Extention of Finite Solvable Torsors over a Curve

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Abstract. Let R be a discrete valuation ring with fraction field K and with algebraically closed residue field of positive characteristic p. Let X be a smooth fibered surface over R with geometrically connected fibers endowed with a section $x \in X(R)$. Let G be a finite solvable K-group scheme and assume that either $|G| = p^n$ or G has a normal series of length 2. We prove that every quotient pointed G-torsor over the generic fiber X_η of X can be extended to a torsor over X after eventually extending scalars and after eventually blowing up X at a closed subscheme of its special fiber X_s.

Mathematics Subject Classification: 14H30, 14L15.
Key words: solvable torsors, fibered surfaces, group schemes.

Contents

1 Introduction

Let S be a connected Dedekind scheme and $\eta = Spec(K)$ its generic point; let X be a scheme, $f : X \to S$ a faithfully flat morphism of finite type and $f_\eta : X_\eta \to \eta$ its generic fiber. Assume we are given a finite K-group scheme G and a G-torsor $Y \to X_\eta$. The problem of extending a torsor $Y \to X_\eta$ consists of searching a finite and flat S-group scheme G' whose generic fiber is isomorphic to G and a G'-torsor $T \to X$ whose generic fiber is isomorphic to $Y \to X_\eta$ as a G-torsor. Some solutions to this problems are known in some particular relevant cases, that we briefly recall hereafter. The first important answer to this problem is due to Grothendieck: he proves that, after eventually extending scalars, the problem has a solution when S is the spectrum of a complete discrete valuation ring with algebraically closed residue field of positive characteristic p, with X proper and smooth over S with geometrically connected fibers and $(|G|, p) = 1$ ([13], Exposé X, or [23], Theorem 5.7.10). When S is the spectrum of a discrete valuation ring of residue characteristic p, X is a proper and smooth curve over S then Raynaud suggests a solution, after eventually extending scalars, for G commutative of order a power of p ([20] §3). A similar problem has been studied by Saïdi in [21], §2.4 for formal curves of finite type and $G = (\mathbb{Z}/p\mathbb{Z})_K$. When S is the spectrum of a d.v.r. R of mixed characteristic $(0, p)$ Tossici provides a solution, after eventually extending scalars, for G commutative when X is a regular scheme, faithfully flat over S, with integral fibers provided that the normalization of X in Y has reduced special fiber ([24], Corollary 4.2.8). Finally in [3], §3.3 we provide a solution for G commutative, when S is a connected Dedekind scheme and $f : X \to S$ is a relative smooth curve with geometrically integral fibers endowed with a section $x : S \to X$ provided that Y is pointed over x_η (or, in higher dimension, a smooth morphism satisfying additional assumptions, cf. [3], §3.2). We stress that in this last case we do not need to extend scalars.

In this paper we study the problem of extending the G-torsor $Y \to X_\eta$ when G is finite and solvable. More precisely the aim of this paper is to prove the following:

Theorem 1.1. *(Theorem 3.20 and Corollary 3.21) Let R be a discrete valuation ring with fraction field K and with algebraically closed residue field of positive characteristic p. Let X be a smooth fibered surface over R with geometrically connected fibers endowed with a section $x \in X(R)$. Let G be a finite and solvable K-group scheme. We prove that every quotient pointed G-torsor over the generic fiber X_η of X can be extended to a torsor over X after eventually extending scalars and after eventually blowing up X at a closed subscheme of its special fiber X_s in the following two cases:*

1. *$|G| = p^n$;*
2. *G has a normal series of length 2.*

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2 Towers of torsors and solvable torsors

Notation 2.1. Throughout the whole paper every scheme will be supposed locally noetherian. Let X be a S-scheme, G a flat S-group scheme and Y a S-scheme endowed with a right action of G. A S-morphism $p : Y \to X$ is said to be a G-torsor if it is affine, faithfully flat, G-invariant and locally trivial for the fpqc topology. We say that a G-torsor is finite if G is finite and flat. Likewise we say that a G-torsor is commutative (resp. solvable) if G is commutative (resp. solvable). When we fix a section $x \in X(S)$ we say that a G-torsor $p : Y \to X$ is pointed if there exists a section $y \in Y_x(S)$.

2.1 Solvable torsors

Let S be any connected scheme, recall that a finite and flat S-group scheme G is said to be solvable if it has a normal series (or solvable series)

$$0 = H_n \triangleleft H_{n-1} \triangleleft .. \triangleleft H_1 \triangleleft H_0 = G \qquad (1)$$

where each H_i is a finite and flat S-group scheme and each quotient H_i/H_{i+1} exists as an S-group scheme and is finite, flat and commutative ($i = 0, .., n-1$). As usual n is called the length of such a normal series.

Remark 2.2. Recall that if the H_i are finite and flat then each H_i/H_{i+1} exists and is a finite and flat S-group scheme ([22], §3, Theorem).

Then observe that for G as in (1) a G-torsor $Y \to X$ can be seen as a tower of commutative torsors, each of them being a H_i/H_{i+1}-torsor: they are called the commutative components of the solvable G-torsor. If for instance $n = 2$ consider the contracted product $Y' := Y \times^G G/H_1$ ([7], III, §4, n° 3) in order to factor Y into a tower of two commutative torsors: a commutative G/H_1-torsor $Y' \to X$ and a commutative H_1-torsor $Y \to Y'$:

$$\begin{array}{ccc} Y & \xrightarrow{\;H_1\;} & Y' \\ & {\scriptstyle G}\searrow \qquad \swarrow{\scriptstyle G/H_1} & \\ & X & \end{array}$$

If $n > 2$ we iterate the process factoring $Y \to Y'$ and so on.

