Dual Raviart-Thomas mixed finite elements

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Résumé

Pour un problème elliptique bidimensionnel, nous proposons de formuler la méthode des volumes finis avec des éléments finis mixtes de Petrov-Galerkin qui s'appuient sur la construction d'une base duale de Raviart-Thomas.

Abstract

For an elliptic problem with two space dimensions, we propose to formulate the finite volume method with the help of Petrov-Galerkin mixed finite elementsthat are based on the building of a dual Raviart-Thomas basis.

Key words : finite volumes, mixed finite elements, Petrov-Galerkin variational formulation, inf-sup condition, Poisson equation.

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Plan

1) Introduction.

• Let Ω be a bidimensional bounded convex domain in $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$. We consider the homogeneous Dirichlet problem for the Laplace operator in the domain Ω :

(1.1) $$-\Delta u \,=\, f \qquad \text{in } \Omega$$

(1.2) $$u \,=\, 0 \qquad \text{on the boundary } \partial\Omega \text{ of } \Omega.$$

We suppose that the datum f belongs to the space $L^2(\Omega)$,

(1.3) $$f \in L^2(\Omega)\,,$$

where this Hilbert space is classically defined according to

(1.4) $$\left\{ \begin{array}{l} L^2(\Omega) \,=\, \Big\{ v \,:\, \Omega \longrightarrow \mathbb{R}\,, \,\displaystyle\int_\Omega |v|^2 \,\mathrm{d}x < \infty \Big\} \\ (u,\, v) \,\equiv\, \displaystyle\int_\Omega u\, v \,\mathrm{d}x \,, \qquad \forall\, u,\, v \,\in L^2(\Omega)\,. \\ \|\, u \,\|_0^2 \,\equiv\, (u,\, u)\,, \qquad \forall\, u \,\in L^2(\Omega)\,. \end{array} \right.$$

We introduce the momentum p defined by

(1.5) $$p \,=\, \nabla u\,.$$

Taking the divergence of both terms arising in equation (1.5), taking into account the relation (1.1) and the hypothesis (1.3), we observe that the divergence of momentum p belongs to the space $L^2(\Omega)$. For this reason, we introduce the vectorial Sobolev space $H(\mathrm{div},\, \Omega)$:

(1.6) $$\left\{ \begin{array}{l} H(\mathrm{div},\, \Omega) \,=\, \big\{\, q \in L^2(\Omega) \times L^2(\Omega)\,,\, \mathrm{div}\, q \in L^2(\Omega) \,\big\} \\ \|\, q \,\|^2_{H(\mathrm{div},\, \Omega)} \,=\, \displaystyle\int_\Omega \big[\, |q|^2 \,+\, |\mathrm{div}\, q|^2 \,\big] \,\mathrm{d}x \,, \qquad \forall\, q \,\in H(\mathrm{div},\, \Omega) \end{array} \right.$$

and we suppose in the following that the momentum p satisfies the condition

(1.7) $$p \in H(\mathrm{div},\, \Omega)\,.$$

• The variational formulation of the problem (1.1) (1.2) with the help of the pair $\xi \,=\, (u,\, p)$ is obtained by testing the definition (1.5) against a vector valued function q and integrating by parts. With the help of the boundary condition, it comes :

$$(1.8) \qquad (p,\, q) + (u,\, \operatorname{div} q) \;=\; 0\,, \qquad \forall\, q \in H(\operatorname{div},\, \Omega)\,.$$

Independently, the relations (1.1) and (1.5) are integrated on the domain Ω after multiplying by a scalar valued function $v \in L^2(\Omega)$. We obtain :

$$(1.9) \qquad (\operatorname{div} p,\, v) + (f,\, v) \;=\; 0\,, \qquad \forall\, v \in L^2(\Omega)\,.$$

The "mixed" variational formulation is obtained by introducing the product space V defined as

$$(1.10) \qquad \left\{ \begin{array}{l} V \;=\; L^2(\Omega) \times H(\operatorname{div},\, \Omega)\,, \\ \|\, (u,\, p)\, \|_V^2 \;\equiv\; \|\, u\, \|_0^2 \;+\; \|\, p\, \|_0^2 \;+\; \|\, \operatorname{div} p\, \|_0^2 \;, \end{array} \right.$$

the following bilinear form $\gamma(\bullet,\, \bullet)$ defined on $V \times V$:

$$(1.11) \qquad \gamma\big((u,\, p),\, (v,\, q)\big) \;=\; (p,\, q) + (u,\, \operatorname{div} q) + (\operatorname{div} p,\, v)$$

and the linear form $\sigma(\bullet)$ defined on V according to :

$$(1.12) \qquad <\sigma,\, \zeta> \;=\; -(f,\, v)\;, \qquad \zeta = (v,\, q) \in V\,.$$

Then the Dirichlet problem (1.1)(1.2) takes the form :

$$(1.13) \qquad \left\{ \begin{array}{l} \xi \in V \\ \gamma(\xi,\, \zeta) \;=\; <\sigma,\, \zeta>\,, \qquad \forall\, \zeta \in V\,. \end{array} \right.$$

Due to classical inf-sup conditions introduced by Babuška [Ba71], the problem (1.13) admits a unique solution $\xi \in V$.

• We introduce a mesh $\mathcal{T}$ that is a bidimensional cellular complex (see *e.g.* Godbillon [Go71]) composed in our case by triangular elements K ($K \in \mathcal{E}_{\mathcal{T}}$), straight edges a ($a \in \mathcal{A}_{\mathcal{T}}$) and ponctual nodes S ($S \in \mathcal{S}_{\mathcal{T}}$). We conside also classical finite dimensional spaces $L^2_{\mathcal{T}}(\Omega)$ and $H_{\mathcal{T}}(\operatorname{div},\, \Omega)$ that approximate the spaces $L^2(\Omega)$ and $H(\operatorname{div},\, \Omega)$ respectively. A scalar valued function $v \in L^2_{\mathcal{T}}(\Omega)$ is constant in each triangle K of the mesh :

$$(1.14) \quad L^2_{\mathcal{T}}(\Omega) \;=\; \big\{ v : \Omega \longrightarrow \mathbb{R},\, \forall\, K \in \mathcal{E}_{\mathcal{T}},\, \exists\, v_K \in \mathbb{R},\, \forall\, x \in K,\, v(x) \;=\; v_K \,\big\}\,.$$

A vector valued function function $q \in H_{\mathcal{T}}(\operatorname{div},\, \Omega)$ is a linear combination of Raviart-Thomas [RT77] basis functions φ_a of lower degree, defined in the forthcoming section.

• Let $a \in \mathcal{A}_{\mathcal{T}}$ be an internal edge of the mesh, we denote by S and N the two vertices that compose its boundary ∂a (see Figure 1) :

$$(1.15) \qquad \partial a \;=\; \{\, \mathrm{S},\, \mathrm{N}\,\}$$

and by K and L the two elements that compose its co-boundary $\partial^c a$

$$(1.16) \qquad \partial^c a \;=\; \{\, K,\, L\,\}$$

in such a way that the normal direction n_a is oriented from K towards L and that the pair of vectors $(n_a, \overrightarrow{\mathrm{SN}})$ is direct, as shown on Figure 1. We denote by W (respectively by E) the third vertex of the triangle K (respectively of the triangle L) :

(1.17) $\quad K \;=\; (\mathrm{S, N, W})\,, \qquad L \;=\; (\mathrm{N, S, E})\,.$

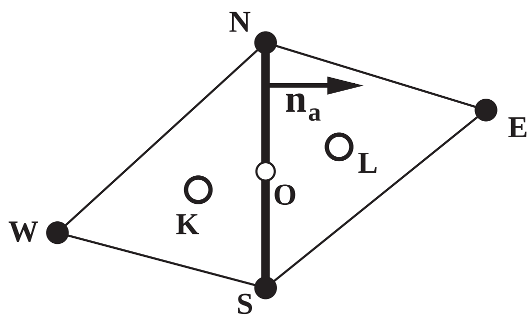

Figure 1. Co-boundary $(K,\, L)$ of the edge $a = (\mathrm{S, N})$.

The vector valued Raviart-Thomas [RT77] basis function φ_a is defined by the relations

$$
(1.18) \quad \varphi_a(x) \;=\; \begin{cases} \dfrac{1}{2\,|K|}\,(x - \mathrm{W})\,, & x \in K \\ -\dfrac{1}{2\,|L|}\,(x - \mathrm{E})\,, & x \in L \\ 0 & \text{elsewhere.} \end{cases}
$$

When the edge a is on the boundary $\partial\Omega$, we suppose that the normal n points towards the exterior of the domain, so the element L is absent. We have in all cases the $H(\mathrm{div},\, \Omega)$ conformity :

(1.19) $\quad \varphi_a \in H(\mathrm{div},\, \Omega)$

and the degrees of freedom are the fluxes of vector field φ_a for all the edges of the mesh (see [RT77]) :

$$
(1.20) \quad \int_b \varphi_a \bullet n_a \,\mathrm{d}\gamma \;=\; \delta_{a,\,b}\,, \qquad \forall\, a,\, b \in \mathcal{A}_{\mathcal{T}}\,.
$$

A vector valued function $q \in H_{\mathcal{T}}(\mathrm{div},\, \Omega)$ is a linear combination of the basis functions φ_a :

$$
(1.21) \quad q \;=\; \sum_{a \in \mathcal{A}_{\mathcal{T}}} q_a\, \varphi_a \;\in\; H_{\mathcal{T}}(\mathrm{div},\, \Omega) \;=\, <\varphi_b\,,\; b \in \mathcal{A}_{\mathcal{T}}>\,.
$$

• The mixed finite element method consists in choosing as discrete linear space the following product :

(1.22) $\quad V_{\mathcal{T}} \;=\; L^2_{\mathcal{T}}(\Omega) \,\times\, H_{\mathcal{T}}(\mathrm{div},\, \Omega)$

and proposes to replace the letter V by $V_{\mathcal{T}}$ inside the variational formulation (1.13) :

$$(1.23) \qquad \begin{cases} \xi_{\mathcal{T}} \in V_{\mathcal{T}} \\ \gamma(\xi_{\mathcal{T}},\, \zeta) \;=\; <\sigma,\, \zeta>\,, \qquad \forall \zeta \in V_{\mathcal{T}} \end{cases}$$

or in other terms

$$(1.24) \qquad \begin{cases} u_{\mathcal{T}} \in L^2_{\mathcal{T}}(\Omega)\,, \quad p_{\mathcal{T}} \in H_{\mathcal{T}}(\mathrm{div},\, \Omega) \\ (p_{\mathcal{T}}\,,\, q) \,+\, (u_{\mathcal{T}}\,,\, \mathrm{div}\, q) \;=\; 0\,, \qquad \forall\, q \in H_{\mathcal{T}}(\mathrm{div},\, \Omega) \\ (\mathrm{div}\, p_{\mathcal{T}}\,,\, v) \,+\, (f\,,\, v) \;\;\;=\; 0\,, \qquad \forall\, v \in L^2_{\mathcal{T}}(\Omega)\,. \end{cases}$$

The numerical analysis of the relations between the continuous problem (1.13) and the discrete problem (1.23) as the mesh $\mathcal{T}$ is more and more refined is classical [RT77]. The above method is popular in the context of petroleum and nuclear industries but suffers from the fact that the associated linear system is quite difficult to solve from a practical point of view. The introduction of supplementary Lagrange multipliers by Brezzi, Douglas and Marini [BDM85] allows a simplification of these algebraic aspects, their interpretation by Croisille in the context of box schemes [Cr2k] gives a good mathematical foundation of a popular numerical method and the possibility to reduce the size of the linear system has also been explored by Younès, Mose, Ackerer and Chavent [YMAC97].

