SPLINE ELEMENT METHOD FOR THE MONGE-AMPÈRE EQUATION

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Abstract. We consider numerical approximations of the Monge-Ampère equation det $D^2u = f, f > 0$ with Dirichlet boundary conditions on a convex bounded domain Ω in $\mathbb{R}^n, n = 2, 3$. We make a comparative study of three existing methods suitable for finite element computations. We construct conforming approximations in the framework of the spline element method where constraints and interelement continuities are enforced using Lagrange multipliers.

1. Introduction

This paper addresses the numerical solution of the Dirichlet problem for the Monge-Ampère equation

$$\det D^2 u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{1.1}$$

where $D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n}$ is the Hessian of u and f, g are given functions with $f > 0$. The domain $\Omega \subset \mathbb{R}^n$ is a convex domain with polygonal boundary $\partial\Omega$.

The above equation is a fully nonlinear equation in the sense that it is nonlinear in the highest order derivatives. Fully nonlinear equations have in general multiple solutions, and even if the domain is smooth, the solution may not be smooth. For the Monge-Ampère equation, the notion of generalized solution in the sense of Alexandrov-Bakelman and that of viscosity solution [39] are the best known to give a meaning to the second derivatives even when the solution is not smooth. To a continuous convex function, one associates the so-called Monge-Ampère measure and (1.1) is said to have a solution in the sense of Alexandrov if the density of that measure with respect to the Lebesgue measure is equal to f. Continuous convex viscosity solutions are defined " in terms of certain inequalities holding wherever the graph of the solution is touched on one side or the other by a smooth test function " [44]. In the case of (1.1) the two notions are equivalent for f continuous, [39]. We will assume throughout this paper that f is continuous on Ω and g continuous on $\partial\Omega$. Equation (1.1) then has at most two solutions when $n = 2$, [24] p. 324. and a unique generalized solution in the class of convex functions, [1, 23]. In general, the theory of viscosity solutions [25, 19, 39] provides a framework for existence and uniqueness of solutions of fully nonlinear equations.

The more general Monge-Ampère equation has form

$$\det D^2 u = H(x, u, Du) \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{1.2}$$

where Du denotes the gradient of u and H is is a given Hamiltonian, at least continuous and nondecreasing in u. They appear in various geometric and variational problems, e.g. the Monge-Kantorovich problem, and in kinetic theory. They also appear in applied fields such as meteorology, fluid mechanics, nonlinear elasticity, antenna design, material sciences and mathematical finance. A huge amount of literature on theoretical questions about these equations is available. A selection in the areas cited above include [2, 58, 12, 3, 19, 49, 38, 26, 21].

Researchers working on the Monge-Kantorovich Problem, MKP, c.f. [37] for background, have noted the problematic lack of good numerical solvers for the Monge-Ampère type equations. Following [29], we quote from [15], "It follows from this theoretical result that a natural computational solution of the L^2 MKP is the numerical resolution of the Monge-Ampère equation" ... "Unfortunately, this fully nonlinear second-order elliptic equation has not received much attention from numerical analysts and, to the best of our knowledge, there is no efficient finite-difference or finite-element methods, comparable to those developed for linear second-order elliptic equations (such as fast Poisson solvers, multigrid methods, preconditioned conjugate gradient methods,...)."

Existing numerical work on the Monge-Ampère type equations included [50, 42, 20] where the generalized solution in the sense of Alexandrov-Bakelman is approximated directly. Other works with proven convergence results is [48, 35] where finite difference schemes satisfying conditions for convergence of [14] were constructed. There have been an explosion of recent numerical results for the Monge-Ampère equations. We have the recent papers [45, 46, 59, 18, 52] which do not address adequately the situations where the Monge-Ampère equation does not have a smooth solution, c.f. Test 2 and Test 3 in Section 4. For progress in this direction we refer to the series of papers [28, 29, 27] and the vanishing moment method in [32, 33, 34]. Finite difference methods which computes viscosity solutions of the Monge-Ampère equation and an iterative method amenable to finite element computations were reported in [16, 36]. See also [17] for an optimization approach.

In this paper, we use the spline element method to compute numerical solution of the Monge-Ampère equation. It is a conforming finite element implementation with Lagrange multipliers. We will obtain conforming approximations for the three dimensional Monge-Ampère equation. We extend the convergence analysis of Newton's method due to [45] to bounded smooth domains using Schauder estimates proved in [56]. However [45] did not address the convergence of the discrete approximations. We give error estimates for conforming approximations of a smooth solution. The Monge-Ampère equation leads to a non-coercive variational problem, a difficulty which is partially handled by the vanishing moment method (the parameter ϵ cannot be taken equal to 0). We show that for smooth solutions, the spline element method is robust for the associated singular perturbation problem. The numerical results mainly examine the performance of three numerical methods, Newton's method, the vanishing moment method and the Benamou-Froese-Oberman iterative method, on three test functions suggested in [27]: a smooth radial solution, a non-smooth solution for which no exact formula is known and a solution not in $H^2(\Omega)$. In this paper, we will refer to the Benamou-Froese-Oberman iterative method as the BFO iterative method, which we extend to three dimensions.

The paper is organized as follows: In the first section, we review the spline element discretization. The following section is devoted to the variational formulations associated to Newton's method, for which we give convergence results for smooth solutions, and the vanishing moment method. Here we introduce the three dimensional version of the BFO iterative method. The last section is devoted to numerical experiments. We will use C for a generic constant but will index specific constants.

2. Spline element discretization

The spline element method has been described in [4, 7, 8, 13, 41] under different names and more recently in [6]. It can be described as a conforming finite element implementation with Lagrange multipliers. We first outline the main steps of the method, discuss its advantages and possible disadvantage. We then give more details of this approach but refer to the above references for explicit formulas.

First, start with a representation of a piecewise discontinuous polynomial as a vector in $\mathbb{R}^N$, for some integer $N > 0$. Then express boundary conditions and constraints including global continuity or smoothness conditions as linear relations. In our work, we use the Bernstein basis representation, [4, 6] which is very convenient to express smoothness conditions and very popular in computer aided geometric design. Hence the term "spline" in the name of the method. Splines are piecewise polynomials with smoothness properties. One then write a discrete version of the equation along with a discrete version of the spaces of trial and test functions. The boundary conditions and constraints are enforced using Lagrange multipliers. We are lead to saddle point problems which are solved by an augmented Lagrangian algorithm (sequences of linear equations with size $N \times N$). The approach here should be contrasted with other approaches where Lagrange multipliers are introduced before discretization, i.e. in [9] or the discontinuous Galerkin methods.

The spline element method, stands out as a robust, flexible, efficient and accurate method. It can be applied to a wide range of PDEs in science and engineering in both two and three dimensions; constraints and smoothness are enforced exactly and there is no need to implement basis functions with the required properties; it is particularly suitable for fourth order PDEs; no inf-sup condition are needed to approximate Lagrange multipliers which arise due to the constraints, e.g. the pressure term in the Navier-Stokes equations; one gets in a single implementation approximations of variable order. Other advantages of the method include the flexibility of using polynomials of different degrees on different elements [41], the facility of implementing boundary conditions and the simplicity of a posteriori error estimates since the method is conforming for many problems. A possible disadvantage of this approach is the high number of degrees of freedom and the need to solve saddle point problems.

Let $\mathcal{T}$ be a conforming partition of Ω into triangles or tetrahedra. We consider a general variational problem: Find $u \in W$ such that

$$a(u, v) = \langle l, v \rangle \quad \text{for all } v \in V, \tag{2.1}$$

where W and V are respectively the space of trial and test functions. We will assume that the form l is bounded and linear and a is a continuous mapping in some sense on $W \times V$ which is linear in the argument v.