2.2 Towers of torsors

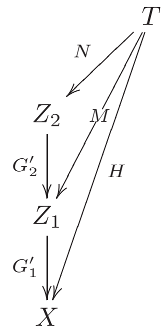
Now, let S be a connected Dedekind scheme and $\eta = Spec(K)$ its generic point, let X be a scheme, $f : X \to S$ a faithfully flat morphism of finite type and $f_\eta : X_\eta \to \eta$ its generic fiber. We assume the existence of a section $x \in X(S)$ and we denote by $x_\eta \in X_\eta(K)$ its generic fiber. We first consider the following general situation: we are given a finite K-group scheme G (here G is not necessarily solvable) and a G-torsor $Y \to X_\eta$ pointed in $y \in Y_{x_\eta}(K)$; we are looking for a model of G, i.e. a finite and flat S-group scheme G' whose generic fiber is isomorphic to G and a model of $Y \to X_\eta$, i.e. a G'-torsor $Z \to X$ whose generic fiber is isomorphic to $Y \to X_\eta$ as a G-torsor. Let G_2 be a non trivial K-finite (but not necessarily commutative) closed, normal subgroup scheme of G and $G_1 := G/G_2$; we can see $Y \to X_\eta$ as a tower of two torsors: a G_2-torsor $Y_2 = Y \to Y_1$ and a G_1-torsor $Y_1 \to X_\eta$ pointed in $y_1 \in Y_{1x}(K)$, image of y. We assume we are able to extend each component, i.e. there exist finite and flat S-group schemes G'_2 and G'_1 models (resp.) of G_2 and G_1, a G'_1-torsor $Z_1 \to X$ extending $Y_1 \to X_\eta$ and a G'_2-torsor $Z_2 \to Z_1$ extending $Y_2 \to Y_1$ (pointed resp. in $z_1 \in Z_{1x}(S)$ and $z_2 \in Z_{2z_1}(S)$, sections extending y_1 and y). Then we are in the situation described by the following diagram:

$$\begin{array}{ccc} Y = Y_2 & \longrightarrow & Z_2 \\ {\scriptstyle G_2}\downarrow & & \downarrow{\scriptstyle G'_2} \\ Y_1 & \longrightarrow & Z_1 \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle G'_1} \\ X_\eta & \longrightarrow & X \\ \downarrow & & \downarrow \\ Spec(K) & \longrightarrow & S \end{array} \tag{2}$$

In general $Z_2 \to X$ need not be a torsor, but from the tower $Z_2 \to Z_1 \to X$ we can obtain a torsor whose generic fiber is isomorphic to the G-torsor $Y \to X_\eta$, this is the object of the following:

Theorem 2.3. *The G-torsor $Y \to X_\eta$ can be extended to a finite pointed G'-torsor $Z \to X$ for some model G' of G if and only if the G_1-torsor $Y_1 \to X_\eta$ and the G_2-torsor $Y \to Y_1$ can be extended.*

Proof. The "only if" part is easy and left to the reader. So consider the tower of torsors $Z_2 \to Z_1 \to X$, that exists by assumption. Then by a result of Garuti ([8], §2, Theorem 1) there exist flat S-group schemes of finite type N, M and H, an S-scheme T (provided with $t \in T_x(S)$) and morphisms $T \to Z_2$, $T \to Z_1$ and $T \to X$ which are respectively a N-torsor, M-torsor and H-torsor (all pointed), such that the following diagram commutes:



then in particular there are canonical faithfully flat group scheme morphisms $\gamma_2 : M \to G'_2$ and $\gamma_1 : H \to G'_1$ over S where $M \simeq ker(\gamma_1)$ and $N \simeq ker(\gamma_2)$. First we observe that N is normal in H: indeed generically $N_\eta \trianglelefteq H_\eta$ because N_η is the kernel of the natural morphism $H_\eta \to G$; but N coincides with the schematic closure of N_η in H then $N \trianglelefteq H$. Hence we can construct the quotient H/N, which is a S-flat group scheme ([1] Théorème 4.C) that fits in the following exact sequence ([7], III, §3, n° 3, 3.7 a))

$$0 \longrightarrow G'_2 \longrightarrow H/N \longrightarrow G'_1 \longrightarrow 0 \tag{3}$$

then it is finite since G'_2 and G'_1 are ([5], Proposition 9.2, (viii)). Let $\gamma : H \to (H/N)$ be the canonical faithfully flat morphism. Thus we construct the contracted product $Z := T \times^H (H/N)$ via γ which is a H/N-torsor. The contracted product commuting with base change ([7], III, §4, n° 3, 3.1), we have $Z_\eta := (T \times^H (H/N))_\eta \simeq T_\eta \times^{H_\eta} H_\eta/N_\eta$ then in particular $T_\eta \times^{H_\eta} H_\eta/N_\eta \simeq Y$ as a G-torsor over X hence Z is a H/N-torsor over X extending the starting one. □

Remark 2.4. Let T_1 and T_2 be (resp.) a G_1-torsor over X pointed in $t_1 \in T_{1x}(S)$ and a G_2-torsor over X pointed in $t_2 \in T_{2x}(S)$. Recall that a X-morphism $T_1 \to T_2$ sending $y_1 \mapsto y_2$ commutes necessarily with the actions of their structural group schemes. We have implicitly used this fact in previous lemma without mentioning it.