- From a theoretical and practical point of view, the resolution of the system (1.24) can be conducted as follows. We introduce the mass-matrix $M_{a,\,b}$ associated with the Raviart-Thomas vector valued functions :

(1.25) $\quad M_{a,\,b} \;=\; (\varphi_a,\, \varphi_b)\,, \qquad a,\, b \in \mathcal{A}_{\mathcal{T}}\,.$

Then the first equation of (1.24) determines the momentum

$$(1.26) \qquad p_{\mathcal{T}} \;\equiv\; \sum_{a\, \in \mathcal{A}_{\mathcal{T}}} p_{\mathcal{T},a}\, \varphi_a$$

as a function of the mean values $u_{\mathcal{T},K}$ for $K \in \mathcal{E}_{\mathcal{T}}$:

$$(1.27) \qquad p_{\mathcal{T},a} \;=\; -\sum_{b\, \in \mathcal{A}_{\mathcal{T}}} \left(M^{-1}\right)_{a,\,b} \sum_{K\, \in \mathcal{E}_{\mathcal{T}}} u_{\mathcal{T},K} \int_K \mathrm{div}\, \varphi_b \, \mathrm{d}x\,.$$

The representation (1.27) suffers at our opinion form a major defect : due to the fact that the matrix M^{-1} is full, the discrete gradient $p_{\mathcal{T}}$ is a **global** function of the mean values $u_{\mathcal{T},K}$ and this property contradicts the mathematical foundations of the derivation operator to be **linear** and **local**. An *a posteriori* correction of this defect has been proposed by Baranger, Maître and Oudin [BMO96] and with an appropriate numerical integration of the mass matrix $M,$ it is possible to lump it and the discrete gradient in the direction n_a of the edge a is represented by a formula of the type :

(1.28) $\quad p_{\mathcal{T},a} \,=\, \dfrac{u_{\mathcal{T},L} - u_{\mathcal{T},K}}{h_a}$

with the notations of Figure 1. The substitution of the relation (1.28) inside the second equation of the formulation (1.24) conducts to a variant of the so-called finite volume method. In an analogous manner, the family of finite volume schemes proposed by Herbin [He95] supposes *a priori* that the discrete gradient in the normal direction admits a representation of the form (1.28). Nevertheless, the engineer intuition is not correctly satisfied by a scheme such that (1.28). The finite difference $\frac{u_{\mathcal{T},L}-u_{\mathcal{T},K}}{h_a}$ must be *a priori* to be a good (strong ?) approximation of the gradient $p_{\mathcal{T}} = \nabla u_{\mathcal{T}}$ in the direction $\overrightarrow{KL}$ whereas the coefficient $p_{\mathcal{T},a}$ is an approximation of $\int_{\rm a} \nabla u_{\mathcal{T}} {\bullet} n \,{\rm d}\tau$ in the **normal** direction (see again the Figure 1). When the mesh $\mathcal{T}$ is composed by general triangles, this approximation is not completely satisfactory at our opinion and contains a real limitation of these variants of the finite volume method.

• We recall here that the finite volume method for the approximation of the diffusion operators has been first proposed from empirical considerations. Following *e.g.* Noh [No64] and Patankar [Pa80], the idea is to represent the normal interface gradient $\int_{\rm a} \nabla u_{\mathcal{T}} {\bullet} n \,{\rm d}\tau$ as a function of **neighbouring** values. Given an edge a, a vicinity $\mathcal{V}(a)$ is **first** determined in order to represent the normal gradient $p_{\mathcal{T},a} = \int_{\rm a} \nabla u_{\mathcal{T}} {\bullet} n \,{\rm d}\tau$ with a "derivation formula" of the type

(1.29) $\quad \displaystyle\int_{\rm a} \nabla u_{\mathcal{T}} {\bullet} n \,{\rm d}\tau \,=\, \sum_{K \in \mathcal{V}(a)} g_{a,K}\, u_{\mathcal{T},K}\,.$

Then the conservation equation

(1.30) $\quad {\rm div}\, p \,+\, f \,=\, 0$

is integrated inside each cell $K \in \mathcal{E}_{\mathcal{T}}$ is order to determine an equation for the mean values $u_{\mathcal{T},K}$ for all $K \in \mathcal{E}_{\mathcal{T}}$. The difficulties of such approches have been presented by Kershaw [Ke81] and a variant of such scheme has been first analysed by Coudière, Vila and Villedieu [CVV99]. The key remark that we have done with F. Arnoux (see [Du89]), also observed by Faille, Gallouët and Herbin [FGH91] is that the representation (1.29) must be **exact** for linear functions $u_{\mathcal{T}}$. We took this remark as a starting point for our tridimensional finite volume scheme proposed in [Du92]. It is also an essential hypothesis for the result proposed by Coudière, Vila and Villedieu.

• In this contribution, we propose to discretize the variational problem (1.13) with the Petrov-Galerkin mixed finite element method, first introduced by Thomas and Trujillo [TT99]. In the way we have proposed in [Du2k], the idea is to construct a discrete functional space $\overset{\star}{H}_{\mathcal{T}}(\mathrm{div}, \Omega)$ generated by vectorial functions $\varphi_a^\star$, $a \in \mathcal{A}_{\mathcal{T}}$, that are conforming in the space $H(\mathrm{div}, \Omega)$

(1.31) $\quad \varphi_a^\star \in H(\mathrm{div}, \Omega)$

and to represent exactly the **dual basis** of the family $\{\varphi_b,\, b \in \mathcal{A}_{\mathcal{T}}\}$ with the L^2 scalar product :

(1.32) $\quad (\varphi_a \,,\, \varphi_b^\star) \,=\, \delta_{a,\,b} \,, \qquad \forall\, a,\, b \in \mathcal{A}_{\mathcal{T}} \,.$

(1.33) $\quad \overset{\star}{H}_{\mathcal{T}}(\mathrm{div}, \Omega) \,=\, <\varphi_b^\star \,,\, b \in \mathcal{A}_{\mathcal{T}}> \,\subset\, H(\mathrm{div}, \Omega) \,.$

Then the mixed Petrov-Galerkin mixed finite element method consists just in replacing the space $H_{\mathcal{T}}(\mathrm{div}, \Omega)$ by the dual space $\overset{\star}{H}_{\mathcal{T}}(\mathrm{div}, \Omega)$ for **test functions** in the first equation of discrete formulation (1.24). We obtain by doing this the so-called **Petrov-Galerkin finite volume** scheme :

(1.34) $$\left\{ \begin{array}{l} u_{\mathcal{T}} \in L^2_{\mathcal{T}}(\Omega) \,, \quad p_{\mathcal{T}} \in H_{\mathcal{T}}(\mathrm{div}, \Omega) \\ (p_{\mathcal{T}} \,,\, q) + (u_{\mathcal{T}} \,,\, \mathrm{div}\, q) \,=\, 0 \,, \qquad \forall\, q \in \overset{\star}{H}_{\mathcal{T}}(\mathrm{div}, \Omega) \\ (\mathrm{div}\, p_{\mathcal{T}} \,,\, v) + (f \,,\, v) \,=\, 0 \,, \qquad \forall\, v \in L^2_{\mathcal{T}}(\Omega) \,. \end{array} \right.$$

We introduce a compact form of the previous mixed Petrov-Galerkin formulation with the help of the product space $V_{\mathcal{T}}^\star$ defined by

(1.35) $\quad V_{\mathcal{T}}^\star \,=\, L^2_{\mathcal{T}}(\Omega) \times \overset{\star}{H}_{\mathcal{T}}(\mathrm{div}, \Omega) \,.$

Then the formulation (1.34) admits the form :

(1.36) $$\left\{ \begin{array}{l} \xi_{\mathcal{T}} \in V_{\mathcal{T}} \\ \gamma(\xi_{\mathcal{T}}, \zeta) \,=\, <\sigma, \zeta> \,, \qquad \forall\, \zeta \in V_{\mathcal{T}}^\star \,. \end{array} \right.$$

By doing this choice, it is easy to check that the scheme (1.34) is in fact a finite volume scheme for the Laplace operator. The key point is to construct the so-called **dual Raviart-Thomas basis functions** $\varphi_a^\star$ in order to guaranty Babuška's [Ba71] inf-sup stability property.

• The plan of the article is the following : we derive in the second part sufficient conditions in order to guaranty the final stability of the finite element scheme. Then we propose a particular family of dual Raviart-Thomas functions and propose by doing this a two-parameter family of finite volumes schemes.

2) Stability analysis

• We suppose in the following that the mesh $\mathcal{T}$ is a bidimensional cellular complex composed by triangles as proposed in the first section. Following the work of Ciarlet and Raviart [CR72], for any element $K \in \mathcal{E}_{\mathcal{T}}$ we denote by h_K the diameter of the triangle K and by ρ_K the diameter of the inscripted ball inside K. We suppose that the mesh $\mathcal{T}$ belongs to a family $\mathcal{U}_\theta$ of meshes that satisfies the following definition.

Definition 1. Family of regular meshes
Let θ be a strictly positive parameter. The family $\mathcal{U}_\theta$ of meshes is defined by the condition

$$(2.1) \qquad \mathcal{T} \in \mathcal{U}_\theta \iff \forall\, K \in \mathcal{E}_\mathcal{T}\,, \; \frac{h_K}{\rho_K} \leq \theta\,.$$

We suppose also that the dual space $H^\star_\mathcal{T}(\mathrm{div},\,\Omega)$ constructed by the conditions (1.31), (1.32), (1.33) satisfies the following hypothesis.

Hypothesis 1. Interpolation operator $H_\mathcal{T}(\mathrm{div},\,\Omega) \longrightarrow H^\star_\mathcal{T}(\mathrm{div},\,\Omega)\,.$
We suppose that the mesh $\mathcal{T}$ belongs to the family $\mathcal{U}_\theta$ of Definition 1 and that the dual basis $\varphi^\star_a$ is constructed in such a way that there exists a linear mapping $H_\mathcal{T}(\mathrm{div},\,\Omega) \ni q \longmapsto \Pi\, q \in H^\star_\mathcal{T}(\mathrm{div},\,\Omega)$ and strictly positive constants A, B, D, E that only depends on the parameter θ such that we have the following estimations :

$$(2.2) \qquad A \,\|\, q \,\|_0^2 \;\leq\; (\, q\,,\, \Pi\, q\,)\,, \qquad \forall\, q \in H_\mathcal{T}(\mathrm{div},\,\Omega)$$

$$(2.3) \qquad \|\, \Pi\, q \,\|_0 \;\leq\; B \,\|\, q \,\|_0\,, \qquad \forall\, q \in H_\mathcal{T}(\mathrm{div},\,\Omega)$$

$$(2.4) \qquad \|\, \mathrm{div}\, \Pi\, q \,\|_0 \;\leq\; D \,\|\, \mathrm{div}\, q \,\|_0\,, \qquad \forall\, q \in H_\mathcal{T}(\mathrm{div},\,\Omega)$$

$$(2.5) \qquad (\, \mathrm{div}\, q\,,\, \mathrm{div}\, \Pi\, q\,) \;\geq\; E \,\|\, \mathrm{div}\, q \,\|_0^2\,, \qquad \forall\, q \in H_\mathcal{T}(\mathrm{div},\,\Omega)\,.$$

Proposition 1. Divergence lifting of scalar fields
Let θ be a strictly positive parameter. We suppose that the dual Raviart-Thomas basis satisfies the Hypothesis 1. Then there exists some strictly positive constant F that only depends on the parameter θ such that for any mesh $\mathcal{T}$ that belongs to the family $\mathcal{U}_\theta$, and for any scalar field u constant in each element $K \in \mathcal{E}_\mathcal{T}$ ($u \in L^2_\mathcal{T}(\Omega)$), there exists some vector field $q \in H^\star_\mathcal{T}(\mathrm{div},\,\Omega)$ such that

$$(2.6) \qquad \|\, q \,\|_{H(\mathrm{div},\,\Omega)} \;\leq\; F \,\|\, u \,\|_0$$

$$(2.7) \qquad (\, u\,,\, \mathrm{div}\, q\,) \;\geq\; \|\, u \,\|_0^2\,.$$

Proof of Proposition 1.