Let W_h and V_h be conforming subspaces of W and V respectively. We can write

$$W_h = \{c \in \mathbf{R}^N, Rc = G\}, \ V_h = \{c \in \mathbf{R}^N, Rc = 0\},$$

for a suitable vector G and R a suitable matrix which encodes the constraints on the solution, e.g. smoothness and boundary conditions. Here h is a discretization parameter which controls the size of the elements in the partition.

The condition $a(u, v) = \langle l, v\rangle$ for all $v \in V$ translates to

$$K(c)d = L^T d \quad \forall d \in V_h, \text{ that is for all } d \text{ with } Rd = 0,$$

for a suitable matrix $K(c)$ which depends on c and L is a vector of coefficients associated to the linear form l. If for example $\langle l, v\rangle = \int_\Omega fv$, then $L^T d = d^T MF$ where M is a mass matrix and F a vector of coefficients associated to the spline interpolant of f. In the linear case $K(c)$ can be written $c^T K$.

Introducing a Lagrange multiplier λ, the functional

$$K(c)d - L^T d + \lambda^T Rd,$$

vanishes identically on V_h. The stronger condition

$$K(c) + \lambda^T R = L^T,$$

along with the side condition $Rc = G$ are the discrete equations to be solved.

By a slight abuse of notation, after linearization by Newton's method, the above nonlinear equation leads to solving systems of type

$$c^T K + \lambda^T R = L^T.$$

The approximation c of $u \in W$ thus is a limit of a sequence of solutions of systems of type

$$\begin{bmatrix} K^T & R^T \\ R & 0 \end{bmatrix} \begin{bmatrix} c \\ \lambda \end{bmatrix} = \begin{bmatrix} L \\ G \end{bmatrix}.$$

It is therefore enough to consider the linear case. If we assume for simplicity that $V = W$ and that the form a is bilinear, symmetric, continuous and V-elliptic, existence of a discrete solution follows from Lax-Milgram lemma. On the other hand, the ellipticity assures uniqueness of the component c which can be retrieved by a least squares solution of the above system [4]. The Lagrange multiplier λ may not be unique. To avoid systems of large size, a variant of the augmented Lagrangian algorithm is used. For this, we consider the sequence of problems

$$\begin{pmatrix} K^T & R^T \\ R & -\mu M \end{pmatrix} \begin{bmatrix} \mathbf{c}^{(l+1)} \\ \lambda^{(l+1)} \end{bmatrix} = \begin{bmatrix} L \\ G - \mu M \lambda^{(l)} \end{bmatrix}, \tag{2.2}$$

where $\lambda^{(0)}$ is a suitable initial guess for example $\lambda^{(0)} = 0$, M is a suitable matrix and $\mu > 0$ is a small parameter taken in practice in the order of 10^{-5}. It is possible to solve for $\mathbf{c}^{(l+1)}$ in terms of $\mathbf{c}^{(l)}$. A uniform convergence rate in μ for this algorithm was shown in [5].

3. Variational formulations

The BFO iterative method for solving (1.1) has a clear variational formulation as it consists in solving a sequence of Poisson problems:

$$\Delta u_{k+1} = \sqrt{(\Delta u_k)^2 + 2(f - \det D^2 u_k)}, \tag{3.1}$$

for the two dimensional case [16] and extended here to three dimensions

$$\Delta u_{k+1} = ((\Delta u_k)^3 + 9(f - \det D^2 u_k))^{\frac{1}{3}}, \tag{3.2}$$

with $u_{k+1} = g$ on $\partial\Omega$. Since

$$\det D^2 u \leq \frac{1}{n^n}(\Delta u)^n, \tag{3.3}$$

the above formula enforces partial convexity since $\Delta u \geq 0$ is a necessary condition for the Hessian of u to be semi positive definite. We note that the constant 2 in (3.1) may be changed to 4. We next discuss Newton's method and the vanishing moment method. The use of Newton's method for proving existence of a solution of (1.1) appeared in [10] combined with a method of continuity argument and more recently for approximation with a direct approach by finite differences in [45] for a Monge-Ampère equation on the torus. We will extend the proof of convergence of Newton's method of [45] in Hölder spaces on bounded smooth domains. We then characterize the Newton's iterates as solutions of variational problems and a solution of (1.1) is also shown to be characterized by a variational formulation for which we derive error estimates for finite element approximations.

3.1. **Newton's method.**

We denote by $C^k(\Omega)$ the set of all functions having all derivatives of order $\leq k$ continuous on Ω where k is a nonnegative integer or infinity and by $C^k(\overline{\Omega})$, the set of all functions in $C^k(\Omega)$ whose derivatives of order $\leq k$ have continuous extensions to $\overline{\Omega}$. The norm in $C^k(\Omega)$ is given by

$$||u||_{C^k(\Omega)} = \sum_{j=0}^{k} \text{sup}_{|\beta|=j} \text{sup}_{\Omega} |D^\beta u(x)|.$$

A function u is said to be uniformly Hölder continuous with exponent $\alpha, 0 < \alpha \leq 1$ in Ω if the quantity

$$\text{sup}_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

is finite. The space $C^{k,\alpha}(\overline{\Omega})$ consists of functions whose k-th order derivatives are uniformly Hölder continuous with exponent α in Ω. It is a Banach space with norm

$$||u||_{C^{k,\alpha}(\overline{\Omega})} = ||u||_{C^k(\Omega)} + \text{sup}_{|\beta|=k} \text{sup}_{x \neq y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

Next note that for any $n \times n$ matrix A, $\det A = 1/n (\operatorname{cof} A) : A$, where $\operatorname{cof} A$ is the matrix of cofactors of A and for two $n \times n$ matrices M, N, $M : N = \sum_{i,j=1}^n M_{ij} N_{ij}$ is the Kronecker product of M and N. For any sufficiently smooth matrix field A and vector field v, $\operatorname{div} A^T v = (\operatorname{div} A) \cdot v + A : Dv$. Here the divergence of a matrix field is the divergence operator applied row-wise. If we put $v = Du$, then $\det D^2 u = 1/n (\operatorname{cof} D^2 u) : D^2 u = 1/n\, (\operatorname{cof} Dv) : Dv$ and $\operatorname{div}(\operatorname{cof} Dv)^T v = \operatorname{div}(\operatorname{cof} Dv) \cdot$

$v + (\text{cof } Dv) : Dv$. But $\text{div}\,\text{cof } Dv = 0$, c.f. for example [31] p. 440. Hence since $D^2 u$ and cof $D^2 u$ are symmetric matrices

$$\det D^2 u = \frac{1}{n} (\text{cof } D^2 u) : D^2 u = \frac{1}{n} \text{div} \big((\text{cof } D^2 u) Du \big). \tag{3.4}$$

Put $F(u) = \det D^2 u - f$. The operator F maps $C^{m,\alpha}$ into $C^{m-2,\alpha}, m \geq 2$. This can be seen from the properties of the Hölder norm of a product [22], p. 18. We have $F'(u)w = (\text{cof } D^2 u) : D^2 w = \text{div} \big((\text{cof } D^2 u) Dw \big)$ for u, w sufficiently smooth. The Newton's iterates can then equivalently be written

$$\begin{aligned} F'(u_k)(u_{k+1} - u_k) &= -F(u_k), \\ \text{div} \big((\text{cof } D^2 u_k)(Du_{k+1} - Du_k) &= -\frac{1}{n} \text{div} \big((\text{cof } D^2 u_k) Du_k \big) + f, \\ (\text{cof } D^2 u_k) : (D^2 u_{k+1} - D^2 u_k) &= -\det D^2 u_k + f. \end{aligned}$$

We will use the last expression as it requires less regularity on u_k. More precisely, we will consider the following damped version of Newton's method: Given an initial guess u_0, we consider the sequence u_k defined by

$$(\text{cof } D^2 u_k) : D^2 \theta_k = \frac{1}{\tau}(f - f_k), \quad f_k = \det D^2 u_k, \quad \theta_k = u_{k+1} - u_k, \tag{3.5}$$

for a parameter $\tau \geq 1$. Our numerical results however use only $\tau = 1$.