Remark 2.5. Keeping notations of theorem 2.3 observe that Z factors through Z_1 and in particular $Z \to Z_1$ is a G'_2-torsor. Indeed

$$Z \times^{H/N} G'_1 \simeq (T \times^H H/N) \times^{H/N} G'_1 \simeq T \times^H G'_1 \simeq Z_1$$

then $Z \to Z_1$ is a $ker(H/N \to G'_1)$-torsor.

Corollary 2.6. *Let G be a finite and solvable K-group scheme and $Y \to X_\eta$ a G-torsor. Then $Y \to X_\eta$ can be extended to a finite solvable G'-torsor $Z \to X$ for some model G' of G if and only if its commutative components can be extended.*

Proof. The case where G has a normal series of length $n = 2$ is exactly theorem 2.3. With a little effort this procedure can be generalized to the case where G has no normal series of length $n = 2$, simply repeating Garuti's construction and theorem 2.3. □

3 Extension of solvable torsors

In the situation of diagram (2) we now assume that G_1 and G_2 are commutative. In [3], Theorem 3.1 we have explained how to extend finite quotient[1] pointed commutative torsors from X_η to X where X needs to satisfy some strong assumptions ([3], Notation 2.20). Thus for such X, it is not difficult to find a finite, flat and commutative S-group scheme G_1' as well as a G_1'-torsor $Z_1 \to X$ that extends the G_1-torsor $Y_1 \to X_\eta$. Unfortunately, even if we can easily find schemes X satisfying these strong conditions (loc. cit. §3.2), it is improbable that $Z_1 \to S$ satisfy the same assumptions, even in the case of curves: for instance it is asked X to be smooth but in general Z_1 is not. So it is necessary to weaken the assumptions on X hoping that $Z_1 \to S$ is nice enough to be able to construct over Z_1 a torsor extending the G_2-torsor $Y \to Y_2$.

3.1 Commutative torsors

For the sake of completeness we recall in a few lines the definition of Néron model and some properties which will be used in this paper. The reader can refer to [6] for a deep discussion on the subject. Here we only consider Néron models of abelian varieties since it is the only case we will use. For the same reason the base scheme S we consider will be the spectrum of a discrete valuation ring R:

Definition 3.1. Let R be a d.v.r. with fraction field K. Let A be an abelian variety over K. A Néron model of A is a smooth and separated R-scheme of finite type $\mathcal{N}_A$ whose generic fiber is isomorphic to A and which satisfies the following universal property (called the Néron mapping property): for each smooth R-scheme Y and each K-morphism $u : Y_\eta \to A$ there exists a unique morphism $u' : Y \to \mathcal{N}_A$ extending u where as usual Y_η denotes the generic fibre of Y.

Proposition 3.2. *We keep notation of definition 3.1. Then A admits a Néron model $\mathcal{N}_A$ over R.*

Proof. See for instance [6], §1.3, Corollary 2. □

By the Néron mapping property the Néron model $\mathcal{N}_A$ of A is unique up to canonical isomorphism and it is a commutative group scheme. Unfortunately in general $\mathcal{N}_A$ is not an abelian scheme and not even a semi-abelian scheme.

[1] Over any base scheme S a pointed G-torsor $Y \to X$ over S is said to be quotient if X has a fundamental group scheme $\pi_1(X, x)$ (cf. for instance [4], where the existence of the fundamental group scheme is studied) and the canonical morphism of S-group schemes $\pi_1(X, x) \to G$ is faithfully flat

When $\mathcal{N}_A$ is an abelian scheme then we simply say that A has abelian (or good) reduction. If $\mathcal{N}_A$ is not an abelian scheme but there exists a finite Galois extension L/K such that the Néron model $\mathcal{N}_{A_L}$ of $A_L := A \times_{Spec(K)} Spec(L)$ is an abelian scheme over the integral closure R' of R in K then we say that A has potentially abelian (or potentially good) reduction.

Let X be an S-scheme and $X \to S$ a proper morphism of finite type, then in what follows we denote by $Pic_{(X/S)(fppf)}$ the sheaf, in the fppf topology, associated to the relative Picard functor given by

$$Pic_{X/S}(T) := Pic(X \times_S T)/Pic(T)$$

for any S-scheme T (see [3], §2 for a brief introduction and [15] for a complete reference on this topic)[2]. It is known that for any $s \in S$ the sheaf $Pic_{(X_s/k(s))(fppf)}$ is represented by a group scheme $\mathbf{Pic}_{X_s/k(s)}$ whose identity component is denoted by $\mathbf{Pic}^0_{X_s/k(s)}$; over S we denote by $Pic^0_{X/S}$ the subfunctor of $Pic_{(X/S)(fppf)}$ which consists to all elements whose restrictions to all fibers $X_s, s \in S$ belong to $\mathbf{Pic}^0_{X_s/k(s)}$. We recall the following result concerning the representability of $Pic^0_{X/S}$:

Theorem 3.3. *Let S be the spectrum of a d.v.r. R and let $\eta := Spec(K)$ be its generic point. Let $f : X \to S$ be a regular fibered surface (i.e. a projective flat morphism with X an integral, regular scheme of dimension 2) with geometrically integral and smooth generic fiber X_η and provided with a section $x \in X(S)$. Then $Pic^0_{X/S}$ is represented by a separated and smooth S-scheme $\mathbf{Pic}^0_{X/S}$ and coincides with the identity component of the Néron model of $J := \mathbf{Pic}^0_{X_\eta/K}$.*

Proof. First we recall that under these assumptions J is an abelian variety. According to [6], §9.5 Remark 5 the existence of a section implies that the greatest common divisor of the geometric multiplicities of the irreducible components of the special fiber X_s of X in X_s is one. Then by loc. cit. §9.5, Theorem 4, $Pic^0_{X/S}$ is represented by a separated and smooth S-scheme $\mathbf{Pic}^0_{X/S}$ which coincides with the identity component of the Néron model of $J := \mathbf{Pic}^0_{X_\eta/K}$. □

Let us denote by $u : X_\eta \to J$ the canonical closed immersion (cf. for instance [15] Exercise 9.4.13) usually known as the Abel-Jacobi map. In next proposition we construct, when possible, a morphism $u' : X \to \mathcal{N}_J$ whose generic fiber is isomorphic to u, where $\mathcal{N}_J$ denotes the Néron model $\mathcal{N}_J$ of J.