• Let $u \in L^2_\mathcal{T}(\Omega)$ be a discrete scalar function supposed to be constant in each triangle K of the mesh $\mathcal{T}$. Let $\psi \in H^1_0(\Omega)$ be the variational solution of the Poisson problem

$$(2.8) \qquad \Delta\, \psi \;=\; u \quad \text{in}\ \Omega\,, \qquad\qquad \psi \;=\; 0 \quad \text{on}\ \partial\Omega\,.$$

Since Ω is convex, the solution ψ of the problem (2.8) belongs to the space $H^2(\Omega)$ and there exists some constant $G > 0$ that only depends on Ω such that

(2.9) $\quad \| \psi \|_2 \leq G \| u \|_0 .$

- Then the field $\nabla\psi$ belongs to the space $H^1(\Omega) \times H^1(\Omega)$. It is in consequence possible to interpolate this field in a continuous way (see *e.g.* Roberts and Thomas [RT91]) in the space $H(\mathrm{div}, \Omega)$ with the help of the fluxes on the edges :

(2.10) $$p_a = \int_a \frac{\partial\psi}{\partial n} \, \mathrm{d}\gamma , \qquad p = \sum_{a \in \mathcal{A}_{\mathcal{T}}} p_a \, \varphi_a \in H_{\mathcal{T}}(\mathrm{div}, \Omega)$$

and there exists a constant $L > 0$ that only depends on the parameter θ such that

(2.11) $\quad \| p \|_{H(\mathrm{div}, \Omega)} \leq L \| u \|_0 .$

- We observe that we have exactly

(2.12) $\quad \mathrm{div}\, p = u \qquad \text{in } \Omega .$

On one hand, the two fields $\mathrm{div}\, p$ and u are constant in each element K of the mesh $\mathcal{T}$. On the other hand, we have :

$$\int_K \mathrm{div}\, p \, \mathrm{d}x = \int_{\partial K} p {\bullet} n \, \mathrm{d}\gamma = \int_{\partial K} \frac{\partial\psi}{\partial n} \, \mathrm{d}\gamma = \int_K \Delta\psi \, \mathrm{d}x = \int_K u \, \mathrm{d}x$$

and the relation (2.12) is a consequence of the above property for the mean values.

- Let Πp be defined according to the Hypothesis 1, and

(2.13) $$q = \frac{1}{E} \Pi p .$$

We have as a consequence of (2.5) and (2.12) :

$$(u , \mathrm{div}\, q) = \frac{1}{E} (\mathrm{div}\, p , \mathrm{div}\, \Pi p) \geq \| \mathrm{div}\, p \|_0^2 = \| u \|_0^2$$

that establishes (2.7). Moreover, we have due to (2.3), (2.4) and (2.11) :

$$\| q \|_0 = \frac{1}{E} \| \Pi p \|_0 \leq \frac{B}{E} \| p \|_0 \leq \frac{BL}{E} \| u \|_0$$

$$\| \mathrm{div}\, q \|_0 = \frac{1}{E} \| \mathrm{div}\, \Pi p \|_0 \leq \frac{D}{E} \| \mathrm{div}\, p \|_0 = \frac{D}{E} \| u \|_0 .$$

Then due to the definition (1.6), the two above inequalities establish the estimate (2.6) with $F = \frac{1}{E} \sqrt{B^2 L^2 + D^2}$ and the Proposition is proven. □

Proposition 2. Discrete stability

Let θ be a strictly positive parameter. We suppose that the dual Raviart-Thomas basis satisfies the Hypothesis 1. Then we have the following discrete stability for the Petrov-Galerkin mixed formulation (1.36) :

(2.14) $$\left\{ \begin{array}{l} \exists \beta > 0 , \; \forall \mathcal{T} \in \mathcal{U}_\theta , \; \forall \xi \in V_{\mathcal{T}} \text{ such that } \| \xi \|_V = 1 , \\ \exists \eta \in V_{\mathcal{T}}^\star , \; \| \eta \|_V \leq 1 \text{ and } \gamma(\xi, \eta) \geq \beta . \end{array} \right.$$

with $\gamma(\bullet,\,\bullet)$ defined at the relation (1.11) and β chosen such that

$$\text{(2.15)}\qquad \sqrt{1-\frac{B+2D}{A}\beta\,-\,\beta^2}\;\geq\;\Big(1+F\Big(1\,+\,\sqrt{\frac{B+2A}{A}}\Big)\Big)\,\sqrt{\beta}\,.$$

Proof of Proposition 2.

- We set $\xi\equiv(u,\,p)$ satisfying the hypothesis (2.14) :

$$\text{(2.16)}\qquad \|\,\xi\,\|_V^2\;\equiv\;\|\,u\,\|_0^2\;+\;\|\,p\,\|_0^2\;+\;\|\,\mathrm{div}\,p\,\|_0^2\;=\;1\,.$$

Then at last one of these terms is not too small and due to the three terms that arise in relation (1.11), the proof is divided into three parts.

- If the following condition

$$\text{(2.17)}\qquad \|\,\mathrm{div}\,p\,\|_0\,\geq\;\beta\,,$$

is satisfied, we set

$$\text{(2.18)}\qquad v\;=\;\frac{\mathrm{div}\,p}{\|\,\mathrm{div}\,p\,\|_0}\,,\quad q\;=\;0\,,\qquad \zeta\;=\;(v,\,q)\,\in\,V_{\mathcal{T}}^{\star}\,.$$

Then $\|\,\mathrm{div}\,v\,\|_0=\;1$ and $\|\,\zeta\,\|_0\leq\;1$. Moreover $\gamma(\xi,\,\zeta)\;=\;(\,\mathrm{div}\,p\,,\,v\,)\;=$ $=\|\,\mathrm{div}\,p\,\|_0\,\geq\;\beta$ by hypothesis (2.17) and the relation (2.14) is satisfied in this particular case.

- Under the following conditions

$$\text{(2.19)}\qquad \|\,\mathrm{div}\,p\,\|_0\,\leq\;\beta\quad\text{and}\quad\|\,p\,\|_0\;\geq\;\sqrt{\frac{B+2D}{A}}\,\sqrt{\beta}\,,$$

we set

$$\text{(2.20)}\qquad v\;=\;0\,,\quad q\;=\;\frac{1}{B+D}\,\Pi\,p\,,\qquad \zeta\;=\;(v,\,q)\,\in\,V_{\mathcal{T}}^{\star}\,.$$

The following inequalities are a direct consequence of (2.3) and (2.4) :

$$\|\,q\,\|_0\;\leq\;\frac{B}{B+D}\;\|\,p\,\|_0\qquad\text{and}\qquad\|\,\mathrm{div}\,q\,\|_0\leq\;\frac{D}{B+D}\;\|\,\mathrm{div}\,p\,\|_0$$

so we deduce :

$$\|\,q\,\|_{H(\mathrm{div},\,\Omega)}^2\;\leq\;\frac{B^2+D^2}{(B+D)^2}\;\|\,\mathrm{div}\,p\,\|_0^2\;\leq\;1$$

because $B>0\,,\;D>0$. Then $\|\,\zeta\,\|_V\,\leq\;1$ and we have also

$$\begin{aligned}\gamma(\xi,\,\zeta)\;&=\;(p,\,q)\,+\,(u,\,\mathrm{div}\,q)\,+\,(\mathrm{div}\,p,\,v)\\ &\geq\;\frac{A}{B+D}\;\|\,p\,\|_0^2\;-\;\frac{D}{B+D}\,\beta\;\|\,u\,\|_0\\ &\geq\;\frac{1}{B+D}\,\big((B+2D)\,\beta\,-\,D\,\beta\big)\qquad\text{because}\quad\|\,u\,\|_0\leq\;1\\ &\geq\;\beta\qquad\text{and the property is established in this case.}\end{aligned}$$

- If the two previous conditions (2.17) and (2.19) are in defect, *i.e.* if we have

$$\text{(2.21)} \qquad \| \operatorname{div} p \|_0 \leq \beta \quad \text{and} \quad \| p \|_0 \leq \sqrt{\frac{B+2D}{A}} \sqrt{\beta} \,,$$

then $\| u \|_0^2 = 1 - \| p \|_0^2 - \| \operatorname{div} p \|_0^2 \geq 1 - \dfrac{B+2D}{A} \beta - \beta^2 \geq \beta > 0$
due to the hypothesis (2.15). From the Proposition 1, there exists some vector field $\widetilde{q} \in H_{\mathcal{T}}^{\star}(\operatorname{div}, \Omega)$ satisfying $\| \widetilde{q} \|_{H(\operatorname{div}, \Omega)} \leq F \| u \|_0$ and $(u, \operatorname{div} \widetilde{q}) \geq \| u \|_0^2$. We set

$$\text{(2.22)} \qquad v = 0 \,, \quad q = \frac{1}{F} \widetilde{q} \,, \qquad \zeta = (v, q) \in V_{\mathcal{T}}^{\star} \,,$$

then

$$\| \zeta \|_V = \frac{1}{F} \| \widetilde{q} \|_{H(\operatorname{div}, \Omega)} \leq \| u \|_0 \leq 1$$

due to the hypothesis (2.14) relative to $\| \xi \|_V$. Moreover, we have

$$\begin{aligned}
\gamma(\xi, \zeta) &= (p, q) + (u, \operatorname{div} q) + (\operatorname{div} p, v) \\
&= \frac{1}{F} (p, \widetilde{q}) + \frac{1}{F} (u, \operatorname{div} \widetilde{q}) \\
&\geq \frac{1}{F} \Big(-\sqrt{\frac{B+2D}{A}} \sqrt{\beta} \, F \| u \|_0 + \| u \|_0^2 \Big) \\
&\geq \| u \|_0 \Big(\frac{1}{F} \| u \|_0 - \sqrt{\frac{B+2D}{A}} \sqrt{\beta} \Big) \\
&\geq \sqrt{\beta} \Big(\frac{1}{F} \Big(1 + F \Big(1 + \sqrt{\frac{B+2A}{A}} \Big) \Big) \sqrt{\beta} - \sqrt{\frac{B+2D}{A}} \sqrt{\beta} \Big) \\
&\geq \sqrt{\beta} \Big(\frac{1}{F} \sqrt{\beta} + \sqrt{\beta} \Big) \\
&\geq \beta
\end{aligned}$$

and the property is satisfied for this last case.

The Proposition 2 is established. □

Theorem 1. Error estimate

Let Ω be a two-dimensional open convex domain of $\mathbb{R}^2$ with a polygonal boundary, $u \in H^2(\Omega)$ be the solution of the problem (1.1)(1.2) considered under variational formulation and $p = \nabla u$ be the associated momentum. Let θ be a strictly positive parameter, $\mathcal{U}_\theta$ a family of meshes $\mathcal{T}$ and $V_{\mathcal{T}}^{\star}$ defined in (1.35) and associated with a choice of a dual Raviart-Thomas basis that satisfies the Hypothesis 1. Let $\xi \equiv (u_{\mathcal{T}}, p_{\mathcal{T}}) \in V_{\mathcal{T}}$ be the solution of the discrete problem (1.34). Then there exists some constant $C > 0$ that only depends on the parameter θ such that

$$(2.23) \qquad \| u - u_{\mathcal{T}} \|_0 + \| p - p_{\mathcal{T}} \|_{H(\mathrm{div},\,\Omega)} \leq C \, h_{\mathcal{T}} \, \| f \|_0 \,.$$

Proof of Theorem 1.