We recall that the domain Ω is uniformly convex [40], if there exists a number $m_0 > 0$ such that through every point $x_0 \in \partial\Omega$, there passes a supporting hyperplane π of Ω satisfying dist $(x, \pi) \geq m_0 |x - x_0|^2$ for $x \in \partial\Omega$. We will need the following global regularity result, [56].

Theorem 3.1. *Let Ω be a uniformly convex domain in R^n, with boundary in C^3. Suppose $\phi \in C^3(\overline{\Omega})$, inf $f > 0$, and $f \in C^\alpha$ for some $\alpha \in (0,1)$. Then* (1.1) *has a convex solution u which satisfies the a priori estimate*

$$||u||_{C^{2,\alpha}(\overline{\Omega})} \leq C,$$

where C depends only on n, α, inf $f, \Omega, ||f||_{C^\alpha(\overline{\Omega})}$ and $||\phi||_{C^3}$.

According to [56], all assumptions in the above theorem are sharp. We have the following analogue of Theorem 2.1 in [45].

Theorem 3.2. *Let Ω be a uniformly convex domain in R^2, with boundary in C^3. Let $0 < m \leq f \leq M, f \in C^\alpha$ for some $m, M > 0$ and $\alpha \in (0,1)$. There exists $\tau \geq 1$ depending on m, f, such that if u_k is the sequence defined by* (3.5)*, it converges in $C^{2,\beta}$ to a solution u of* (1.1)*, for every $\beta < \alpha$.*

Proof. The proof essentially depends on global Hölder regularity of the solution of the Monge-Ampère equation. Theorem (3.1) essentially gives the conditions under which such a regularity result holds on a bounded domain. Part of the proof has been more or less repeated in [35]. We note that the iterates may converge to either a concave or a convex solution if both exists. □

3.2. **Variational formulation.** Using the divergence form of (3.5), the iterates can be characterized as solutions of the problems: Find $u_{k+1} \in H^1(\Omega), u_{k+1} = g$ on $\partial\Omega$ and

$$\begin{aligned} \int_\Omega (\mathrm{cof}\ D^2 u_k) Du_{k+1} \cdot Dw\ \ dx = \frac{n-1}{n} \int_\Omega (\mathrm{cof}\ D^2 u_k) Du_k \cdot Dw\ \ dx \\ + \int_\Omega fw\ \ dx, \quad \forall w \in H_0^1(\Omega). \end{aligned} \tag{3.6}$$

With an initial guess u_0 which solves $\Delta u = nf^{1/n}$, for f in $C^\alpha, 0 < \alpha < 1$, we have a sequence of uniformly elliptic problems, (see proof of Theorem 2.1 in [45]) and the problems (3.5) and (3.6) are equivalent. We then know that the iterates converge to a solution of (1.1) which solves the formal variational limit of (3.6): Find $u \in H^n(\Omega)$, $u = g$ on $\partial\Omega$ such that

$$\int_\Omega (\mathrm{cof}\ D^2 u) Du \cdot Dw\ \ dx = -n \int_\Omega fw\ \ dx, \quad \forall w \in H^n(\Omega) \cap H_0^1(\Omega). \tag{3.7}$$

The problem (3.7) is not well posed in general. For example if $g = 0$, then both u and $-u$ are solutions.

To see that the left hand side is bounded for $u \in H^n(\Omega)$, notice that for $n = 2$

$$\left| \int_\Omega (\mathrm{cof}\ D^2 u) Du \cdot Dw \right| \leq C ||D^2 u||_{L^2(\Omega)} ||Du||_{L^4(\Omega)} ||Dw||_{L^4(\Omega)}.$$

Next for $u \in H^2(\Omega), \partial u/\partial x_i \in H^1(\Omega), i = 1, \ldots, n$ and by the compactness of the embedding of $H^1(\Omega)$ in $L^q(\Omega)$ for $q \geq 1$ when $n = 2$, the right hand side above is bounded by $C||D^2 u||_{L^2(\Omega)} ||u||_{H^2(\Omega)} ||w||_{H^2(\Omega)}$. However in three dimensions, the term cof $D^2 u$ involves the product of two second order derivatives. For it to be bounded, we will need $u \in H^3(\Omega)$ so that $\partial^2 u/\partial x_i \partial x_j, i, j, = 1, \ldots, n \in H^1(\Omega)$ and we can use the embedding of $H^1(\Omega)$ in $L^q(\Omega)$ for $1 \leq q \leq 6$ when $n = 3$. We give below error estimates only for the two dimensional case using techniques borrowed from [32]. We note that in the spline element method, it is possible to impose the C^2 continuity requirements for a conforming finite element subspace of $H^3(\Omega)$. However for a smooth solution, C^1 continuity was enough for numerical evidence of convergence.

3.3. **Error estimate for 2D conforming approximation of a smooth solution.** In this section, we will assume that Ω is a two dimensional domain. Put $V = H^2(\Omega)$ and $V_0 = H^2(\Omega) \cap H_0^1(\Omega)$. Let V^h be a conforming finite element subspace of $H^2(\Omega)$, V_0^h be a conforming finite element subspace of $H^2(\Omega) \cap H_0^1(\Omega)$ with approximation properties

$$\inf_{v_h \in V^h} ||v - v_h||_j \leq C_1 h^{p-j} ||v||_4, j = 0, 1, 2 \tag{3.8}$$

for all $v \in H^4(\Omega)$ for some $p \geq 4$.

For example, in this paper, we take V_h as the spline space of degree d and smoothness $r \geq 1$

$$S_d^r(\mathcal{T}) = \{p \in C^r(\Omega),\ p_{|t} \in P_d,\ \forall t \in \mathcal{T}\},$$

where P_d denotes the space of polynomials of degree d in two variables and $\mathcal{T}$ denotes the triangulation of the domain Ω. It is known that [43], for $d \geq 3r+2$ and $0 \leq m \leq d$, there exists a linear quasi-interpolation operator Q mapping $L_1(\Omega)$ into the spline space $S^r_d(\mathcal{T})$ and a constant C such that if f is in the Sobolev space $H^{m+1}(\Omega)$,

$$|f - Qf|_k \leq Ch^{m+1-k}|f|_{m+1}, \tag{3.9}$$

for $0 \leq k \leq m$. For our purposes, $m = 3$ and $p = 4$. If Ω is convex, the constant C depends only on d, m and on the smallest angle $\theta_\triangle$ in $\mathcal{T}$. In the non convex case, C depends only on the Lipschitz constant associated with the boundary of Ω. It is also known c.f. [30] that the full approximation property for spline spaces holds for certain combinations of d and r on special triangulations. We have the following theorem

Theorem 3.3. *Assume that problem* (1.1) *has a solution $u \in H^4(\Omega)$ hence in $C^2(\Omega)$ by Sobolev embedding, then the discrete problem, find $u_h \in V_h$, $u_h = g$ on $\partial\Omega$ such that*

$$-\frac{1}{2}\int_\Omega (cof\ D^2 u_h) Du_h \cdot Dw_h \ dx = \int_\Omega f w_h \ dx, \quad \forall w_h \in V_0^h, \tag{3.10}$$

has a unique solution in a neighborhood of $I_h u$ where I_h is an interpolation operator associated with V_h. Moreover $||u_h - I_h(u)||_2$ is at least O(h).

The proof of the above theorem follows the combined fixed point and linearization method used in [32]. The difference here is the assumption that the solution is smooth and the use of an inverse inequality. We start with some preliminaries.

We consider the linear problem: Find $v \in H^1_0(\Omega)$ such that

$$\int_\Omega (\text{cof } D^2 u) Dv \cdot Dw = \int_\Omega \phi\, w, \forall w \in H^1_0(\Omega), \tag{3.11}$$

for $\phi \in L^2(\Omega)$.