Proposition 3.4. *Let S, R, K be as in theorem 3.3. Let $f : X \to S$ be a regular fibered surface with geometrically integral and smooth generic fiber X_η and provided with a section $x \in X(S)$. Let J be the Jacobian of X_η, $\mathcal{N}_J$ its Néron model and $u : X_\eta \to J$ the canonical closed immersion. Assume moreover that J has abelian reduction. Then there exists a morphism $u' : X \to \mathcal{N}_J$ whose generic fiber is isomorphic to u.*

[2]N.B.: here we have used Kleiman's notation. In [6], §8.1, Definition 2, however, our $Pic_{(X/S)(fppf)}$ is called "the relative Picard functor" and denoted $Pic_{X/S}$.

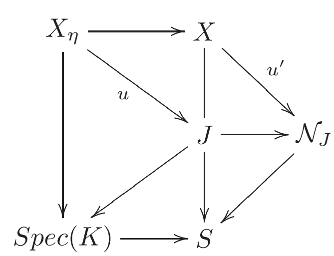
Proof. If X were smooth this would be the Néron mapping property of the Néron model $\mathcal{N}_J$. Since in general this does not happen then we argue as follows: by assumption $\mathcal{N}_J$ is an abelian scheme (thus proper), then construct the schematic closure $C := \overline{X_\eta}$ of X_η in $\mathcal{N}_J$, i.e. the only closed subscheme of $\mathcal{N}_J$, flat over S with generic fiber isomorphic to X_η. It is an integral scheme ([10] Proposition 9.5.9), proper over S (because $\mathcal{N}_J$ is) whose special fiber is equidimensional of dimension one ([16], Ch. 4, Proposition 4.16). Now we desingularize C, i.e. we construct a projective ([16], Ch. 8, Theorem 3.16) regular model $\widetilde{C}$ of X_η and a morphism $\widetilde{C} \to C$ which is generically an isomorphism. In particular $\mathcal{N}_J \simeq \mathcal{N}_J^0 \simeq \mathbf{Pic}^0_{X/S} \simeq \mathbf{Pic}^0_{\widetilde{C}/S}$ by theorem 3.3 and from $\widetilde{C} \to \mathbf{Pic}^0_{X/S}$ one obtains the desired morphism $u' : X \to \mathcal{N}_J$. Indeed the morphism $\widetilde{C} \to \mathbf{Pic}^0_{X/S}$ is an element of $\mathbf{Pic}^0_{X/S}(\widetilde{C})$, then in particular this corresponds to an element

$$\xi \in \frac{Pic(X \times \widetilde{C})}{Pic(\widetilde{C})} = Pic_{X/S}(\widetilde{C}) = Pic_{(X/S)(fppf)}(\widetilde{C})$$

(use [16], Ch. 8, Corollary 3.6, (c) then apply [6], §8.1 Proposition 4) but since $\widetilde{C}$ and X are both regular then $Pic(X) \simeq Pic(X_\eta) \simeq Pic(\widetilde{C})$ and consequently

$$\frac{Pic(X \times \widetilde{C})}{Pic(\widetilde{C})} \simeq \frac{Pic(X \times \widetilde{C})}{Pic(X)} = Pic_{\widetilde{C}/S}(X) = Pic_{(\widetilde{C}/S)(fppf)}(X).$$

Hence, starting from ξ, we get a morphism $X \to Pic_{(\widetilde{C}/S)(fppf)}$ that on the generic and special fibers factors (resp.) through $u : X_\eta \to J$ and $X_s \to \mathbf{Pic}^0_{\widetilde{C}_s/k(s)}$ (here s denotes the special point) since X has geometrically connected fibers, thus obtaining a morphism $X \to \mathbf{Pic}^0_{\widetilde{C}/S}$ that composed with $\mathbf{Pic}^0_{\widetilde{C}/S} \simeq \mathcal{N}_J^0 \simeq \mathcal{N}_J$, gives the desired morphism $u' : X \to \mathcal{N}_J$ extending $u : X_\eta \to J$ as described by the following diagram:



□

A result due to Raynaud, that we state in our setting in the following theorem, shows that the hypothesis of proposition 3.4 are satisfied in many relevant cases after eventually extending scalars:

Theorem 3.5. *Let R be a complete d.v.r. with residue characteristic $p > 0$ and fraction field K. Let X be a smooth fibered surface over R with geometrically* *connected generic fiber and provided with a section $x \in X(S)$. Let G be a finite and étale K-group scheme of order p^n and $Y \to X_\eta$ a quotient G-torsor over the generic fiber of X, then the Jacobian J_Y of Y has potential abelian reduction. In particular every commutative component Y_i of Y has a Jacobian J_{Y_i} with potential abelian reduction.*