• On one hand, it is sufficient to apply the general approximation Theorem established by Babuška's for continuous (respectively discrete) variational mixed systems (1.13) (respectively (1.36)) *i.e.* to verify that the bilinear form $\gamma(\bullet, \bullet)$ defined in (1.11) is continuous on the Hilbert space $V = L^2(\Omega) \times H(\mathrm{div}, \Omega)$, which is clear. It is also necessary to verify the so-called discrete inf-sup condition (2.14), that has been established at the Proposition 2. Last but not least, it is necessary to satisfy the following infinity condition :

$$(2.24) \qquad \forall \eta \in V_{\mathcal{T}}^{\star}, \quad \eta \neq 0 \implies \sup_{\xi \in V_{\mathcal{T}}} \gamma(\xi, \eta) = +\infty \,.$$

• The infinity condition (2.24) is established as follows. Let $\zeta \equiv (v, q) \in V_{\mathcal{T}}^{\star}$ be a "test vector" different from zero. If there exists some mesh element $K \in \mathcal{E}_{\mathcal{T}}$ such that $\int_K \mathrm{div}\, q \, \mathrm{d}x \neq 0$, then we consider $\xi = (u, p)$ chosen according to $u = \lambda \widetilde{u}$ and $p = 0$. We suppose that the field $\widetilde{u} \in L^2_{\mathcal{T}}$ is null for all the elements of the mesh $\mathcal{T}$ except for the particular element K where we suppose $\widetilde{u}_K = \int_K \mathrm{div}\, q \, \mathrm{d}x$. Then we have $\gamma(\xi, \zeta) \equiv (p, q) + (u, \mathrm{div}\, q) + (\mathrm{div}\, p, v) = \lambda \left(\int_K \mathrm{div}\, q \, \mathrm{d}x \right)^2$, which tends to infinity as λ tends to infinity. If $\int_K \mathrm{div}\, q \, \mathrm{d}x = 0$ for all mesh elements $K \in \mathcal{E}_{\mathcal{T}}$ and if the field q is not null, we can write it on the form $q = \Pi \widetilde{p}$ with $\widetilde{p} \in H_{\mathcal{T}}(\mathrm{div}, \Omega)$ because the mapping Π is clearly bijective due to the property (2.2). We set $p = \lambda \widetilde{p}$ and $u = 0$. Then $\gamma(\xi, \zeta) = (p, q) = \lambda (\widetilde{p}, q) \geq \lambda A \| q \|_0^2$ due to the hypothesis (2.2) ; the infinity property (2.24) is established in this second particular case because $q \neq 0$. If $q = 0$, then v is not null due to the left hand side of (2.24). Following the proof of Proposition 1, we introduce the vector field $\widetilde{p} \in H_{\mathcal{T}}(\mathrm{div}, \Omega)$ satisfying the relations (2.11) and (2.12) : $\mathrm{div}\, \widetilde{p} \equiv v$ and $\| \widetilde{p} \|_{H(\mathrm{div},\,\Omega)} \leq L \, \| v \|_0$. We set $p = \lambda \widetilde{p}$, $u = 0$ and $\xi = (u, p)$. Then $\gamma(\xi, \zeta) = \lambda (\mathrm{div}\, \widetilde{p}, v) = \lambda \| v \|_0^2$ tends to infinity when λ tends to infinity, and the infinity condition (2.24) is established.

• The conclusion of the Babuška's Theorem [Ba71] assures the existence of some constant $C > 0$ that only depends on θ such that the error between the solution of the continuous problem (1.13) and the discrete problem (1.23) is majorated by the interpolation error :

$$(2.25) \qquad \left\{ \begin{array}{l} \| u - u_{\mathcal{T}} \|_0 + \| p - p_{\mathcal{T}} \|_{H(\mathrm{div},\,\Omega)} \leq \\ \leq C \left(\inf\limits_{v \in L^2_{\mathcal{T}}} \| u - v \|_0 + \inf\limits_{q \in H_{\mathcal{T}}(\mathrm{div},\,\Omega)} \| p - q \|_{H(\mathrm{div},\,\Omega)} \right). \end{array} \right.$$

Then following classical interpolation results for scalar [CR72] and vectorial [RT77] fields, we deduce from (2.25) :

$$\| u - u_{\mathcal{T}} \|_0 + \| p - p_{\mathcal{T}} \|_{H(\mathrm{div},\,\Omega)} \leq C \, \big(h_{\mathcal{T}} \, \| u \|_1 + h_{\mathcal{T}} \, \| p \|_1 \big)$$
$$\leq C \, h_{\mathcal{T}} \, \| u \|_2 \qquad \leq C \, h_{\mathcal{T}} \, \| f \|_0$$

and the Theorem 1 is proven. □

3) Towards a first Petrov-Galerkin finite volume scheme

• We propose in this section to formulate some ideas in order to construct a dual Raviart-Thomas basis $\varphi_a^\star$ where a is an internal edge of the mesh $\mathcal{T}$ $(a \in \mathcal{A}_{\mathcal{T}})$. Following (1.15) and (1.16), we denote by $a \equiv (\mathrm{S},\, \mathrm{N})$ this edge, by O the middle of SN and by K, L the two triangles that compose the co-boundary. The normal n_a is supposed to be oriented from the element K towards the element L and there exists two vertices W and E such that the relation (1.17) holds (see the Figure 1). We consider the four edges (N, W), (W, S), (S, E) and (E, N) that compose the boundary of the union $K \cup L$. We define four new triangles M, P, Q and R and four new vertices A, B, C and D in the mesh $\mathcal{T}$ by the relations

$$\text{(3.1)} \qquad \left\{ \begin{array}{lll} \partial^c(\mathrm{E},\,\mathrm{N}) \equiv (L,\, M) \,, & & M \equiv (\mathrm{N},\,\mathrm{E},\,\mathrm{A}) \\ \partial^c(\mathrm{N},\,\mathrm{W}) \equiv (K,\, P) \,, & & P \equiv (\mathrm{W},\,\mathrm{N},\,\mathrm{B}) \\ \partial^c(\mathrm{W},\,\mathrm{S}) \equiv (K,\, Q) \,, & & Q \equiv (\mathrm{S},\,\mathrm{W},\,\mathrm{C}) \\ \partial^c(\mathrm{S},\,\mathrm{E}) \equiv (L,\, R) \,, & & R \equiv (\mathrm{E},\,\mathrm{S},\,\mathrm{D}) \end{array} \right.$$

as illustrated on the Figure 2.

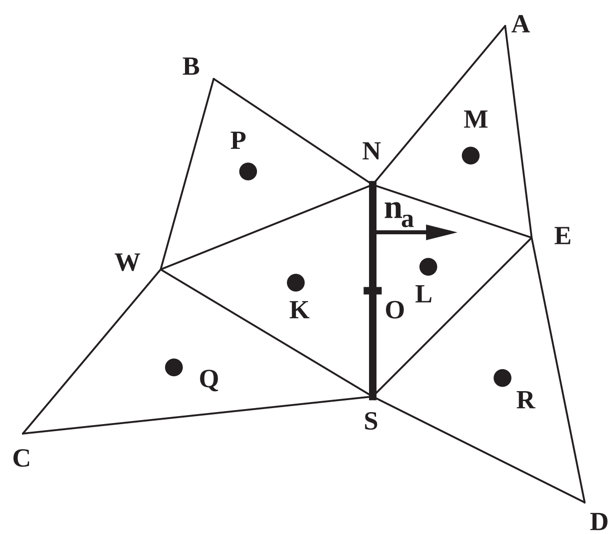

Figure 2 : support $\mathcal{V}(\mathrm{S},\,\mathrm{N})$ of the dual Raviart-Thomas basis function $\varphi_{\mathrm{SN}}^\star$.

Hypothesis 2. A simple choice for dual Raviart Thomas basis functions

We suppose first that the Raviart Thomas dual basis $\varphi_b^\star$, $(b \in \mathcal{A}_\mathcal{T})$ satisfies the $H(\mathrm{div})$ conformity property (1.31) and the orthogonality (1.32). Moreover, we suppose that for each internal edge $a \equiv (\mathrm{S}, \mathrm{N})$, the support of the dual Raviart-Thomas basis function $\varphi_{\mathrm{SN}}^\star$ is included in a vicinity $\mathcal{V}(a) = \mathcal{V}(\mathrm{S}, \mathrm{N})$ composed by the six triangles K, L, M, P, Q and R introduced previously (see the Figure 2) :

$$
(3.2) \qquad \left\{ \begin{array}{l} \mathcal{V}(\mathrm{S}, \mathrm{N}) \ \subset \ K \cup L \cup M \cup P \cup Q \cup R \,, \\ \mathrm{supp}\left(\varphi_{\mathrm{SN}}^\star \right) \ \subset \ \mathcal{V}(\mathrm{S}, \mathrm{N}) \,. \end{array} \right.
$$

We suppose also that the divergence field $\mathrm{div}\, \varphi_a^\star$ is **constant in each triangle** of the mesh :

$$
(3.3) \qquad \mathrm{div}\, \varphi_a^\star \ \in \ L_\mathcal{T}^2(\Omega) \,, \qquad \forall\, a \in \mathcal{A}_\mathcal{T} \,.
$$

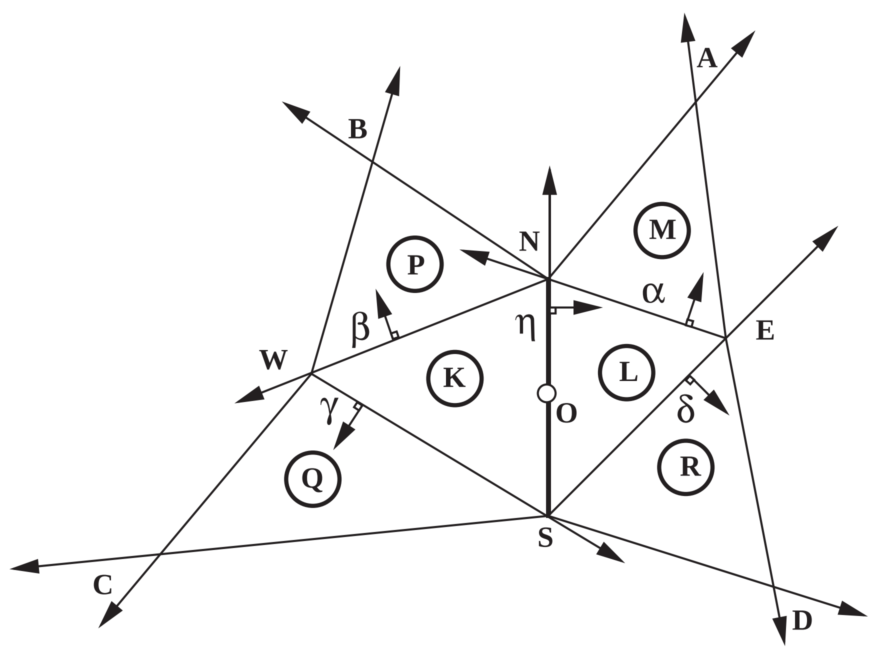

Figure 3 : Notations and orientations.

Theorem 2. Necessary condition for a dual Raviart-Thomas basis.