Since $u \in C^2(\Omega), \exists\, m, M > 0$ such that

$$m \leq \frac{\partial u}{\partial x_i \partial x_j}(x) \leq M,\ i, j = 1, \ldots, 2, \forall x \in \Omega.$$

The existence and uniqueness of a solution to (3.11) follows immediately from Lax-Milgram lemma. Similarly, there exists a unique solution to the problem: Find $v_h \in V_0^h$ such that

$$\begin{aligned} \int_\Omega (\text{cof } D^2 u) Dv_h \cdot Dw_h &= \int_\Omega \phi\, w_h, \forall w_h \in V_0^h. \\ v &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.12}$$

We note that the constant C above depends on u and that since Ω is assumed convex, $v \in H^2(\Omega)$ by elliptic regularity.

We define a bilinear form on $H^1_0(\Omega) \times H^1_0(\Omega)$ by

$$B[v, w] = \int_\Omega (\text{cof } D^2 u) Dv \cdot Dw, \tag{3.13}$$

and for any $v_h \in V^h$, $v_h = g$ on $\partial\Omega$, we define $T(v_h)$ by

$$B[v_h - T(v_h), w_h] = \frac{1}{2}\int_\Omega (\text{cof } D^2 v_h) Dv_h \cdot D\psi_h \ dx + \int_\Omega f\psi_h \ dx, \quad \forall \psi_h \in V_0^h. \tag{3.14}$$

Since the solution of the above problems exists and is in V_0^h, $T(v_h) \in V^h$, $T(v_h) = g$ on $\partial\Omega$. A fixed point of the nonlinear operator T corresponds to a solution of (3.10) and conversely if v_h is a solution of (3.10), then v_h is a fixed point of T. We will show that T has a unique fixed point in a neighborhood of $I_h(u)$. Put

$$B_h(\rho) = \{v_h \in V_h, v_h = g \,\text{on}\, \partial\Omega, ||v_h - I_h u||_2 \leq \rho\}.$$

First, we note that

$$B[v_h - T(v_h), w_h] = -\int_\Omega \det D^2 v_h \psi_h \; dx + \int_\Omega f \psi_h \; dx, \quad \forall \psi_h \in V_0^h. \tag{3.15}$$

Lemma 3.4. *There exists $C_2 > 0$ such that*

$$||I_h u - T(I_h u)||_2 \leq C_2 h^{p-3} ||u||_4.$$

Proof. Put $w_h = I_h u - T(I_h u)$. We have using $\det D^2 u = f$,

$$B[w_h, w_h] = \int_\Omega \big(\det D^2 u - \det D^2 I_h u\big) w_h \, dx.$$

Now, let $I_h^\epsilon(u)$ be a mollifier of $I_h u$. We have

$$\begin{aligned} B[w_h, w_h] &= \int_\Omega \big(\det D^2 u - \det D^2 I_h^\epsilon(u)\big) w_h \, dx + \int_\Omega \big(\det D^2 I_h^\epsilon(u) - \det D^2 I_h u\big) w_h \, dx \\ &= \int_\Omega \Big((\text{cof } t D^2 u + (1-t) D^2 I_h^\epsilon(u)) : D^2(u - I_h^\epsilon(u)) \Big) w_h \, dx \\ &\qquad + \int_\Omega \big(\det D^2 I_h^\epsilon(u) - \det D^2 I_h u\big) w_h \, dx \,\text{for some}\, t \in [0,1] \\ &\leq ||\text{cof } t D^2 u + (1-t) D^2 I_h^\epsilon(u))||_\infty ||u - I_h^\epsilon(u)||_2 ||w_h||_0 \\ &\qquad + ||\det D^2 I_h^\epsilon(u) - \det D^2 I_h u||_0 ||w_h||_0 \\ &\leq C ||D^2 u||_\infty ||u - I_h(u)||_2 ||w_h||_0 \leq C M h^{p-2} ||u||_4 ||w_h||_0, \end{aligned}$$

since $||\det D^2 I_h^\epsilon(u) - \det D^2 I_h u||_0 \to 0$ as $\epsilon \to 0$. We conclude that

$$||w_h||_1^2 \leq C h^{p-2} ||u||_4 ||w_h||_0, \text{ and } ||w_h||_2 \leq \frac{C}{h} ||w_h||_1 \leq C h^{p-3} ||u||_4,$$

using the coercivity of the bilinear form B with a constant C which depends on m and M and an inverse estimate. □

Lemma 3.5. *There exists $h_0 > 0$ and $0 < \rho_0(h_0)$ such that T is a contraction mapping in the ball $B_h(\rho_0)$ with a contraction factor 1/2.*

Proof. For $v_h, w_h \in B_h(\rho_0)$, and $\psi_h \in V_0^h$, let v_h^ϵ and w_h^ϵ denotes mollifiers for v_h and w_h respectively.

$$\begin{aligned}
B[T(v_h) - T(w_h), \psi_h] &= B[T(v_h) - v_h, \psi_h] + B[v_h - w_h, \psi_h] + B[w_h - T(w_h), \psi_h] \\
&= \int_\Omega (\det D^2 v_h - \det D^2 w_h)\psi_h \, dx \\
&\qquad + \int_\Omega (\text{cof } D^2 u)(Dv_h - Dw_h) \cdot D\psi_h \ dx \\
&= A_\epsilon + \int_\Omega (\text{cof } D^2 u)(Dv_h - Dw_h) \cdot D\psi_h \ dx \\
&\qquad + \int_\Omega (\det D^2 v_h^\epsilon - \det D^2 w_h^\epsilon)\psi_h \, dx,
\end{aligned}$$

where $A_\epsilon = \int_\Omega (\det D^2 v_h - \det D^2 v_h^\epsilon)\psi_h \, dx + \int_\Omega (\det D^2 w_h - \det D^2 w_h^\epsilon)\psi_h \, dx \to 0$ as $\epsilon \to 0$. We have for some $t \in [0,1])$,

$$\begin{aligned}
B[T(v_h) - T(w_h), \psi_h] &= A_\epsilon + \int_\Omega (\text{cof } D^2 u)(Dv_h - Dw_h) \cdot D\psi_h \ dx \\
&\qquad - \int_\Omega (\text{cof } tD^2 v_h^\epsilon + (1-\tau) D^2 w_h^\epsilon)(Dv_h^\epsilon - Dw_h^\epsilon)\psi_h \, dx \\
&= \int_\Omega (\text{cof } D^2 u - (\text{cof } tD^2 v_h^\epsilon + (1-\tau) D^2 w_h^\epsilon))(Dv_h - Dw_h) \cdot D\psi_h \ dx \\
&\qquad + A_\epsilon + B_\epsilon,
\end{aligned}$$

where $B_\epsilon = \int_\Omega (\text{cof } tD^2 v_h^\epsilon + (1-\tau) D^2 w_h^\epsilon)(Dv_h - Dw_h - Dv_h^\epsilon + Dw_h^\epsilon)\psi_h \, dx \to 0$ as $\epsilon \to 0$. Put $\Psi_\epsilon = \text{cof } D^2 u - (\text{cof } tD^2 v_h^\epsilon + (1-\tau) D^2 w_h^\epsilon)$. We have

$$\begin{aligned}
||\Psi_\epsilon||_0 &= ||D^2 u - D^2 v_h^\epsilon + \tau(D^2 w_h^\epsilon - D^2 v_h^\epsilon)||_0 \leq ||D^2 u - D^2 v_h^\epsilon||_0 + ||D^2 w_h^\epsilon - D^2 v_h^\epsilon||_0 \\
&\leq ||D^2 u - D^2 I_h u||_0 + ||D^2 I_h u - D^2 v_h||_0 + ||D^2 v_h - D^2 v_h^\epsilon||_0 + ||D^2 w_h^\epsilon - D^2 w_h||_0 \\
&\qquad + ||D^2 w_h - D^2 v_h||_0 + ||D^2 v_h - D^2 v_h^\epsilon||_0 \\
&\leq C h^{p-2} ||u||_4 + 3\rho_0 + 2||v_h - v_h^\epsilon||_2 + ||w_h - w_h^\epsilon||_2.
\end{aligned}$$