Proof. It is known that G becomes constant after a finite Galois extension L of K. Moreover since X_η is geometrically connected then so is X_s ([16], Ch. 8, Corollary 3.6, (c)), then finally use [20], Théorème 1. □

We conclude this section with a result that will be used later:

Theorem 3.6. *Let S be the spectrum of a d.v.r. R and let $\eta := Spec(K)$ be its generic point. Let $f : X \to S$ be a regular fibered surface provided with a section $x \in X(S)$. Assume that f has smooth generic fiber $X_\eta \to \eta$. Assume moreover that the Jacobian J of X_η has abelian reduction. Then every finite, quotient, commutative, pointed torsor over X_η can be extended to a finite commutative pointed torsor over X.*

Proof. Let $\mathcal{N}_J$ be the Néron model of J and $u' : X \to \mathcal{N}_J$ the morphism obtained in proposition 3.4. Let G be a finite and flat K-group scheme, then according to [3], Corollary 3.8 we know that every finite, quotient and commutative G-torsor $T' \to X_\eta$ (pointed over $x_\eta \in X_\eta(K)$, generic fiber of x) is the pull back of a finite, quotient and commutative G-torsor $T \to J$ (pointed over 0_J). Now it is easy to find an R-model H of G (commutative, finite and flat) and a pointed (over $0_{\mathcal{N}_J}$) H-torsor $Y \to \mathcal{N}_J$ whose generic fiber is isomorphic to $T \to J$ (cf. for instance [2], §2.2). Then finally $Y' := Y \times_{\mathcal{N}_J} X$, the pull back over u', is a finite, commutative H-torsor over X (pointed over x) extending $T' \to X_\eta$. □

3.2 Solvable torsors over curves

Notation 3.7. From now on S will be the spectrum of a complete discrete valuation ring R with algebraically closed residue field k of positive characteristic p and with fraction field K. We will denote by $\eta := Spec(K)$ and $s := Spec(k)$ respectively the generic and special points. Moreover $f : X \to S$ will be a regular fibered surface provided with a section $x \in X(S)$ with smooth and geometrically connected (then geometrically integral) generic fiber $X_\eta \to Spec(K)$, pointed in x_η. Using a standard convention we say that a S-morphism of schemes $Z_1 \to Z_2$ is a model map if it is generically an isomorphism.

Remark 3.8. Let Y be any fibered surface over S. If we have a section $y \in Y(S)$ its generic fiber Y_η is geometrically connected if and only if it is connected; moreover if Y_η is geometrically reduced (resp. geometrically irreducible) then Y_η is reduced (resp. irreducible) ([16], Ch. 3, §2, ex. 2.11 and 2.13). Of course the same is true for the special fiber Y_s. Finally we recall that if Y_η is integral then so is Y ([10] Proposition 9.5.9). The Néron blowing up of Y at a closed subscheme C of Y_s will be denoted by Y^C: the reader should refer to [6], §3.2,

[25], §1 or [1], §2.1 for the definition and properties. We are not making any assumption on the characteristic of K.

Before stating the principal result we need some preliminary lemmas. Lemma 3.9, as recalled in its proof, slightly generalizes [25] Theorem[3] 1.4, that we strongly use.

Lemma 3.9. *Let Y and $\widetilde{Y}$ be two schemes faithfully flat and of finite type over S and $h : \widetilde{Y} \to Y$ an affine model map. Then h is isomorphic to a composite of a finite number of Néron blowing ups.*

Proof. First we observe that if the special fiber $h_s : \widetilde{Y}_s \to Y_s$ of h is a schematically dominant morphism (i.e. $\mathcal{O}_{Y_s} \hookrightarrow h_{s*}(\mathcal{O}_{\widetilde{Y}_s})$ is injective) then h is an isomorphism: indeed let $U = Spec(A)$ be any open affine subset of Y and $V = Spec(A') := h^{-1}(U)$ then consider $h_{|V} : V \to U$ and its special fiber $(h_{|V})_s : V_s \to U_s$ where $V_s = Spec(A'_k) = Spec(A' \otimes_R k)$ and $U_s = Spec(A_k) = Spec(A' \otimes_R k)$. We are thus reduced to consider the affine case, then one just needs to argue as in [25], Lemma 1.3.
Now we prove the statement of the lemma: if $h_s : \widetilde{Y}_s \to Y_s$ is schematically dominant there is nothing to do, otherwise consider the scheme theoretic image $C_1 := h_s(\widetilde{Y}_s)$ of $\widetilde{Y}_s$ in Y_s. It is a closed subscheme of Y_s ([10], §9.5). Now consider the Néron blowing up Y^{C_1} of Y in C_1 then h factors through $Y_1 := Y^{C_1}$. Denote by $h_1 : Y \to Y_1$ the S-morphism obtained. If its special fiber $(h_1)_s$ is schematically dominant then $h_1 : Y \simeq Y_1$ otherwise we set $C_2 := (h_1)_s(\widetilde{Y}_s)$, $Y_2 := Y_1^{C_2}$ and we continue as before. Hence we conclude that $\widetilde{Y} \simeq \varprojlim_i Y_i$. If Y and $\widetilde{Y}$ are affine then one argues as in [25] Theorem 1.4 to conclude that we can stop after a finite number of steps, i.e. there exist $n \geq 0$ such that $\widetilde{Y} \simeq Y_n$. If Y and $\widetilde{Y}$ are not affine then let $\{U_j\}_{j\in J}$ be an affine open cover of Y and $\left\{V_j := h^{-1}(U_j)\right\}_{j\in J}$ the induced affine open cover of $\widetilde{Y}$. Since Y is quasi compact we can take $|J| < \infty$. To give h is equivalent to give the family of morphisms