Let $\varphi_{\mathrm{SN}}^\star$ be a dual Raviart Thomas basis function satisfying the Hypothesis 2. We introduce the following fluxes accross the internal edges SN, EN, NW, WS and SE respectively :

$$(3.4)\quad \left\{ \begin{array}{l} \eta \equiv \displaystyle\int_{\rm SN} \varphi^\star_{\rm SN} \bullet n_{\rm SN} \,\, {\rm d}\gamma \,, \\ \alpha \equiv \displaystyle\int_{\rm EN} \varphi^\star_{\rm SN} \bullet n_{\rm EN} \,\, {\rm d}\gamma \,, \qquad \beta \equiv \displaystyle\int_{\rm NW} \varphi^\star_{\rm SN} \bullet n_{\rm NW} \,\, {\rm d}\gamma \,, \\ \gamma \equiv \displaystyle\int_{\rm WS} \varphi^\star_{\rm SN} \bullet n_{\rm WS} \,\, {\rm d}\gamma \,, \qquad \delta \equiv \displaystyle\int_{\rm SE} \varphi^\star_{\rm SN} \bullet n_{\rm SE} \,\, {\rm d}\gamma \,. \end{array} \right.$$

Then we have the necessary conditions :

$$(3.5)\quad \eta \, \overrightarrow{\rm KL} + \alpha \, \overrightarrow{\rm LM} + \beta \, \overrightarrow{\rm KP} + \gamma \, \overrightarrow{\rm KQ} + \delta \, \overrightarrow{\rm LR} \,=\, |\overrightarrow{\rm SN}| \,\, n_{\rm SN}$$

$$(3.6)\quad \left\{ \begin{array}{l} \alpha \, \overrightarrow{\rm LM} \bullet \overrightarrow{\rm WA} + \beta \, \overrightarrow{\rm KP} \bullet \overrightarrow{\rm EB} + \gamma \, \overrightarrow{\rm KQ} \bullet \overrightarrow{\rm EC} + \delta \, \overrightarrow{\rm LR} \bullet \overrightarrow{\rm WD} \,= \\ \qquad\qquad\qquad\qquad = \, -3 \, |\overrightarrow{\rm SN}| \,\, n_{\rm SN} \bullet \big(\overrightarrow{\rm OL} + \overrightarrow{\rm OK} \big) \,. \end{array} \right.$$

• The finite volume approach is then obained with a six point scheme for the mean gradient in the normal direction in the manner of (1.29) thanks to the first equation of the mixed variational formulation (1.24) :

$$(3.7)\quad \left\{ \begin{array}{l} \displaystyle\int_{\rm SN} \nabla u_{\mathcal T} \bullet n \,\, {\rm d}\gamma \,=\, \eta \, (u_{\rm L} - u_{\rm K}) + \\ \quad + \alpha \, (u_{\rm M} - u_{\rm L}) + \beta \, (u_{\rm P} - u_{\rm K}) + \gamma \, (u_{\rm Q} - u_{\rm K}) + \delta \, (u_{\rm R} - u_{\rm L}). \end{array} \right.$$

We remark that the constraints (3.5) express that the relation (3.7) is **exact** if the field $u_{\mathcal T}$ is an affine function.

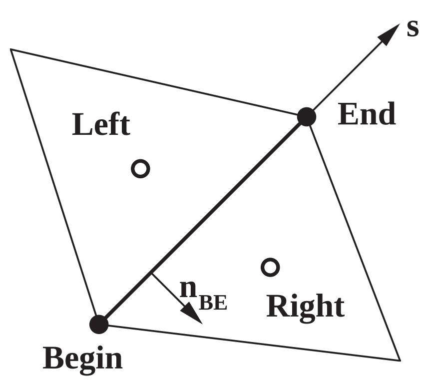

Figure 4 : Notations for an arbitrary edge BE.

• We precise some notations that we will use in the next pages. Let (B, E) be an edge of the mesh (see *e.g.* the Figure 4), and (L, R) its co-boundary. If the edge is directed from B towards E, the axis s has its origin at vertex B and the normal $n_{\rm BE}$ is oriented from L to R in such a way that the pair of vectors $(n_{\rm BE}, \overrightarrow{\rm BE})$ is direct. If $\xi \equiv \int_{\rm BE} \varphi^\star \bullet n_{\rm BE} \, {\rm d}s$ is the flux of the function $\varphi^\star$ accross the edge (B, E), we will denote by ξ_1 , $\widetilde{\xi}_1$, ξ_2 , and $\widetilde{\xi}_2$ the following momenta :

$$(3.8) \qquad \left\{ \begin{array}{l} \xi_1 \,=\, \displaystyle\int_{\rm BE} \varphi^\star \bullet n_{\rm BE} \;\, s \, {\rm d}s \,, \quad \widetilde{\xi}_1 \,=\, \int_{\rm BE} \varphi^\star \bullet n_{\rm BE} \;\, ({\rm BE} - s) \, {\rm d}s \,, \\ \xi_2 \,=\, \displaystyle\int_{\rm BE} \varphi^\star \bullet n_{\rm BE} \;\, s^2 \, {\rm d}s \,, \quad \widetilde{\xi}_2 \,=\, \int_{\rm BE} \varphi^\star \bullet n_{\rm BE} \;\, ({\rm BE} - s)^2 \, {\rm d}s \,. \end{array} \right.$$

The proof of Theorem 2 needs a certain number of technical lemmae and preliminary propositions.

Lemma 1. Radius of giration

Let $M = ({\rm N, E, A})$ be a triangle of the mesh $\mathcal{T}$ and M its associated center of gravity (see the Figure 4). We will denote by ρ_M the radius of giration :

$$(3.9) \qquad \rho_M \,=\, \sqrt{\frac{1}{36} \, ({\rm AN}^2 \,+\, {\rm NE}^2 \,+\, {\rm EA}^2 \,)}$$

Then we have

$$(3.10) \qquad \frac{1}{|\,M\,|} \int_M |\,x - {\rm N}\,|^2 \;\, {\rm d}x \,=\, \rho_M^2 \,+\, {\rm NM}^2 \,.$$

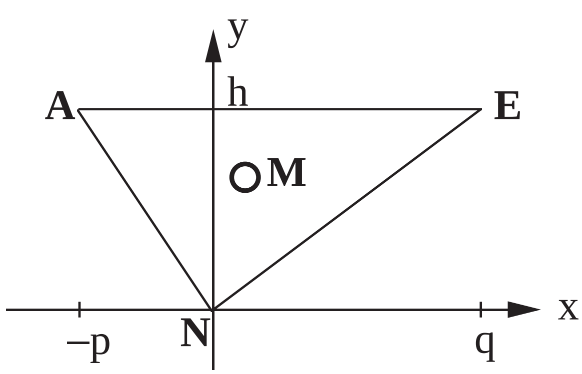

Figure 5 : About the radius of giration.

Proof of Lemma 1.

We have on one hand :

$$|\,M\,| = \frac{1}{2} \, h \, (p+q) \,,$$

$$\begin{aligned} \int_M |\,x - {\rm N}\,|^2 \;\, {\rm d}x \,&=\, \int_0^h {\rm d}y \, \int_{-p \, \frac{y}{h}}^{q \, \frac{y}{h}} {\rm d}x \, \big(x^2 + y^2\big) \\ &=\, \int_0^h {\rm d}y \, \bigg(\frac{1}{3} \, (p^3 + q^3) \, \Big(\frac{y}{h}\Big)^3 \,+\, (p+q) \, \frac{y}{h} \, y^2 \bigg) \\ &=\, \frac{1}{12} \, (p+q) \, (p^2 - p\,q + q^2) \, h \,+\, \frac{1}{4} \, (p+q) \, h^3 \end{aligned}$$

$$= |\,M\,| \,\Big(\frac{1}{6}\,(p^2 - p\,q + q^2) \,+\, \frac{1}{2}\,h^2 \Big)$$

and on the other hand :

$$\frac{1}{36}\,(\mathrm{AN}^2 \,+\, \mathrm{NE}^2 \,+\, \mathrm{EA}^2\,) \,+\, \mathrm{NM}^2 \;=$$

$$\begin{aligned} &= \frac{1}{36}\,\big[\,p^2 + h^2 + q^2 + h^2 + (p+q)^2\,\big] \,+\, \frac{1}{9}\,\big[\,(p-q)^2 \,+\, 4\,h^2\,\big] \\ &= \frac{1}{36}\,\big(\,6\,p^2 \,+\, 6\,q^2 \,-\, 6\,p\,q \,+\, 18\,h^2\,\big) \\ &= \frac{1}{6}\,\big(\,p^2 \,+\, q^2 \,-\, p\,q \,+\, 3\,h^2\,\big)\,. \end{aligned}$$

So the relation (3.10) is established. □

Proposition 3. First relations between momenta
The Hypothesis 2 implies the following relations inside the triangle $M = (\mathrm{N},\, \mathrm{E},\, \mathrm{A})$:

$$(3.11)\qquad \left\{ \begin{array}{llll} \alpha_1 \,=\, \overrightarrow{\mathrm{EM}} \bullet \dfrac{\overrightarrow{\mathrm{EN}}}{\mathrm{EN}}\,\alpha\,, & & \widetilde{\alpha}_1 \,=\, -\overrightarrow{\mathrm{NM}} \bullet \dfrac{\overrightarrow{\mathrm{EN}}}{\mathrm{EN}}\,\alpha\,, \\ \alpha_2 \,=\, (\,\rho_M^2 \,+\, \mathrm{EM}^2\,)\,\alpha\,, & & \widetilde{\alpha}_2 \,=\, (\,\rho_M^2 \,+\, \mathrm{NM}^2\,)\,\alpha\,, \end{array} \right.$$

and the analogous ones obtained from the Figure 3 in the triangles $P = (\mathrm{W},\, \mathrm{N},\, \mathrm{B})$, $Q = (\mathrm{S},\, \mathrm{W},\, \mathrm{C})$ and $R = (\mathrm{E},\, \mathrm{S},\, \mathrm{D})$:

$$(3.12)\qquad \left\{ \begin{array}{llll} \beta_1 \,=\, \overrightarrow{\mathrm{NP}} \bullet \dfrac{\overrightarrow{\mathrm{NW}}}{\mathrm{NW}}\,\beta\,, & & \widetilde{\beta}_1 \,=\, -\overrightarrow{\mathrm{WP}} \bullet \dfrac{\overrightarrow{\mathrm{NW}}}{\mathrm{NW}}\,\beta\,, \\ \beta_2 \,=\, (\,\rho_P^2 \,+\, \mathrm{NP}^2\,)\,\beta\,, & & \widetilde{\beta}_2 \,=\, (\,\rho_P^2 \,+\, \mathrm{WP}^2\,)\,\beta\,, \end{array} \right.$$

$$(3.13)\qquad \left\{ \begin{array}{llll} \gamma_1 \,=\, \overrightarrow{\mathrm{WQ}} \bullet \dfrac{\overrightarrow{\mathrm{WS}}}{\mathrm{WS}}\,\gamma\,, & & \widetilde{\gamma}_1 \,=\, -\overrightarrow{\mathrm{SQ}} \bullet \dfrac{\overrightarrow{\mathrm{WS}}}{\mathrm{WS}}\,\gamma\,, \\ \gamma_2 \,=\, (\,\rho_Q^2 \,+\, \mathrm{WQ}^2\,)\,\gamma\,, & & \widetilde{\gamma}_2 \,=\, (\,\rho_Q^2 \,+\, \mathrm{SQ}^2\,)\,\gamma\,, \end{array} \right.$$

$$(3.14)\qquad \left\{ \begin{array}{llll} \delta_1 \,=\, \overrightarrow{\mathrm{SR}} \bullet \dfrac{\overrightarrow{\mathrm{SE}}}{\mathrm{SE}}\,\delta\,, & & \widetilde{\delta}_1 \,=\, -\overrightarrow{\mathrm{ER}} \bullet \dfrac{\overrightarrow{\mathrm{SE}}}{\mathrm{SE}}\,\delta\,, \\ \delta_2 \,=\, (\,\rho_R^2 \,+\, \mathrm{SR}^2\,)\,\delta\,, & & \widetilde{\delta}_2 \,=\, (\,\rho_R^2 \,+\, \mathrm{ER}^2\,)\,\delta\,. \end{array} \right.$$

Proof of Proposition 3.