As $\epsilon \to 0$, we obtain

$$B[T(v_h) - T(w_h), \psi_h] \leq C(h^{p-2}||u||_4 + \rho_0)||v_h - w_h||_2 ||\psi_h||)2.$$

By coercivity and an inverse estimate,

$$||T(v_h) - T(w_h)||_2 \leq C(h^{p-3}||u||_4 + \frac{\rho_0}{h})||v_h - w_h||_2.$$

First choose h such that $h^{p-3}||u||_4 \leq 1/4$ then $\rho_0 \leq h/4$. The result follows $\square$

Proof. (of Theorem 3.3) Let $\rho_1 = 2C_2 h^{p-3}||u||_4$. We first show that T maps $B_h(\rho_1)$ into itself. For $v_h \in B_h(\rho_1)$,

$$\begin{aligned}
||I_h u - T(v_h)||_2 &\leq ||I_h u - T(I_h u)||_2 + ||T(I_h u) - T(v_h)||_2 \leq \frac{\rho_1}{2} + \frac{1}{2}||I_h u - v_h||_2 \\
&\leq \frac{\rho_1}{2} + \frac{\rho_1}{2} = \rho_1.
\end{aligned}$$

By the Brouwer's fixed point theorem, T has a unique fixed point in $B_h(\rho_1)$ which is u_h, i.e. $T(u_h) = u_h$. Next,

$$\begin{aligned} ||u - u_h||_2 &\leq ||u - I_h u||_2 + ||I_h u - u_h||_2 \leq C_1 h^{p-2} ||u||_4 + ||I_h u - T(u_h)||_2 \\ &\leq C_1 h^{p-2} ||u||_4 + 2C_2 h^{p-3} ||u||_4 \leq C h^{p-3} ||u||_4, \end{aligned}$$

for h sufficiently small. We have the result. □

3.4. **Vanishing moment methodology.** The vanishing moment methodology approach to (1.1), consists in computing a solution of the singular perturbation problem

$$-\epsilon \Delta^2 u + \det\, D^2 u = f, \text{ in } \Omega, \quad u = g, \ \Delta u = \epsilon^2 \text{ on } \partial\Omega. \tag{3.16}$$

It is an analogue of a singular perturbation problem

$$\epsilon \Delta^2 u - \Delta u = f \text{ in } \Omega, u = 0, \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega$$

which was addressed numerically in [47] and also in the spline element method [6]. The analogy holds as many properties of the Laplace operator have a counterpart for the Monge-Ampère operator.

The Newton's iterates in the vanishing moment formulation consisting in solving the sequence of problems: Find $u_{k+1} \in H^n(\Omega)$ with $u_{k+1} = g$ on $\partial\Omega$

(3.17)

$$\begin{aligned} \epsilon \int_\Omega \Delta u_{k+1} \Delta v \, dx + \int_\Omega (\text{cof } D^2 u_k) D u_{k+1} \cdot Dv \ \ dx = \frac{n-1}{n} \int_\Omega (\text{cof } D^2 u_k) D u_k \cdot Dv \ \ dx \\ + \epsilon^3 \int_{\partial\Omega} \frac{\partial v}{\partial n} \, ds - \int_\Omega f v \ \ dx, \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

Formally as ϵ approaches 0, the solution of the above problem degenerates to the solution of (3.6), a result which will be illustrated numerically in the next section.

4. Numerical results

The first two subsections are devoted to two dimensional and three dimensional numerical results respectively. The three methods are compared on three test functions for 2D experiments.

Test 1: A smooth solution $u(x, y) = e^{(x^2+y^2)/2}$ so that $f(x, y) = (1 + x^2 + y^2)e^{(x^2+y^2)}$ and $g(x, y) = e^{(x^2+y^2)/2}$ on $\partial\Omega$.

Test 2: A solution not in $H^2(\Omega)$, $u(x, y) = -\sqrt{2 - x^2 - y^2}$ so that $f(x, y) = 2/(2 - x^2 - y^2)^2$ and $g(x, y) = -\sqrt{2 - x^2 - y^2}$ on $\partial\Omega$.

Test 3: No exact solution is known. Solutions are either convex or concave. Here $f(x, y) = 1$ and $g(x, y) = 0$.

We give numerical evidence of the robustness of the spline element method for the singular perturbation problem associated to the vanishing moment methodology. Formally as ϵ approaches 0, the problem (3.16) degenerates to the problem (1.1), which can be solved by Newton's method when a smooth solution exists. We show here numerically that the solution of (3.17) converges to that of (3.6) as ϵ approaches 0.

Unlike [34], there is no boundary layers issue with the spline element method. These results are illustrated in Tables 4.1 and 4.2.

In general, vanishing moment (with Newton) gives Newton's method result for $\epsilon = 10^{-9}$. In particular, Newton's method diverges for the non smooth solutions of Test 2 and 3. However with ϵ large, which implies that the equation solved is much further from the actual problem, divergence can be avoided in the vanishing moment methodology. We refer to the problem (1.1) as reduced in the tables.

In the two dimensional case of Test 3, both concave and convex solutions can be computed by either changing the initial guess or the structure of the approximations.

(1) Newton's method: initial guess $\pm u_0$,
(2) Vanishing moment: $\pm\epsilon$ and initial guess $\pm u_0$,
(3) BFO iterative method: $u_{k+1} = \pm\sqrt{(\Delta u_k)^2 + 2(f - \det D^2 u_k)}$.

Since Newton's method diverges for Test 3, we illustrate this feature of the method on a fourth test on a non-square domain. This also helps contrast with finite difference methods.

Test 4: The domain is the unit circle which is discretized with a Delanauy triangulation with 824 triangles and the test functions are $u(x,y) = e^{(x^2+y^2)/2}$ (convex) and $u(x,y) = -e^{(x^2+y^2)/2}$ (concave) with the corresponding right hand side and boundary conditions.

Since none of the methods perform convincingly on Test 2 in the spline element framework, the methods are tested for the three dimensional case on two other test functions analogues of Test 1 and Test 3. We only consider the performance of the BFO and vanishing moment method.

Test 5: $u(x,y,z) = e^{(x^2+y^2+z^2)/3}$ so that $f(x,y,z) = 8/81(3+2(x^2+y^2+z^2))e^{(x^2+y^2+z^2)}$ and $g(x,y,z) = e^{(x^2+y^2+z^2)/3}$ on $\partial\Omega$.

Test 6: $f(x,y,z) = 1$ and $g(x,y,z) = 0$.

The initial guess of the Newton's iterations is the solution of the Poisson equation $\Delta u = n f^{1/n}\, n = 2,3 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega$.

We also illustrate the performance of the 3D BFO method on a degenerate Monge-Ampère equation,

Test 7: $f(x,y,z) = 0$ and $g(x,y,z) = |x - 1/2|$.