$$\{h_j := (V_j \xrightarrow{h_{|V_j}} U_j \longrightarrow Y)\}_{j\in J}$$

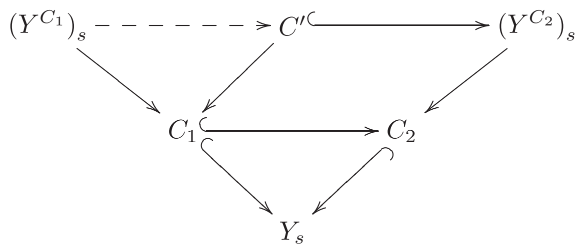
where we have given the V_j and U_j the induced subscheme structure ([14], II, Theorem 3.3, Step 3). For any $j \in J$ set $C_j^1 := C_1 \times_Y U_j$ (the scheme theoretic image $h_{|V_j}(V_j)$), $U_j^1 := {U_j}^{C_j^1}$ and so on: it follows that $V_j \simeq \varprojlim_i {U_j}^i$ but since U_j and V_j are affine then the projective limits become stable after $n(j) \geq 0$ steps. Take $n := max_{j\in J}\{n(j)\}$: this is the number of steps after which we can stop. ☐

Lemma 3.10. *Let Y be a scheme faithfully flat and of finite type over S, C_2 a closed subscheme of Y_s and C_1 a closed subscheme of C_2. Denote by Y^{C_i} the*

[3] This result is stated by Waterhouse and Weisfeiler only for affine group schemes but, as observed by the authors, the group structure is never used ([25], page 552, Remark (4)).

Néron blowing up of Y *in* C_i $(i = 1, 2)$*. Let* $C' := Y^{C_2} \times_{C_2} C_1$ *the induced closed subscheme of* $(Y^{C_2})_s$ *then* $Y^{C_1} \simeq (Y^{C_2})^{C'}$*.*

Proof. This follows directly from the universal property of the Néron blowing up and the following diagram:



□

Lemma 3.11. *Let* Y *be an integral fibered surface. Let* $f : Y \to X$ *be a finite and flat morphism,* C *a closed subscheme of* Y_s *and* Y^C *the Néron blowing up of* Y *in* C*. Assume that the canonical morphism* $h : Y^C \to Y$ *is a finite model map. Then there exist a regular fibered surface* X' *and a finite model map* $X' \to X$ *such that* $Y^C \simeq Y \times_X X'$*.*

Proof. Let $f_s : Y_s \to X_s$ be the special fiber of f and $D_1 := f_s(C)$ the scheme theoretic image of C: it is a closed subscheme of X_s. Now consider the fiber product $C_1 := D_1 \times_{X_s} Y_s$ and the natural closed immersion $C \hookrightarrow C_1$: if it is an isomorphism then, by the universal propertiy of the Néron blowing up, $Y^C \simeq X^{D_1} \times_X Y$ hence $X' := X^{D_1}$ is the required solution. Otherwise let $Y_1 := Y^{C_1}$, $X_1 := X^{D_1}$ and $f_1 : Y_1 \to X_1$ the pull back of f over $X_1 \to X$. The morphism $Y^C \to Y$ now factors through Y_1; then we analyze the morphism $Y^C \to Y_1$: by lemma 3.10 $Y^C \simeq Y_1^{C_1'}$ where $C_1' := C \times_{C_1} (Y_1)_s$, thus we are in the same situation as before: let $D_2 := (f_1)_s(C_1')$, $C_2 := D_2 \times_{(X_1)_s} (Y_1)_s$, $Y_2 := {Y_1}^{C_2}$, $X_2 := X_1^{D_2}$, $C_2' := C_1' \times_{C_2} (Y_2)_s$ and so on. We finally obtain the isomorphism $Y^C \simeq \varprojlim_i Y_i$ (where $Y_0 := Y$). Now using arguments similar to those used in the last part of the proof of lemma 3.9 we are reduced to study the case where X (then also Y and Y^C) is affine: so let us set $Y_i := Spec(A_i)$ and $Y^C = Spec(B)$ then since every A_i is integral the morphisms $Y_i \to Y_{i-1}$ induce a sequence of inclusions

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq .. \subseteq A_i \subseteq .. \subseteq B;$$

since B, as a A_0-module, is finite then it is generated by a finite number of elements $\{b_j\}_j \subset B$ so there exists an integer $n \geq 0$ such that $\{b_j\}_j \subset A_n$. Hence $Y^C \simeq Y_n$ and $X' := X_n$ allows us to conclude. □

Remark 3.12. In lemma 3.11 we never use the assumption that the absolute dimension of X is 2, but it is the only case of interest in this paper. The assumption that the residue field is algebraically closed will be used in lemma 3.15 and not before.