- We write the orthogonality (1.32) between the two edges $a = (\mathrm{S},\, \mathrm{N})$ and the edge $b = (\mathrm{A},\, \mathrm{N})$ (see the Figure 3). Inside the triangle $M = (\mathrm{N},\, \mathrm{E},\, \mathrm{A})$, we have

$$\varphi_{\mathrm{AN}} \;=\; \frac{1}{2\,|\,M\,|}\,(x - \mathrm{E}) \;=\; \frac{1}{4\,|\,M\,|}\;\nabla(|\,x - \mathrm{E}\,|^2)$$

then

$$
\begin{aligned}
0 \;&=\; \int_\Omega \varphi^\star_{\rm SN} \bullet \varphi_{\rm AN} \, {\rm d}x \;=\; \int_M \varphi^\star_{\rm SN} \bullet \varphi_{\rm AN} \, {\rm d}x \\
&=\; - \int_M \big({\rm div}\, \varphi^\star_{\rm SN} \big) \, \frac{1}{4\,|M|} \;\; |x - {\rm E}|^2 \;\, {\rm d}x \;\; + \;\; \int_{\partial M} \big(\varphi^\star_{\rm SN} \bullet n \big) \, \frac{1}{4\,|M|} \;\; |x - {\rm E}|^2 \;\, {\rm d}\gamma \\
&=\; - \big({\rm div}\, \varphi^\star_{\rm SN} \big)(M) \int_M \frac{1}{4\,|M|} \;\; |x - {\rm E}|^2 \;\, {\rm d}x \\
&\qquad\qquad\qquad\qquad - \; \frac{1}{4\,|M|} \int_{\rm NE} \big(\varphi^\star_{\rm SN} \bullet n_{\rm NE} \big) \;\, |x - {\rm E}|^2 \;\, {\rm d}\gamma \\
&=\; \frac{1}{4\,|M|} \bigg(\frac{\alpha}{|M|} \int_M \;\, |x - {\rm E}|^2 \;\, {\rm d}x \;\; - \;\; \int_{\rm EN} \big(\varphi^\star_{\rm SN} \bullet n_{\rm NE} \big) \;\, |x - {\rm E}|^2 \;\, {\rm d}s \; \bigg) \\
&=\; \frac{1}{4\,|M|} \Big(\alpha \; (\, \rho_M^2 \; + \; {\rm EM}^2 \,) \;\; - \;\; \alpha_2 \; \Big)
\end{aligned}
$$

and the third relation of (3.11) is proven.

- We write now the orthogonality (1.32) between the two edges $a = ({\rm S},\,{\rm N})$ and $b = ({\rm E},\,{\rm A})$ inside the triangle $M = ({\rm N},\,{\rm E},\,{\rm A})$. When we exchange the roles of the two vertices N and E in the previous relations, we obtain the same result, excepts that α_2 has to be replaced by $\widetilde{\alpha}_2$. So the fourth relation of (3.11) is established.

- We have from the relation (3.8) : $\widetilde{\alpha}_2 \;=\; {\rm NE}^2 \, \alpha \;-\; 2 \, {\rm NE} \; \alpha_1 \;+\; \alpha_2$. Then

$$
\begin{aligned}
\alpha_1 \;\; &= \;\; \frac{ \big({\rm EM}^2 \; - \; {\rm NM}^2 \; + \; {\rm NE}^2 \big) }{2 \, {\rm NE}} \;\; \alpha \;\; = \;\; \frac{ \overrightarrow{\rm EN} \bullet \big(\overrightarrow{\rm EM} \, + \, \overrightarrow{\rm NM} \, + \, \overrightarrow{\rm EN} \big) }{2 \, {\rm NE}} \;\; \alpha \\
&= \;\; \frac{ \overrightarrow{\rm EN} \bullet \overrightarrow{\rm EM} }{\rm NE} \;\, \alpha
\end{aligned}
$$

and the first relation of (3.11) is established. As previously, the exchange of the vertices N and E induces the change of α_1 into $\widetilde{\alpha}_1$ that establishes the second relation of (3.11).

- The relations (3.12), (3.13) and (3.14) are obtained by circular permutation, following the rules that are natural when viewing the Figure 3 :

$\alpha \longrightarrow \beta \longrightarrow \gamma \longrightarrow \delta$, $\quad {\rm E} \longrightarrow {\rm N} \longrightarrow {\rm W} \longrightarrow {\rm S}$, $\quad {\rm N} \longrightarrow {\rm W} \longrightarrow {\rm S} \longrightarrow {\rm E}$ and $\quad {\rm M} \longrightarrow {\rm P} \longrightarrow {\rm Q} \longrightarrow {\rm R}$. $\square$

Lemma 2. A mean value of the dual Raviart-Thomas basis function
Let $M = ({\rm N},\,{\rm E},\,{\rm A})$ be a triangle of the mesh $\mathcal{T}$ associated to the edge $a = ({\rm S},\,{\rm N})$ as in Figure 3 and $\varphi^\star_{\rm SN}$ a dual Raviart-Thomas basis function satisfying the Hypothesis 2. Then for each constant vector ξ, we have :

$$
(3.15) \qquad \xi \bullet \int_M \varphi^\star_{\rm SN} \; {\rm d}x \;\; = \;\; (\xi \bullet n_{\rm EN}) \, (\overrightarrow{\rm ME} \bullet n_{\rm EN}) \; \alpha \, .
$$

Proof of Lemma 2.

We have :

$$\xi \bullet \int_M \varphi^\star_{\rm SN} \, {\rm d}x \;=\; \int_M \nabla \big(\xi \bullet (x - {\rm M}) \big) \, \varphi^\star_{\rm SN} \, {\rm d}x$$

$$= \; - \int_M \xi \bullet (x - {\rm M}) \, {\rm div} \, \varphi^\star_{\rm SN} \, {\rm d}x \; + \; \int_{\partial M} \big(\varphi^\star_{\rm SN} \bullet n \big) \, \xi \bullet (x - {\rm M}) \, {\rm d}\gamma$$

$$= \; - \big({\rm div} \, \varphi^\star_{\rm SN} \big) \int_M \xi \bullet (x - {\rm M}) \, {\rm d}x \; + \; \int_{\rm NE} \big(\varphi^\star_{\rm SN} \bullet (-n_{\rm EN}) \big) \, \xi \bullet (x - {\rm M}) \, {\rm d}\gamma$$

$$= \; 0 \; + \; \int_{\rm EN} \big(\varphi^\star_{\rm SN} \bullet (-n_{\rm EN}) \big) \, \xi \bullet \big[x - {\rm E} + {\rm E} - {\rm M} \big] \, {\rm d}\gamma$$

$$= \; - \int_{\rm EN} \big(\varphi^\star_{\rm SN} \bullet n_{\rm EN} \big) \, \xi \bullet \Big[s \, \frac{\overrightarrow{\rm EN}}{\rm EN} \; + \; \overrightarrow{\rm ME} \Big] \, {\rm d}\gamma$$

$$= \; -\xi \bullet \Big(\frac{\overrightarrow{\rm EN}}{\rm EN} \, \alpha_1 \; + \; \overrightarrow{\rm ME} \, \alpha \Big) \;\; = \;\; -\xi \bullet \Big[\Big(\overrightarrow{\rm EM} \bullet \frac{\overrightarrow{\rm EN}}{\rm EN} \Big) \frac{\overrightarrow{\rm EN}}{\rm EN} \; + \; \overrightarrow{\rm ME} \Big] \, \alpha$$

$$= \; -\xi \bullet \Big[\big(\overrightarrow{\rm ME} \bullet n_{\rm EN} \big) \, n_{\rm EN} \Big] \, \alpha$$

and the relation (3.15) is established. □

Lemma 3. A simple relation between two triangles

Let $L = ({\rm S, E, N})$ and $M = ({\rm N, E, A})$ be the two triangles of the mesh $\mathcal{T}$ associated to the edge $a = ({\rm S, N})$ as in Figure 3. Let $\varphi^\star_{\rm SN}$ be the dual Raviart-Thomas basis function satisfying the Hypothesis 2. Then we have :

$$\frac{1}{2\,|M|} \int_M (A - x) \bullet \varphi^\star_{\rm SN} \, {\rm d}x \; = \; \frac{1}{2\,|L|} \big(\overrightarrow{\rm SN} \bullet n_{\rm EN} \big) \big(\overrightarrow{\rm EM} \bullet n_{\rm EN} \big) \, \alpha \, . \tag{3.16}$$

Proof of Lemma 3.

We denote by h the height of the triangle NEA and by h^* the height of ENS chosen such that $|M| = \frac{1}{2} \, h \, {\rm NE}$ and $|L| = \frac{1}{2} \, h^* \, {\rm NE}$. We have from the Lemma 2 :

$$\frac{1}{2\,|M|} \int_M (A - x) \bullet \varphi^\star_{\rm SN} \, {\rm d}x \; = \; \frac{1}{2\,|M|} \int_M \big(N - x + \overrightarrow{\rm NA} \big) \bullet \varphi^\star_{\rm SN} \, {\rm d}x$$

$$= \; \frac{1}{2\,|M|} \, \overrightarrow{\rm NA} \bullet \int_M \varphi^\star_{\rm SN} \, {\rm d}x \; = \; \frac{1}{h \, {\rm NE}} \, \big(\overrightarrow{\rm NA} \bullet n_{\rm EN} \big) \big(\overrightarrow{\rm EM} \bullet n_{\rm EN} \big) \, \alpha$$

$$= \; \frac{1}{h \, {\rm NE}} \, h \, \big(\overrightarrow{\rm EM} \bullet n_{\rm EN} \big) \, \alpha \;\; = \;\; \frac{1}{h^* \, {\rm NE}} \, h^* \, \big(\overrightarrow{\rm EM} \bullet n_{\rm EN} \big) \, \alpha$$

$$= \; \frac{1}{2\,|L|} \, \big(\overrightarrow{\rm SN} \bullet n_{\rm EN} \big) \, \big(\overrightarrow{\rm EM} \bullet n_{\rm EN} \big) \, \alpha$$

and the relation (3.16) is proven. □

Proposition 4. Second relation between momenta
The Hypothesis 2 inside the triangles $L = (\mathrm{S},\, \mathrm{E},\, \mathrm{N})$ and $M = (\mathrm{N},\, \mathrm{E},\, \mathrm{A})$ implies the following relation :

$$(3.17)\qquad \eta_2 \;=\; \left\{ \begin{array}{l} (\rho_L^2 + \mathrm{SL}^2)\, \eta \;+\; \dfrac{1}{6}\, \overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{SN}})\, \alpha \,+ \\ \qquad\qquad\qquad +\; \dfrac{1}{6}\, \overrightarrow{\mathrm{ND}} \bullet (\overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SN}})\, \delta \,. \end{array} \right.$$

We have also :

$$(3.18)\qquad \widetilde{\eta}_2 \;=\; \left\{ \begin{array}{l} (\rho_L^2 + \mathrm{NL}^2)\, \eta \;+\; \dfrac{1}{6}\, \overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{NE}} + \overrightarrow{\mathrm{NA}})\, \alpha \,+ \\ \qquad\qquad\qquad +\; \dfrac{1}{6}\, \overrightarrow{\mathrm{ND}} \bullet (\overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{ND}} + \overrightarrow{\mathrm{NE}})\, \delta \;, \end{array} \right.$$

$$(3.19)\qquad \widetilde{\eta}_2 \;=\; \left\{ \begin{array}{l} (\rho_K^2 + \mathrm{NK}^2)\, \eta \;-\; \dfrac{1}{6}\, \overrightarrow{\mathrm{SB}} \bullet (\overrightarrow{\mathrm{NB}} + \overrightarrow{\mathrm{NW}} + \overrightarrow{\mathrm{NS}})\, \beta \\ \qquad\qquad\qquad -\; \dfrac{1}{6}\, \overrightarrow{\mathrm{NC}} \bullet (\overrightarrow{\mathrm{NW}} + \overrightarrow{\mathrm{NC}} + \overrightarrow{\mathrm{NS}})\, \gamma \;, \end{array} \right.$$

$$(3.20)\qquad \eta_2 \;=\; \left\{ \begin{array}{l} (\rho_K^2 + \mathrm{SK}^2)\, \eta \;-\; \dfrac{1}{6}\, \overrightarrow{\mathrm{SB}} \bullet (\overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{SB}} + \overrightarrow{\mathrm{SW}})\, \beta \\ \qquad\qquad\qquad -\; \dfrac{1}{6}\, \overrightarrow{\mathrm{NC}} \bullet (\overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{SW}} + \overrightarrow{\mathrm{SC}})\, \gamma \;. \end{array} \right.$$

Proof of Proposition 4.