In the two dimensional case, to approximate a concave solution, one should solve $\Delta u = -2\sqrt{f}$. But unless $u = 0$ on $\partial\Omega$, as in Test 3, it is not clear which boundary condition to use. Note that if u is a smooth convex function, $\Delta u \geq 0$. To compute the concave solution of Test 4, we first solved the Monge-Ampère equation with the negative of the boundary condition, then use the negative of that solution as an initial guess. However, a good initial guess could not be found if we uniformly refine a Delanauy triangulation of the circle with 143 triangles, but convergence was obtained with the choice in Test 4, perhaps because the domain is closer to being smooth. For the vanishing moment calculations, the initial guess was taken as the biharmonic regularization of a suitable Poisson equation, for example, $-\epsilon\Delta^2 u + \Delta u = n f^{1/n}\, n = 2,3 \text{ in } \Omega, \quad u = g, \Delta u = \epsilon^2 \text{ on } \partial\Omega$. We simply took the zero function as initial guess

d	L^2 norm	H^1 norm	H^2 norm
3	1.0610 10^{-3}	1.1101 10^{-2}	1.6383 10^{-1}
4	3.5127 10^{-5}	4.8553 10^{-4}	9.0596 10^{-3}
5	4.1572 10^{-6}	6.5142 10^{-5}	1.9364 10^{-3}
6	1.9685 10^{-7}	3.6401 10^{-6}	1.4774 10^{-4}
7	2.2699 10^{-8}	4.1498 10^{-7}	2.2424 10^{-5}
8	1.2430 10^{-9}	2.2586 10^{-8}	1.5479 10^{-6}

TABLE 1. 2D Newton's method for Test 1, $h = 1/2$

d	L^2 norm	H^1 norm	H^2 norm
3	1.2809 10^{-4}	2.6554 10^{-3}	8.9587 10^{-2}
4	1.6278 10^{-6}	4.5619 10^{-5}	1.7395 10^{-3}
5	1.1531 10^{-7}	2.3916 10^{-6}	1.3444 10^{-4}
6	1.7705 10^{-9}	6.8506 10^{-8}	5.5403 10^{-6}
7	1.4548 10^{-10}	3.7545 10^{-9}	3.9490 10^{-7}
8	8.1014 10^{-12}	5.3353 10^{-10}	7.2159 10^{-8}

TABLE 2. 2D Newton's method for Test 1, $h = 1/4$

in the BFO method. Unless otherwise stated, we use C^1 splines for all the numerical experiments. Even for the BFO iterative method which requires only solving Poisson equations as in that case we obtained better numerical results (smooth graphs) for Test 3. We listed n_{it}, the number of iterations of the BFO method. We do not claim that our numerical solutions are convex but that they approximate convex functions. Convexity (or concavity) is not explicitly enforced in the numerical iterations.

4.1. **Two-dimensional Monge-Ampère equation.** The computational domain is the unit square $[0,1]^2$ which is first divided into squares of side length h. Then each square is divided into two triangles by the diagonal with negative slope. As evidenced in the last line of Table 4.1, we noted a degradation of the performance of the BFO iterative method even for a smooth solution when the number of degrees of freedom is large, either for smaller h or higher degree. This may be an indication that the method is not suitable for a general finite element implementation but is more likely a consequence of the conditioning properties of the iterative method (2.2). For Test 2, we display the results for C^1 cubic splines, but much higher order accuracy, of the order of 10^{-5} was obtained with C^0 splines. We caution that in our implementation, this may lead to non smooth numerical solutions.

For Test 3, we displayed both the graph and the contour of both convex and concave solutions. To get good results with the vanishing moment method, we experimented with a combination of the parameters ϵ and h.

4.2. **Three-dimensional Monge-Ampère equation.** We used two computational domains both on the unit cube $[0,1]^3$ which is first divided into six tetrahedra (Domain 1 for Test 4) or twelve tetrahedra (Domain 2 for Test 5) forming a tetrahedral partition

ϵ	L^2 norm	H^1 norm	H^2 norm
10^{-3}	1.727110^{-3}	2.091010^{-2}	8.633810^{-1}
10^{-4}	1.856310^{-4}	3.558110^{-3}	2.005010^{-1}
10^{-5}	1.891710^{-5}	4.070010^{-4}	2.411910^{-2}
10^{-6}	1.818210^{-6}	4.077510^{-5}	2.438810^{-3}
10^{-7}	1.244110^{-7}	4.095110^{-6}	2.494910^{-4}
10^{-8}	1.011910^{-7}	2.296210^{-6}	1.302910^{-4}
10^{-9}	1.138410^{-7}	2.379010^{-6}	1.338210^{-4}
10^{-10}	1.151610^{-7}	2.390310^{-6}	1.343810^{-4}
10^{-11}	1.153010^{-7}	2.391410^{-6}	1.344310^{-4}
10^{-14}	1.153110^{-7}	2.391610^{-6}	1.344410^{-4}
Reduced	$1.1531\ 10^{-7}$	$2.3916\ 10^{-6}$	$1.3444\ 10^{-4}$

TABLE 3. 2D numerical robustness Test 1, $h = 1/4$, $d = 5$.

h	n_{it}	L^2 norm	H^1 norm	H^2 norm
$1/2^1$	41	$2.8275\ 10^{-6}$	$6.1372\ 10^{-5}$	$1.8845\ 10^{-3}$
$1/2^2$	37	$5.4642\ 10^{-8}$	$2.1971\ 10^{-6}$	$1.2972\ 10^{-4}$
$1/2^3$	38	$8.3164\ 10^{-10}$	$7.2252\ 10^{-8}$	$8.4790\ 10^{-6}$
$1/2^4$	37	$2.7871\ 10^{-9}$	$1.4089\ 10^{-8}$	$1.0809\ 10^{-6}$

TABLE 4. BFO iterative method for Test 1, $d = 5$

h	L^2 norm	H^1 norm
$1/2^1$	2.195410^{-2}	1.640910^{-1}
$1/2^2$	3.609710^{-3}	6.140510^{-2}
$1/2^3$	1.068510^{-3}	4.097810^{-2}
$1/2^4$	5.083810^{-3}	2.804810^{-1}
$1/2^5$	2.579710^{+3}	2.268810^{+5}
$1/2^6$	1.845210^{+4}	3.592210^{+6}

h	n_{it}	L^2 norm	H^1 norm
$1/2^1$	50	$2.3921\ 10^{-1}$	1.1900
$1/2^2$	159	$1.2585\ 10^{-1}$	$7.1292\ 10^{-1}$
$1/2^3$	151	$1.0341\ 10^{-1}$	$6.4299\ 10^{-1}$
$1/2^4$	160	$9.6031\ 10^{-2}$	$6.2088\ 10^{-1}$
$1/2^5$	199	$9.4551\ 10^{-2}$	$6.2453\ 10^{-1}$
$1/2^6$	8	$1.6977\ 10^{-2}$	$2.2925\ 10^{-1}$

TABLE 5. Newton's method and BFO iterative method for Test 2, $d = 3$

h	L^2 norm	H^1 norm
$1/2^1$	7.668010^{-3}	7.449110^{-2}
$1/2^2$	1.453610^{-3}	3.924410^{-2}
$1/2^3$	9.872710^{-3}	2.511210^{-1}
$1/2^4$	5.681910^{-3}	2.492710^{-1}
$1/2^5$	$1.9830\ 10^{+4}$	$1.1812\ 10^{+6}$

h	L^2 norm	H^1 norm
$1/2^1$	7.825410^{-3}	9.318410^{-2}
$1/2^2$	1.064610^{-2}	9.520110^{-2}
$1/2^3$	1.130610^{-2}	9.615410^{-2}
$1/2^4$	1.150010^{-2}	9.133610^{-2}
$1/2^5$	1.162510^{-2}	8.778510^{-2}
$1/2^6$	1.168110^{-2}	8.563210^{-2}

TABLE 6. Vanishing moment Test 2 $\epsilon = 10^{-3}$ and $\epsilon = 10^{-2}$, $d = 5$

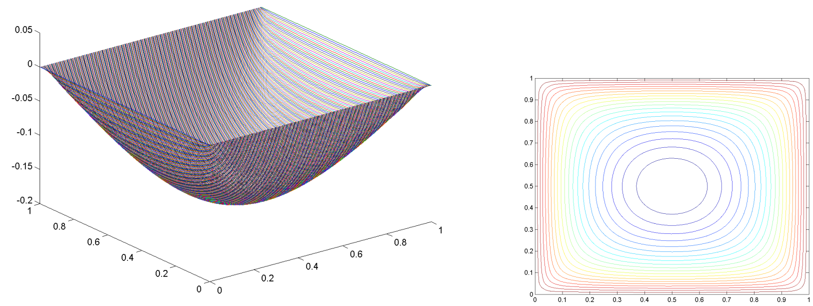

FIGURE 1. Vanishing moment on Test 3, $h = 1/2^4, d = 3, \epsilon = 10^{-3}$.