Lemma 3.13. *Let $f : Y \to X$ be a finite and flat morphism with Y integral. Let $h : \widetilde{Y} \to Y$ be a finite model map. Then there exist a regular fibered surface $\widetilde{X}$ and a model map $\widetilde{X} \to X$ such that $\widetilde{Y} \simeq Y \times_X \widetilde{X}$. Moreover $\widetilde{X} \to X$ is isomorphic to a composite of a finite number of Néron blowing ups.*

Proof. This is just a consequence of lemmas 3.9 and 3.11. □

Corollary 3.14. *Let $f : Y \to X$ be a finite and flat morphism with Y integral. Let $h : \widetilde{Y} \to Y$ be the normalization morphism. Assume that the generic fiber Y_η of Y is smooth and geometrically integral. Then there exist a regular fibered surface $\widetilde{X}$ and a model map $\widetilde{X} \to X$ such that $\widetilde{Y} \simeq Y \times_X \widetilde{X}$.*

Proof. In this context the normalization morphism $h : \widetilde{Y} \to Y$ is a finite (then affine) model map ([16], Ch 8, Lemma 3.49). Then the result is just a consequence of lemma 3.13. □

Lemma 3.15. *Let G be a finite and flat S-group scheme with infinitesimal special fiber G_s and $f : Y \to X$ a G-torsor. Assume that the generic fiber Y_η of Y is smooth and geometrically integral. Let moreover $\widetilde{Y} \to Y$ be the blowing-up of Y centered at a point q of the special fiber Y_s of Y. Then $\widetilde{Y} \simeq Y \times_X \widetilde{X}$ where $\widetilde{X} \to X$ is the blowing up centered at $p := f_s(q)$.*

Proof. The residue field k being algebraically closed then $q : Spec(k) \to Y$ and also $p : Spec(k) \to X$ ([16], Ch.2 ex. 5.9). Thus, since $f_s : Y_s \to X_s$ is a G_s-torsor, $p \times_X Y \simeq G_s$ and the canonical closed immersion $q \to G_s$ identifies q with $(G_s)_{\text{red}}$ (recall that G_s is infinitesimal). Then the blowing up Y' of Y centered at $p \times_X Y$ is isomorphic to the blowing up of Y centered at q ([16], Ch. 2, ex. 3.11 (a)). But since $Y' \simeq Y \times_X \widetilde{X}$ ([16], Ch. 8, Proposition 1.12 (c)) then $\widetilde{Y} \simeq Y \times_X \widetilde{X}$, as required. □

Remark 3.16. Let Y be any fibered surface over S with smooth generic fiber Y_η: the canonical desingularization of Y is the sequence of blowing ups

$$.. \to Y_i \to Y_{i-1} \to Y_{i-2} \to .. \to Y_1 \to Y_0 = Y \tag{4}$$

where for each i, the morphism $Y_i \to Y_{i-1}$ denotes

- the normalization morphism if i is odd (it can eventually be an isomorphism if Y_{i-1} is already normal);
- the blowing up at the singular points of Y_{i-1} if i is even.

Recall that at each step, when i is even, the set $Sing(Y_{i-1})$ of singular points of Y_{i-1} is a finite set of points contained in the special fiber $(Y_{i-1})_s$. According to [16], Ch.8, Corollary 3.51, there exists an integer $n \geq 0$ such that $\widetilde{Y} := Y_n$ is regular and the morphism $\widetilde{Y} \to Y$ is a model map.

Proposition 3.17. *Let G be a finite and flat S-group scheme with infinitesimal special fiber G_s and $f : Y \to X$ a G-torsor. Assume that the generic fiber Y_η of Y is smooth and geometrically integral. Let moreover $\widetilde{Y} \to Y$ be the canonical desingularization of Y. Then there exist a regular fibered surface $\widetilde{X}$ and a morphism $\widetilde{X} \to X$ such that $\widetilde{Y} \simeq Y \times_X \widetilde{X}$. In particular $\widetilde{Y} \to \widetilde{X}$ is a G-torsor.*

Proof. According to previous discussion $\widetilde{Y} \to Y$ is a sequence of normalization morphisms (which are finite morphisms) and blowing ups centered at a finite set of singular points. Then in order to conclude it is sufficient to use lemma 3.15 and corollary 3.14. □

Before stating the main theorem of this paper we need a last lemma:

Lemma 3.18. *Let $Z \to X$ be a finite $(\mathbb{Z}/p\mathbb{Z})_R$-torsor. Then there exist a finite and flat R-group scheme G with infinitesimal special fiber, a G-torsor $Y \to X$ and a model map $\varphi : Z \to Y$ commuting with the actions of $(\mathbb{Z}/p\mathbb{Z})_R$ and G.*

Proof. That a model map $\rho : (\mathbb{Z}/p\mathbb{Z})_R \to G$ such that G_s is infinitesimal exists is clear from [17], §3.2 when $char(K) = p$ and from [19], I, §2, when $char(K) = 0$, then the model map $\varphi : Z \to Y$ is given by the contracted product (through ρ) $Y = Z \times^{(\mathbb{Z}/p\mathbb{Z})_R} G$. □

Remark 3.19. The G-torsor $Y \to X$ obtained in lemma 3.18 has trivial special fiber but this will not affect the following discussion.

Theorem 3.20. *Let X be a proper and smooth fibered surface over R with geometrically connected fibers and provided with a section $x \in X(S)$. Let G be a finite, étale, solvable K-group scheme of order p^n and $Y \to X_\eta$ a quotient G-torsor, pointed in $y \in Y_x(K)$. Then, after eventually a finite extension of scalars, there exist a regular fibered surface $\widetilde{X}$, a model map $\widetilde{X} \to X$, a finite flat and solvable R-group scheme G' of order p^n such that $Y \to X_\eta$ can be extended to a G'-torsor $Y' \to \widetilde{X}$. Moreover we can construct Y' in such a way to make it regular.*