• We write the orthogonality (1.32) between the two edges $a = (\mathrm{S},\, \mathrm{N})$ and the edge $b = (\mathrm{E},\, \mathrm{N})$ (see the Figure 3). We have

$$\varphi_{\mathrm{EN}} \;=\; \left\{ \begin{array}{ll} \dfrac{1}{2\,|L|}\,(x - \mathrm{S}) \;=\; \dfrac{1}{4\,|L|}\; \nabla(|\,x - \mathrm{S}\,|^2) & \text{inside } L = (\mathrm{E},\, \mathrm{N},\, \mathrm{S}) \\ \dfrac{1}{2\,|M|}\,(\mathrm{A} - x) & \text{inside } M = (\mathrm{N},\, \mathrm{E},\, \mathrm{A})\,. \end{array} \right.$$

Then

$$\begin{aligned} 0 \;&=\; \int_\Omega \varphi^\star_{\mathrm{SN}} \bullet \varphi_{\mathrm{EN}}\, \mathrm{d}x \;=\; \int_L \varphi^\star_{\mathrm{SN}} \bullet \varphi_{\mathrm{EN}}\, \mathrm{d}x \;+\; \int_M \varphi^\star_{\mathrm{SN}} \bullet \varphi_{\mathrm{EN}}\, \mathrm{d}x \\ &=\; \frac{1}{4\,|L|} \int_L \varphi^\star_{\mathrm{SN}} \bullet \nabla(|\,x - \mathrm{S}\,|^2)\, \mathrm{d}x \;+\; \frac{1}{2\,|M|} \int_M \varphi^\star_{\mathrm{SN}} \bullet (\mathrm{A} - x)\, \mathrm{d}x \\ &=-\; \frac{1}{4\,|L|} \big(\mathrm{div}\, \varphi^\star_{\mathrm{SN}}\big)(L) \bullet \int_L |\,x - \mathrm{S}\,|^2 \;\mathrm{d}x \;+\; \frac{1}{4\,|L|} \int_{\partial L} (\varphi^\star_{\mathrm{SN}} \bullet n)\; |\,x - \mathrm{S}\,|^2 \;\mathrm{d}\gamma \\ &\qquad +\; \frac{1}{2\,|M|} \int_M \varphi^\star_{\mathrm{SN}} \bullet (\mathrm{A} - x)\, \mathrm{d}x \\ &=-\; \frac{1}{4\,|L|} \bigg(\frac{\delta \,+\, \alpha \,-\, \eta}{|\,L\,|} \bigg) \int_L |\,x - \mathrm{S}\,|^2 \;\mathrm{d}x \;+ \\ &\qquad +\; \frac{1}{4\,|L|} \Big(\, \delta_2 \;+\; \mathrm{SE}^2\, \alpha \;+\; 2\, \overrightarrow{\mathrm{SE}} \bullet \frac{\overrightarrow{\mathrm{EN}}}{\mathrm{EN}}\, \alpha_1 \;+\; \alpha_2 \;-\; \eta_2 \Big) \;+ \\ &\qquad +\; \frac{1}{2\,|L|}\, (\overrightarrow{\mathrm{SN}} \bullet n_{\mathrm{EN}})\, (\overrightarrow{\mathrm{EM}} \bullet n_{\mathrm{EN}})\; \alpha \end{aligned}$$

and we have, thanks to the relations (3.10), (3.11) and (3.14) :

$$(3.21) \qquad \eta_2 = \left\{ \begin{array}{l} -(\delta + \alpha - \eta)\,(\rho_L^2 + \mathrm{SL}^2) + (\rho_R^2 + \mathrm{SR}^2)\,\delta + \mathrm{SE}^2\,\alpha + \\ + 2\left(\overrightarrow{\mathrm{SE}} \bullet \dfrac{\overrightarrow{\mathrm{EN}}}{\mathrm{EN}}\right)\left(\overrightarrow{\mathrm{EM}} \bullet \dfrac{\overrightarrow{\mathrm{EN}}}{\mathrm{EN}}\right)\alpha + (\rho_M^2 + \mathrm{EM}^2)\,\alpha + \\ + 2\,(\overrightarrow{\mathrm{SN}} \bullet n_{\mathrm{EN}})\,(\overrightarrow{\mathrm{EM}} \bullet n_{\mathrm{EN}})\;\alpha\,. \end{array} \right.$$

- The α coefficient in the right hand side of the relation (3.21) is equal to :

$$-\frac{1}{36}\,(\mathrm{SE}^2 + \mathrm{EN}^2 + \mathrm{NS}^2) - \mathrm{SL}^2 + \mathrm{SE}^2 + 2\left(\overrightarrow{\mathrm{SE}} \bullet \overrightarrow{\mathrm{EM}} - (\overrightarrow{\mathrm{SE}} \bullet n_{\mathrm{EN}})\,(\overrightarrow{\mathrm{EM}} \bullet n_{\mathrm{EN}})\right) +$$

$$+ \frac{1}{36}\,(\mathrm{EA}^2 + \mathrm{AN}^2 + \mathrm{NE}^2) + \mathrm{EM}^2 + 2\,(\overrightarrow{\mathrm{SN}} \bullet n_{\mathrm{EN}})\,(\overrightarrow{\mathrm{EM}} \bullet n_{\mathrm{EN}})$$

$$= \frac{1}{36}\left((\overrightarrow{\mathrm{EA}} + \overrightarrow{\mathrm{SE}}) \bullet (\overrightarrow{\mathrm{EA}} - \overrightarrow{\mathrm{SE}}) + (\overrightarrow{\mathrm{AN}} + \overrightarrow{\mathrm{NS}}) \bullet (\overrightarrow{\mathrm{AN}} - \overrightarrow{\mathrm{NS}})\right) - \mathrm{SL}^2 + (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{EM}})^2$$

$$= \frac{1}{36}\,\overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{EA}} + \overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{NA}} + \overrightarrow{\mathrm{NS}}) + (\overrightarrow{\mathrm{SM}} + \overrightarrow{\mathrm{SL}}) \bullet (\overrightarrow{\mathrm{SM}} - \overrightarrow{\mathrm{SL}})$$

$$= \frac{1}{36}\,\overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{EA}} + \overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{NA}} + \overrightarrow{\mathrm{NS}}) + (2\,\overrightarrow{\mathrm{SL}} + \overrightarrow{\mathrm{LM}}) \bullet \overrightarrow{\mathrm{LM}}$$

$$= \frac{1}{36}\,\overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{EA}} + \overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{NA}} + \overrightarrow{\mathrm{NS}}) + \left(\frac{2}{3}\,(\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SN}}) + \frac{1}{3}\,\overrightarrow{\mathrm{SA}}\right) \bullet \left(\frac{1}{3}\,\overrightarrow{\mathrm{SA}}\right)$$

$$= \frac{1}{36}\,\overrightarrow{\mathrm{SA}} \bullet \left(\overrightarrow{\mathrm{EA}} + \overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{NA}} + \overrightarrow{\mathrm{NS}} + 8\,(\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SN}}) + 4\,\overrightarrow{\mathrm{SA}}\right)$$

$$= \frac{1}{36}\,\overrightarrow{\mathrm{SA}} \bullet \left(\overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{NS}} + 8\,\overrightarrow{\mathrm{SE}} + 8\,\overrightarrow{\mathrm{SN}} + 4\,\overrightarrow{\mathrm{SA}}\right)$$

$$= \frac{1}{6}\,\overrightarrow{\mathrm{SA}} \bullet \left(\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{SN}}\right)$$

in coherence with the right hand side of the relation (3.17).

- In a similar way, the coefficient of δ in the right hand side of the relation (3.21) is equal to :

$$-\frac{1}{36}\,(\mathrm{SE}^2 + \mathrm{EN}^2 + \mathrm{NS}^2) + \frac{1}{36}\,(\mathrm{DE}^2 + \mathrm{ES}^2 + \mathrm{SD}^2) + (\overrightarrow{\mathrm{SR}} + \overrightarrow{\mathrm{SL}}) \bullet (\overrightarrow{\mathrm{SR}} - \overrightarrow{\mathrm{SL}})$$

$$= \frac{1}{36}\left((\overrightarrow{\mathrm{DE}} + \overrightarrow{\mathrm{EN}}) \bullet (\overrightarrow{\mathrm{DE}} - \overrightarrow{\mathrm{EN}}) + (\overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{NS}}) \bullet (\overrightarrow{\mathrm{SD}} - \overrightarrow{\mathrm{NS}})\right) +$$

$$+ \frac{1}{3}\,(\overrightarrow{\mathrm{SD}} + 2\,\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SN}}) \bullet \left(\frac{1}{3}\,\overrightarrow{\mathrm{ND}}\right)$$

$$= \frac{1}{36}\,\overrightarrow{\mathrm{ND}} \bullet \left(\overrightarrow{\mathrm{ED}} + \overrightarrow{\mathrm{EN}} + \overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SN}} + 4\,\overrightarrow{\mathrm{SD}} + 8\,\overrightarrow{\mathrm{SE}} + 4\,\overrightarrow{\mathrm{SN}}\right)$$

$$= \frac{1}{36}\,\overrightarrow{\mathrm{ND}} \bullet \left(\overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SN}} + 4\,\overrightarrow{\mathrm{SD}} + 8\,\overrightarrow{\mathrm{SE}} + 4\,\overrightarrow{\mathrm{SN}}\right)$$

$$= \frac{1}{6}\,\overrightarrow{\mathrm{ND}} \bullet \left(\overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SN}}\right)$$

as proposed in the right hand side of the relation (3.17). Then the relation (3.17) is a direct consequence of (3.21).

- The proof of the relation (3.18) is obtained from the previous relation (3.17) with the following changes : $\mathrm{E} \longleftrightarrow \mathrm{E}$, $\eta \longleftrightarrow \eta$, $\mathrm{A} \longleftrightarrow \mathrm{D}$, $\mathrm{N} \longleftrightarrow \mathrm{S}$, $M \longleftrightarrow R$, $\alpha \longleftrightarrow \delta$ and $\eta_2 \longleftrightarrow \widetilde{\eta}_2$. In a similar way, the relations (3.19) and (3.20) are a straightforward consequence of the relations (3.17) and (3.18) with a vision of the Figure 3 "from the top to the bottom", *id est* with the following changes : $\mathrm{E} \longleftrightarrow \mathrm{W}$, $\mathrm{N} \longleftrightarrow \mathrm{S}$, $\mathrm{D} \longleftrightarrow \mathrm{B}$, $\mathrm{A} \longleftrightarrow \mathrm{C}$, $L \longleftrightarrow K$, $M \longleftrightarrow Q$, $R \longleftrightarrow P$, $\eta \longleftrightarrow -\eta$, $\alpha \longleftrightarrow \gamma$, $\delta \longleftrightarrow \beta$ and $\eta_2 \longleftrightarrow -\widetilde{\eta}_2$. So the proposition is established. □

Proposition 5. Two expressions for the first order momentum
Under the Hypothesis 2 and the notations proposed at the Figure 3, we have :

$$(3.22) \qquad \eta_1 \;=\; \big(\overrightarrow{\mathrm{SL}}\, \eta + \overrightarrow{\mathrm{LM}}\, \alpha + \overrightarrow{\mathrm{LR}}\, \delta \big) \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}} \,,$$

$$(3.23) \qquad \eta_1 \;=\; \big(\overrightarrow{\mathrm{SK}}\, \eta - \overrightarrow{\mathrm{KP}}\, \beta - \overrightarrow{\mathrm{KQ}}\, \gamma \big) \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}} \,.$$

Proof of Proposition 5 .