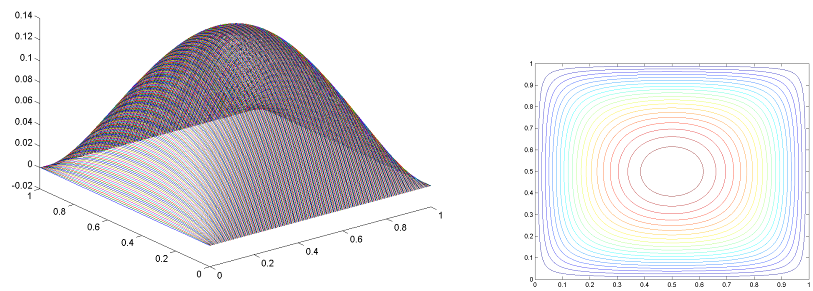

FIGURE 2. Vanishing moment on Test 3, $h = 1/2^4, d = 5, \epsilon = -10^{-2}$.

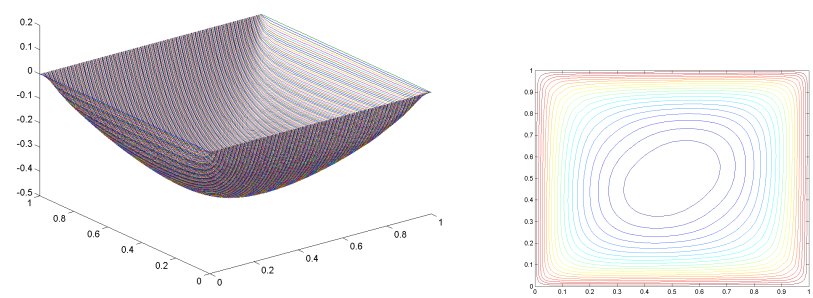

FIGURE 3. BFO iterative method on Test 3, $h = 1/2^4, d = 3$.

$\mathcal{T}_1$. This partition is uniformly refined following a strategy introduced in [4] similar to the one of [51] resulting in successive level of refinements $\mathcal{T}_k$, $k = 2, 3, \ldots$. For Test 6, we plot the graph of the function as well as its contour in the plane $x = 1/2$ as well as slices in the $x-$ direction.

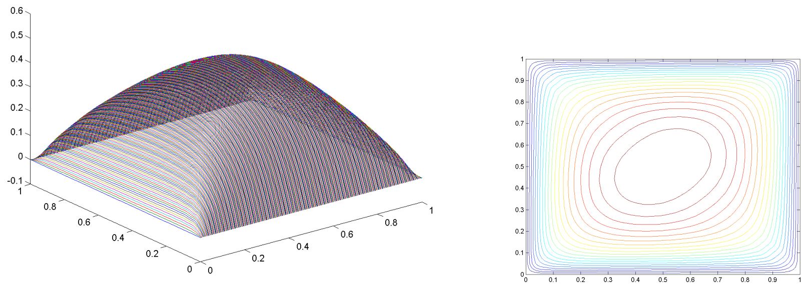

FIGURE 4. BFO iterative method on Test 3, $h = 1/2^4, d = 3$.

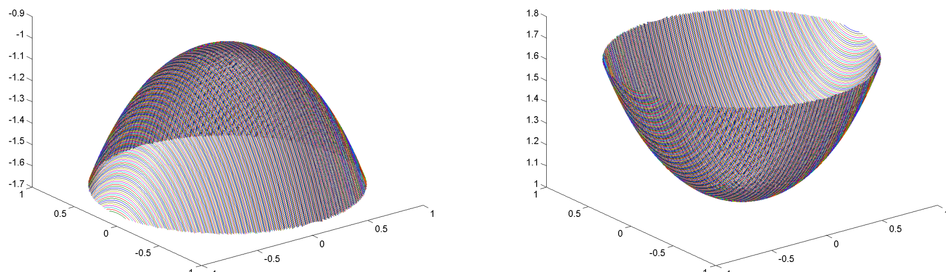

FIGURE 5. Approximations of smooth concave and convex solutions on a non rectangular domain, Test 4, $d = 5, r = 1$

d	L^2 norm	H^1 norm	H^2 norm
3	$1.2338\ 10^{-2}$	$7.6984\ 10^{-2}$	$4.4411\ 10^{-1}$
4	$1.6289\ 10^{-3}$	$1.4719\ 10^{-2}$	$1.3983\ 10^{-1}$
5	$1.5333\ 10^{-3}$	$8.7312\ 10^{-3}$	$6.0412\ 10^{-2}$
6	$1.2324\ 10^{-4}$	$9.7171\ 10^{-4}$	$1.0584\ 10^{-2}$

TABLE 7. Newton's method Test 5, Domain 1 on $\mathcal{I}_1$

d	L^2 norm	H^1 norm	H^2 norm
3	$3.1739\ 10^{-3}$	$2.3005\ 10^{-2}$	$2.4496\ 10^{-1}$
4	$3.2786\ 10^{-4}$	$3.5626\ 10^{-3}$	$5.2079\ 10^{-2}$
5	$2.4027\ 10^{-5}$	$3.9210\ 10^{-4}$	$8.8868\ 10^{-3}$
6	$1.3821\ 10^{-6}$	$2.2369\ 10^{-5}$	$6.0918\ 10^{-4}$