Proof. First of all we observe that we can decompose $Y \to X_\eta$ into a tower of n torsors $Y_1 \to X_\eta$, $Y_i \to Y_{i-1}$ (for $i = 2, .., n$, where $Y_n = Y$) each one being a quotient pointed G_i-torsor where $|G_i| = p$. After eventually extending scalars (as explained in theorem 3.5), we can assume that $G_i \simeq (\mathbb{Z}/p\mathbb{Z})_K$ (for all $i = 1, .., n$) and that the Jacobian J_{Y_i} has abelian reduction. Assume first that $n = 2$: according to theorem 3.6 there exist a finite and flat R-group scheme G_1' of order p, generically isomorphic to G_1, and a G_1'-torsor $Z_1 \to X$ extending $Y_1 \to X_\eta$. We can assume by lemma 3.18 that $(G_1)_s$ is infinitesimal. If Z_1 is regular we go on extending $Y_2 \to Y_1$, otherwise we desingularize Z_1 as recalled in remark 3.16, i.e. we find a regular fibered surface Z_1' and a model map $Z_1' \to Z_1$. Moreover by proposition 3.17 there exist a regular fibered surface X' and a model map $X' \to X$ such that $Z_1' \to X'$ is a G_1'-torsor. Now we proceed as before: there exist a finite and flat R-group scheme G_2' of order p, generically isomorphic to G_2, and a G'_2-torsor $Z_2 \to Z'_1$ extending $Y_2 \to Y_1$. Again we can assume that $(G_2)_s$ is infinitesimal. Then by theorem 2.3 there exist a finite, flat, infinitesimal S-group scheme G' generically isomorphic to G and a G'-torsor $Z \to X'$ extending $Y \to X_\eta$ and we are done setting $Y' := Z$. We only mention how to proceed when $n > 2$: we start from Z and, as before, we desingularize it, i.e. we find a regular fibered surface Z'_2 and a model map $Z'_2 \to Z$. As before there exist a regular fibered surface $X'' \to X'$ such that $Z'_2 \to X''$ is a G'_1-torsor; then we can extend $Y_3 \to Y_2$ to a torsor over Z'_2 and so on. We argue in the same way to prove that we can find a regular Y' (if it is not we desingularize, etc.). □

Corollary 3.21. *Let X be a proper and smooth fibered surface over R with geometrically connected fibers and provided with a section $x \in X(S)$. Let G be a finite, étale, K-group scheme having a normal series of length $n = 2$. Let $Y \to X_\eta$ be a quotient G-torsor, pointed in $y \in Y_x(K)$. Then, after eventually extending scalars, there exist a regular fibered surface $\widetilde{X}$, blowing up of X at a closed subcheme of X_s, a finite flat and solvable R-group scheme G' such that $Y \to X_\eta$ can be extended to a G'-torsor over $\widetilde{X}$.*

Proof. We can assume that the K-group scheme G is constant (it is always true after eventually extending scalars). Let us decompose $Y \to X_\eta$ into a tower of two commutative torsors: a G_1-torsor $Y_1 \to X_\eta$ and a G_2-torsor $Y \to Y_1$. If $p \nmid |G_1|$ then the problem has an easy answer, otherwise let p^n be the maximal p-power dividing $|G_1|$ and ${}^pG_1$ a (normal) K-subgroup of G_1 of order p^n. Then the Jacobian J_{Y_1} of Y_1 has potentially abelian reduction. Indeed Y_1 can be decomposed into a tower of two torsors: a ${}^pG_1$-torsor $Y_1 \to T$ and a $G_1/{}^pG_1$-torsor $T \to X_\eta$. The latter can be extended, after eventually extending scalars, to a finite and étale torsor $T' \to X$ (we refer the reader to the introduction of this paper) then we apply theorem 3.5 to $Y_1 \to T$. Now we forget this decomposition for $Y_1 \to X_\eta$ and we assume that over K the Jacobian J_{Y_1} has abelian reduction (we have seen it is always true after eventually extending scalars). We would rather consider the following decomposition for $Y_1 \to X_\eta$ as a tower of two torsors: a ${}^pG_1$-torsor $P \to X_\eta$ and a $G_1/{}^pG_1$-torsor $Y_1 \to P$. Theorem 3.5 tells us that the Jacobian J_P of P has potentially abelian reduction; again we can assume that it has in fact abelian reduction. Hence according to theorem 3.20 there exist a regular fibered surface $\widetilde{X}$, a model map $\widetilde{X} \to X$, a finite flat and commutative R-group scheme H_1 such that $P \to X_\eta$ can be extended to a H_1-torsor $P' \to \widetilde{X}$ with P' regular. Furthermore by theorem 3.6 there exists a finite flat and commutative R-group scheme H_2 such that $Y_1 \to P$ can be extended to a H_2-torsor $Y'_1 \to P'$; by theorem 2.3 there exist a finite and flat S-group scheme H generically isomorphic to G_1 and a H-torsor $Z \to \widetilde{X}$ extending $Y_1 \to X_\eta$. Since $p \nmid |H_2|$ then H_2 is étale; moreover $Z \to \widetilde{X}$ factors through P', more precisely $Z \to P'$ is a H_2-torsor (remark 2.5) so $Z \to P'$ is smooth, then Z is regular as P' is (see for instance [6], §2.3 Proposition 9). Finally we can apply again theorem 3.6 to $Y \to Y_1$ and 2.3 in order to conclude. □

Remark 3.22. It is obvious that the tools we have presented allows us to extend solvable torsors even if they do not have a normal series of length 2 but only in some particular cases, for example if every commutative component Y_i of the torsor $Y \to X_\eta$ has a Jacobian J_{Y_i} that has potentially abelian reduction. As clear from the proof of corollary 3.21 this condition is satisfied, for instance, when all the G_i but G_1 have order not divisible by p.

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