) \circ T = (\mathrm{id} \underset{T}{\otimes} \Delta) \circ T, \quad (\varepsilon \underset{T}{\otimes} \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \underset{T}{\otimes} \varepsilon) \circ \Delta$$

and

$$(\mathit{mult}) \circ (\eta \underset{T}{\otimes} \mathrm{id}) \circ \Delta = \varepsilon \circ \iota_T = (\mathit{mult}) \circ (\mathrm{id} \underset{T}{\otimes} \eta) \circ \Delta,$$

where (mult) is the multiplication morphism (i.e. the morphism inducing $\mathrm{id}_{T[M]}$ on each components of $T[M] \underset{T}{\otimes} T[M]$.)

$$\begin{array}{ccc} T[M] & \xrightarrow{\Delta} & T[M] \otimes_T T[M] \\ {\scriptstyle\Delta}\downarrow & \circlearrowright & \downarrow{\scriptstyle \Delta \underset{T}{\otimes} \mathrm{id}} \\ T[M] \otimes_T T[M] & \xrightarrow[\mathrm{id} \underset{T}{\otimes} \Delta]{} & T[M] \otimes_T T[M] \otimes_T T[M] \end{array}$$



Proof. We define $\Delta, \varepsilon, \eta$ to be those morphisms corresponding to

$$M \xrightarrow{\Delta_M} M \oplus M, \quad M \xrightarrow{0} 0, \quad M \xrightarrow{(-)^{-1}} M$$

which are the diagonal morphism for M, the zero morphism, and the inverse, respectively (defined in an obvious way). Then the required compatibility conditions immediately follows from the functoriality of $\mathcal{F}(T, -)$. $\square$

3. Polynomial Tambara functors

In this section, we consider G-bivariant analogs of the polynomial ring. Remark that, in the trivial group case, the polynomial ring satisfies the following properties.

Remark 3.1. Let R be a ring, and let $R[\mathbf{X}]$ be the polynomial ring over R with one variable. Then we have the following.

(1) (Existence of the indeterminate element) For any R-algebra S, we have a natural bijection

$$R\text{-}Alg(R[\mathbf{X}], S) \xrightarrow{\cong} S \; ; \; \varphi \mapsto \varphi(\mathbf{X}),$$

where R-*Alg* denotes the category of R-algebras.

(2) (Structural isomorphism) We have a natural isomorphism of rings

$$R[\mathbf{X}] \cong R \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\mathbf{X}] \cong R \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\mathbb{N}] \tag{3.1}$$

We propose two types of 'polynomial' Tambara functors, which satisfy analogous properties to those in Remark 3.1.

Theorem 3.2. *Let G be a finite group.*

(1) *There exists a functor*

$$po\ell_{\mathbf{x}} \colon Tam(G) \to Tam(G) \; ; \; T \mapsto T[\mathbf{x}],$$

which admits a natural bijection

$$T\text{-}Tam(G)(T[\mathbf{x}], S) \cong S^{\mu}(G/e)^G$$

for each $T \in \mathrm{Ob}(Tam(G))$ and $S \in \mathrm{Ob}(T\text{-}Tam(G))$.

(2) *There exists a functor*

$$po\ell_{\mathfrak{X}} \colon Tam(G) \to Tam(G) \; ; \; T \mapsto T[\mathfrak{X}],$$

which admits a natural bijection

$$T\text{-}Tam(G)(T[\mathfrak{X}], S) \cong S^{\mu}(G/G)$$

for each $T \in \mathrm{Ob}(Tam(G))$ and $S \in \mathrm{Ob}(T\text{-}Tam(G))$.

Moreover if G is trivial, each of these agrees with the functor taking the polynomial ring $po\ell \colon Ring \to Ring \; ; \; R \mapsto R[\mathbf{X}]$.

Proof. If we follow the analogy of (3.1), we can expect that each of the desired functors is of the form

$$\mathcal{F}(-,M)\colon \mathit{Tam}(G)\to \mathit{Tam}(G)$$

for some semi-Mackey functor M, which can be regarded as a 'G-bivariant analog of $\mathbb{N}$'.

To show (1), first we remark the following.

Remark 3.3 (Claim 3.8 in [4]). If M is a semi-Mackey functor on G, then $M(G/e)$ carries a natural G-monoid structure. The functor taking its G-fixed part

$$ev^G\colon \mathit{SMack}(G)\to \mathit{Mon}\ ;\ M\mapsto M(G/e)^G$$

admits a left adjoint functor

$$\mathcal{L}\colon \mathit{Mon}\to \mathit{SMack}(G)\ ;\ Q\mapsto \mathcal{L}_Q.$$

Combining this with Theorem 2.5, we obtain:

Corollary 3.4. *Let T be a Tambara functor on G. For any monoid Q and any T-Tambara functor S, we have an isomorphism*

$$T\text{-}\mathit{Tam}(G)(T[\mathcal{L}_Q],S)\cong \mathit{Mon}(Q,S^{\mu}(G/e)^G)$$

which is natural in Q and S.