TABLE 8. Newton's method Test 5, Domain 1 on $\mathcal{T}_2$

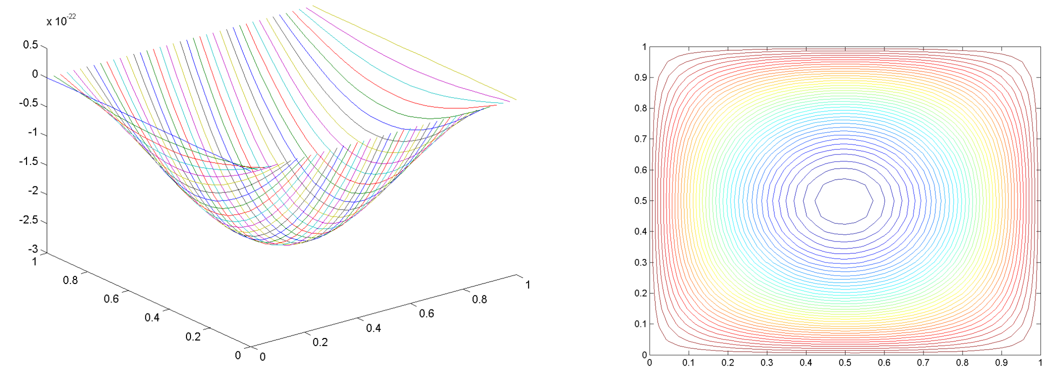

FIGURE 6. Vanishing moment Test 6, on $\mathcal{I}_3$, $r = 2, d = 3$

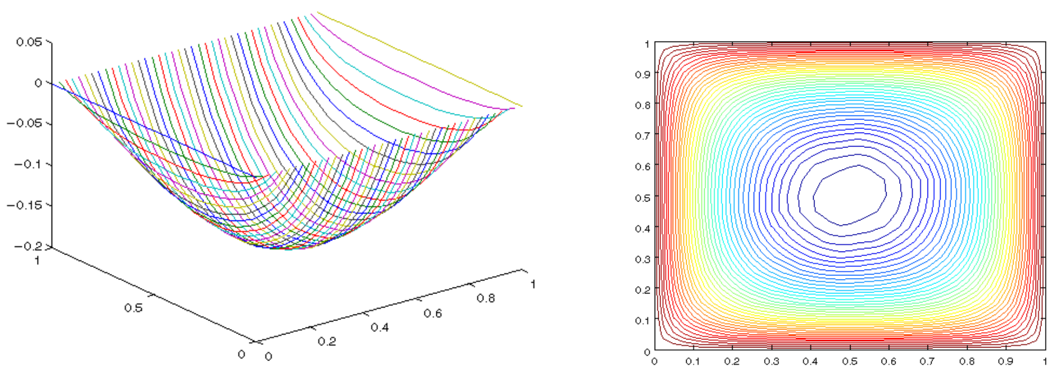

FIGURE 7. BFO iterative method Test 6, on $\mathcal{I}_3$, $d = 5, r = 1$

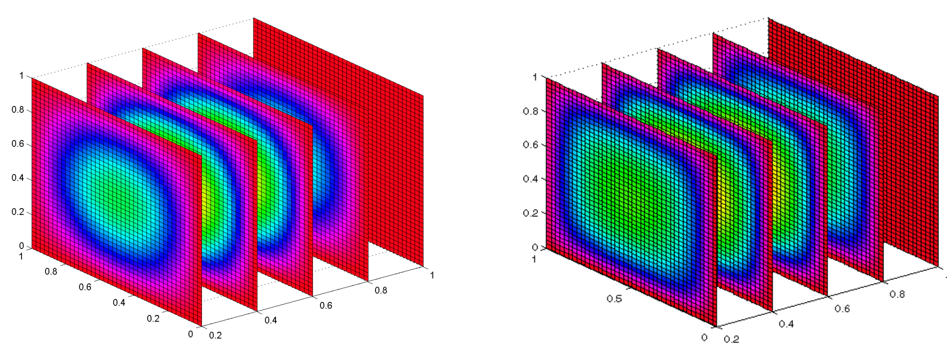

FIGURE 8. Slices in the $x-$direction Test 6 on Domain 2 and $\mathcal{I}_3, d = 3$
, Vanishing moment $r = 2, \epsilon = 10^{-5}$ and BFO $d = 5, r = 1$

ϵ	L^2 norm	H^1 norm	H^2 norm
10^{-1}	$6.6870\ 10^{-2}$	$3.9292\ 10^{-1}$	2.8852
10^{-2}	$1.8832\ 10^{-2}$	$1.3137\ 10^{-1}$	1.5882
10^{-3}	$2.4237\ 10^{-3}$	$2.5273\ 10^{-2}$	$5.3206\ 10^{-1}$
10^{-4}	$2.5661\ 10^{-4}$	$3.2633\ 10^{-3}$	$7.9936\ 10^{-2}$
10^{-5}	$3.1058\ 10^{-5}$	$5.0367\ 10^{-4}$	$1.2543\ 10^{-2}$
10^{-6}	$2.3519\ 10^{-5}$	$3.9165\ 10^{-4}$	$8.9744\ 10^{-3}$
10^{-7}	$2.3964\ 10^{-5}$	$3.9193\ 10^{-4}$	$8.8921\ 10^{-3}$
10^{-10}	$2.4027\ 10^{-5}$	$3.9210\ 10^{-4}$	$8.8868\ 10^{-3}$
Reduced	$2.4027\ 10^{-5}$	$3.9210\ 10^{-4}$	$8.8868\ 10^{-3}$

TABLE 9. 3D numerical robustness Test 5, Domain 1 on $\mathcal{T}_2$, $d = 5$

2	$3.1739\ 10^{-3}$	$2.3005\ 10^{-2}$	$2.4496\ 10^{-1}$
3	$1.6859\ 10^{-2}$	$1.0519\ 10^{-1}$	$9.1615\ 10^{-1}$
59	$1.1283\ 10^{-3}$	$7.1385\ 10^{-3}$	$7.3671\ 10^{-2}$
38	$2.1423\ 10^{-4}$	$1.4452\ 10^{-3}$	$1.8083\ 10^{-2}$
35	$4.5582\ 10^{-5}$	$3.0440\ 10^{-4}$	$4.0506\ 10^{-3}$

TABLE 10. BFO iterative method for Test 5, on $\mathcal{T}_2$, $d = 5$

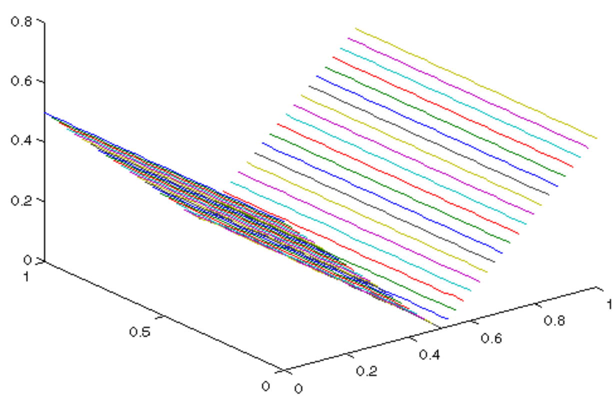

FIGURE 9. BFO Test 7 on Domain 2 and $\mathcal{I}_3$, $d = 5, r = 1$

5. Concluding Remarks

Remark 5.1. *For the finite element approximation of* (1.1)*, we note the remark in* [27]*, "Newton's and conjugate gradients methods may be well-suited for the solution of . . . combines the difficulty of both harmonic and bi-harmonic problems, making the* *approximation a delicate matter, albeit solvable ... If ... has no solution, we can expect the divergence of the Newton ... ” We have established that Newton’s method performed well for smooth solutions. Another problem which also combines the difficulty of both the harmonic and biharmonic problem is a singular perturbation problem we addressed in* [6] *by the spline element method. Here it is seen that for the singular perturbation problem arising from the vanishing moment methodology, the spline element method is robust. Moreover we note that numerical results for smooth solutions using Newton’s method in the spline element method are more accurate than what can be achieved using Argyris elements and the vanishing moment methodology* [34].

Remark 5.2. *It is still not known whether the BFO iterative method always converges even in the case of smooth solution. Nor is known whether Newton’s method always converges on a non-smooth domain. We have not addressed the convergence of the vanishing moment methodology to viscosity solutions as these results have been announced in* [34].

Remark 5.3. *We used C^1 cubic splines on most of the approximations even though they do not have full approximation power on general meshes. This reduced the computational cost. One may use for full approximation power, special meshes as in* [30] *or* [53].

Remark 5.4. *The BFO iterative method introduced in* [16] *was tested on some very singular right hand sides. It was noted that it is slower than another iterative method based on a different algebraic manipulation of the Monge-Ampère equation. The latter does not seem directly amenable to finite element computations. We note that for singular sources, specialized finite elements may have to be used and even in the finite difference context, specialized finite difference methods* [55] *or fast Poisson solvers* [54] *or preconditioners could have been used.*

Remark 5.5. *The problem: Find u such that*

$$\int_\Omega (cof\ D^2u)Du \cdot Dv \ \ dx = -n \int_\Omega fv \ \ dx, \tag{5.1}$$

is the Euler-Lagrange equation of the functional

$$\mathcal{J}(v) = \int_\Omega (cof\ D^2v)Dv \cdot Dvdx + 2n \int_\Omega fvdx. \tag{5.2}$$

If $v = 0$ on $\partial\Omega$, we have

$$\mathcal{J}(v) = -n \int_\Omega (\det D^2v)v + 2n \int_\Omega fvdx,$$

and a generalized solution of (1.1) *has been shown in* [11, 57] *to be a minimizer of a related functional on the set of convex functions.*

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