Especially when $Q=\mathbb{N}$, then we obtain a natural bijection

$$T\text{-}\mathit{Tam}(G)(T[\mathcal{L}_{\mathbb{N}}],S)\cong S^{\mu}(G/e)^G.$$

Thus if we denote $T[\mathcal{L}_{\mathbb{N}}]$ by $T[\mathbf{x}]$, then $T[\mathbf{x}]$ satisfies the desired property in (1). $po\ell_{\mathbf{x}}$ is given by $po\ell_{\mathbf{x}}=\mathcal{F}(-,\mathcal{L}_{\mathbb{N}})$.

If G is trivial, (and thus T is identified with the ring $R=T(G/e)$,) then $T[\mathbf{x}]$ is naturally isomorphic to the polynomial ring $R[\mathbf{x}]$ over R, with an indeterminate element $\mathbf{x}$.

To show (2), we remark the following.

Remark 3.5. For any $X\in \mathrm{Ob}({}_G\mathit{set})$, the set of isomorphism classes $c\ell({}_G\mathit{set}/X)$ of the category ${}_G\mathit{set}/X$ forms a semi-ring. If we define $\mathfrak{A}$ by $\mathfrak{A}(X)=c\ell({}_G\mathit{set}/X)$, then $\mathfrak{A}$ becomes a semi-Tambara functor on G, called the *Burnside semi-ring functor*, with appropriately defined structure morphisms.

If we denote the isomorphism class of $(G/G\xrightarrow{\mathrm{id}}G/G)$ in $\mathfrak{A}(G/G)$ by $\mathfrak{X}$, then we have a natural isomorphism

$$\mathit{SMack}(G)(\mathfrak{A}^{\alpha},M)\cong M(G/G)\ ;\ \varphi\mapsto\varphi_{G/G}(\mathfrak{X})$$

for any $M\in \mathit{SMack}(G)$.

As a corollary of Theorem 2.5 and Remark 3.5, we obtain:

Corollary 3.6. *Let T be a Tambara functor on G. For any T-Tambara functor S, we have an isomorphism*

$$T\text{-}\mathit{Tam}(G)(T[\mathfrak{A}^{\alpha}],S)\cong S^{\mu}(G/G)$$

which is natural in S.

Thus if we denote $T[\mathfrak{A}^\alpha]$ abbreviately by $T[\mathfrak{X}]$, then $T[\mathfrak{X}]$ satisfies the desired property in (2). $po\ell_{\mathfrak{X}}$ is given by $po\ell_{\mathfrak{X}} = \mathcal{F}(-, \mathfrak{A}^\alpha)$.

If G is trivial, (and thus T is identified with the ring $R = T(G/G)$,) then $T[\mathfrak{X}]$ is naturally isomorphic to the polynomial ring $R[\mathfrak{X}]$ over R, with the indeterminate element $\mathfrak{X}$. □

Remark 3.7. We remark also that $T[\mathcal{L}_Q]$ is closely related to the Witt-Burnside ring. In fact, we have a natural isomorphism of commutative rings

$$T[\mathcal{L}_Q](G/G) \cong \mathbb{W}_G(\mathbb{Z}[Q]),$$

where the right hand side is the Witt-Burnside ring of the semi-group ring $\mathbb{Z}[Q]$ over G. (Theorem 3.9 in [4], Theorem 1.7 in [3]. See also [2].)

References

[1] S. Bouc.: *Green functors and G-sets*, Lecture Notes in Mathematics, 1671, Springer-Verlag, Berlin (1977).

[2] Brun, M.: *Witt vectors and Tambara functors*. Adv. in Math. **193** (2005) 233–256.

[3] Elliott, J.: *Constructing Witt-Burnside rings*. Adv. in Math. **203** (2006) 319–363.

[4] Nakaoka, H.: *Tambarization of a Mackey functor and its application to the Witt-Burnside construction*. arXiv:1010.0812.

[5] Oda, F; Yoshida, T: *Crossed Burnside rings III: The Dress construction for a Tambara functor*, preprint.

[6] Tambara, D.: *On multiplicative transfer*. Comm. Algebra **21** (1993), no. 4, 1393–1420.

[7] Tambara, D.: *Multiplicative transfer and Mackey functors*. manuscript.

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