- We deduce from (3.8), (3.17) and (3.18) :

$$\begin{aligned}
\eta_1 \;&=\; \frac{1}{2\,\mathrm{NS}} \, (\eta_2 - \widetilde{\eta}_2 + \mathrm{SN}^2\, \eta) \\
&=\; \frac{1}{2\,\mathrm{NS}} \Big((\mathrm{SL}^2 - \mathrm{NL}^2 + \mathrm{SN}^2)\, \eta \;+\; \frac{1}{6}\, \overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{SN}})\; \alpha \;+ \\
&\qquad + \; \frac{1}{6}\, \overrightarrow{\mathrm{ND}} \bullet (\overrightarrow{\mathrm{SD}} \;+\; \overrightarrow{\mathrm{SE}} \;+\; \overrightarrow{\mathrm{SN}})\; \delta \;-\; \frac{1}{6}\, \overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{NE}} + \overrightarrow{\mathrm{NA}})\; \alpha \\
&\qquad - \; \frac{1}{6}\, \overrightarrow{\mathrm{ND}} \bullet (\overrightarrow{\mathrm{NS}} \;+\; \overrightarrow{\mathrm{ND}} \;+\; \overrightarrow{\mathrm{NE}})\; \delta \Big) \\
&=\; \frac{1}{2\,\mathrm{NS}} \Big(\big[(\overrightarrow{\mathrm{SL}} + \overrightarrow{\mathrm{NL}}) \bullet (\overrightarrow{\mathrm{SL}} - \overrightarrow{\mathrm{NL}}) + \mathrm{SN}^2 \big]\; \eta \\
&\qquad + \; \frac{1}{6}\, \overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{EN}} + \overrightarrow{\mathrm{AN}})\; \alpha \\
&\qquad + \; \frac{1}{6}\, \overrightarrow{\mathrm{ND}} \bullet (\overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{DN}} + \overrightarrow{\mathrm{EN}})\; \delta \Big) \\
&=\; \frac{1}{2\,\mathrm{NS}} \Big(\overrightarrow{\mathrm{SN}} \bullet (\overrightarrow{\mathrm{SL}} + \overrightarrow{\mathrm{NL}} + \overrightarrow{\mathrm{SN}})\; \eta \;+\; \frac{2}{3}\, \overrightarrow{\mathrm{SA}} \bullet \overrightarrow{\mathrm{SN}}\; \alpha \;+\; \frac{2}{3}\, \overrightarrow{\mathrm{ND}} \bullet \overrightarrow{\mathrm{SN}}\; \delta \Big) \\
&=\; \frac{1}{\mathrm{NS}} \, (\overrightarrow{\mathrm{SL}}\; \eta \;+\; \overrightarrow{\mathrm{LM}}\; \alpha \;+\; \overrightarrow{\mathrm{LR}}\; \delta) \bullet \overrightarrow{\mathrm{SN}}
\end{aligned}$$

and the relation (3.22) is established.

• The proof of the relation (3.23) is analogous. It is a consequence of the relations (3.8), (3.19) and (3.20) :

$$\begin{aligned}
\eta_1 \;&=\; \frac{1}{2\,\mathrm{NS}}\,(\,\eta_2 - \widetilde{\eta}_2 + \mathrm{SN}^2\,\eta\,) \\
&=\; \frac{1}{2\,\mathrm{NS}}\Big(\,(\mathrm{SK}^2 - \mathrm{NK}^2 + \mathrm{SN}^2)\,\eta \;+\; \frac{1}{6}\,\overrightarrow{\mathrm{SB}}\bullet(\overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{BS}} + \overrightarrow{\mathrm{WS}})\;\beta \;+ \\
&\qquad +\; \frac{1}{6}\,\overrightarrow{\mathrm{NC}}\bullet(\overrightarrow{\mathrm{NS}} \;+\; \overrightarrow{\mathrm{WS}} \;+\; \overrightarrow{\mathrm{CS}})\;\gamma \;+\; \frac{1}{6}\,\overrightarrow{\mathrm{SB}}\bullet(\overrightarrow{\mathrm{NB}} + \overrightarrow{\mathrm{NW}} + \overrightarrow{\mathrm{NS}})\;\beta \;+ \\
&\qquad +\; \frac{1}{6}\,\overrightarrow{\mathrm{NC}}\bullet(\overrightarrow{\mathrm{NW}} \;+\; \overrightarrow{\mathrm{NC}} \;+\; \overrightarrow{\mathrm{NS}})\;\gamma\Big) \\
&=\; \frac{1}{2\,\mathrm{NS}}\Big(\,\big[(\overrightarrow{\mathrm{SK}} + \overrightarrow{\mathrm{NK}})\bullet(\overrightarrow{\mathrm{SK}} - \overrightarrow{\mathrm{NK}}) + \mathrm{SN}^2\big]\;\eta \;+ \\
&\qquad +\; \frac{1}{6}\,\overrightarrow{\mathrm{SB}}\bullet(\overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{BS}} + \overrightarrow{\mathrm{WS}} + \overrightarrow{\mathrm{NB}} + \overrightarrow{\mathrm{NW}} + \overrightarrow{\mathrm{NS}})\;\beta \;+ \\
&\qquad +\; \frac{1}{6}\,\overrightarrow{\mathrm{NC}}\bullet(\overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{WS}} + \overrightarrow{\mathrm{CS}} + \overrightarrow{\mathrm{NW}} + \overrightarrow{\mathrm{NC}} + \overrightarrow{\mathrm{NS}})\;\gamma\Big) \\
&=\; \frac{1}{2\,\mathrm{NS}}\Big(\,\overrightarrow{\mathrm{SN}}\bullet(\overrightarrow{\mathrm{SK}} + \overrightarrow{\mathrm{NK}} + \overrightarrow{\mathrm{SN}})\;\eta \;+\; \frac{2}{3}\,\overrightarrow{\mathrm{SB}}\bullet\overrightarrow{\mathrm{NS}}\;\beta \;+\; \frac{2}{3}\,\overrightarrow{\mathrm{NC}}\bullet\overrightarrow{\mathrm{NS}}\;\gamma\Big) \\
&=\; \frac{1}{\mathrm{NS}}\,(\,\overrightarrow{\mathrm{SK}}\;\eta \;-\; \overrightarrow{\mathrm{KP}}\;\beta \;-\; \overrightarrow{\mathrm{KQ}}\;\gamma\,)\bullet\overrightarrow{\mathrm{SN}}\,.
\end{aligned}$$

The relation (3.23) is established and the Proposition 5 is proven. □

Lemma 4. Two usefull integrals

Let $K = (\mathrm{S, N, W})$ and $L = (\mathrm{N, S, E})$ be the two triangles of the mesh $\mathcal{T}$ that compose the co-boundary of the edge $a = (\mathrm{S, N})$ as in Figure 3. Let $\varphi^\star_{\mathrm{SN}}$ be the a Raviart-Thomas basis function satisfying the Hypothesis 2. Then we have :

$$\text{(3.24)}\qquad \frac{1}{2\,|L|}\int_L \varphi^\star_{\mathrm{SN}}\bullet(\mathrm{E} - x)\;\mathrm{d}x \;=\; \frac{1}{\mathrm{NS}}\,(\,\overrightarrow{\mathrm{SL}}\;\eta \;+\; \overrightarrow{\mathrm{LM}}\;\alpha \;+\; \overrightarrow{\mathrm{LR}}\;\delta\,)\bullet n_{\mathrm{SN}}$$

$$\text{(3.25)}\qquad \frac{1}{2\,|K|}\int_K \varphi^\star_{\mathrm{SN}}\bullet(\mathrm{W} - x)\;\mathrm{d}x \;=\; \frac{1}{\mathrm{NS}}\,(\overrightarrow{\mathrm{NK}}\;\eta \;-\; \overrightarrow{\mathrm{KP}}\,\beta \;-\; \overrightarrow{\mathrm{KQ}}\,\gamma\,)\bullet n_{\mathrm{SN}}\,.$$

Proof of Lemma 4.

• We establish the relation (3.24) by integrating by parts and using the relations (3.10), (3.11) and (3.14) :

$$\begin{aligned}
&\frac{1}{2\,|L|}\int_L \varphi^\star_{\mathrm{SN}}\bullet(\mathrm{E} - x)\;\mathrm{d}x \;=\; -\frac{1}{4\,|L|}\int_L \varphi^\star_{\mathrm{SN}}\bullet\nabla(|x - \mathrm{E}|^2)\;\mathrm{d}x \\
&=\; \frac{1}{4\,|L|}\,(\mathrm{div}\,\varphi^\star_{\mathrm{SN}})(L)\int_L |x - \mathrm{E}|^2\;\mathrm{d}x \;-\; \frac{1}{4\,|L|}\int_{\partial L}(\varphi^\star_{\mathrm{SN}}\bullet n)\;|x - \mathrm{E}|^2\;\mathrm{d}\gamma
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\,|L|}\ \Big(\ \frac{\delta+\alpha-\eta}{|L|} \int_L \ |x-\mathrm{E}|^2 \ \mathrm{d}x \\
&\qquad\qquad - \Big[\ \widetilde{\delta}_2 \ + \ \alpha_2 \ - \ \big(\ \eta_2 \ + \ 2\,\overrightarrow{\mathrm{ES}} \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}}\,\eta_1 \ + \ \mathrm{ES}^2\ \eta\ \big)\Big]\Big) \\
&= \frac{1}{4\,|L|}\bigg(\,(\delta+\alpha-\eta)\,(\rho_L^2 \ + \ \mathrm{EL}^2) \ - \ (\rho_R^2 \ + \ \mathrm{ER}^2)\,\delta \ - \ (\rho_M^2 \ + \ \mathrm{EM}^2)\,\alpha \, + \\
&+ (\rho_L^2 + \mathrm{SL}^2)\,\eta \ + \ \frac{1}{6}\,\overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{SN}})\ \alpha \ + \ \frac{1}{6}\,\overrightarrow{\mathrm{ND}} \bullet (\overrightarrow{\mathrm{SD}} \ + \ \overrightarrow{\mathrm{SE}} \ + \ \overrightarrow{\mathrm{SN}})\,\delta \\
&\quad + 2\Big(\overrightarrow{\mathrm{ES}} \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}}\Big)\ \big(\overrightarrow{\mathrm{SL}}\,\eta + \overrightarrow{\mathrm{LM}}\,\alpha + \overrightarrow{\mathrm{LR}}\,\delta\,\big) \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}} \ + \ \mathrm{ES}^2\,\eta\bigg) \\
&= \frac{1}{4\,|L|}\bigg(\ \bigg[-\mathrm{EL}^2 + \mathrm{SL}^2 + 2\Big(\overrightarrow{\mathrm{ES}} \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}}\Big)\Big(\overrightarrow{\mathrm{SL}} \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}}\Big) + \mathrm{ES}^2\bigg]\ \eta \, + \\
&\quad + \bigg[\ \frac{1}{36}\,\big((\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{EA}}) \bullet (\overrightarrow{\mathrm{SE}} - \overrightarrow{\mathrm{EA}}) + (\overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{AN}}) \bullet (\overrightarrow{\mathrm{SN}} - \overrightarrow{\mathrm{AN}})\big) \ + \ \mathrm{EL}^2 - \mathrm{EM}^2 \, + \\
&\qquad + \ \frac{1}{6}\,\overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{SN}}) \ + \ 2\Big(\overrightarrow{\mathrm{ES}} \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}}\,\Big)\Big(\overrightarrow{\mathrm{LM}} \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}}\,\Big)\ \bigg]\ \alpha \ + \\
&\quad + \bigg[\ \frac{1}{36}\,\big((\overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{SD}}) \bullet (\overrightarrow{\mathrm{NS}} - \overrightarrow{\mathrm{SD}}) + (\overrightarrow{\mathrm{EN}} + \overrightarrow{\mathrm{DE}}) \bullet (\overrightarrow{\mathrm{EN}} - \overrightarrow{\mathrm{DE}})\big) \ + \ \mathrm{EL}^2 - \mathrm{ER}^2 \, + \\
&\qquad + \ \frac{1}{6}\,\overrightarrow{\mathrm{ND}} \bullet (\overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SN}}) \ + \ 2\Big(\overrightarrow{\mathrm{ES}} \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}}\,\Big)\Big(\overrightarrow{\mathrm{LR}} \bullet \frac{\overrightarrow{\mathrm{SN}}}{\mathrm{SN}}\,\Big)\ \bigg]\ \delta\ \bigg) \\
&= \frac{1}{4\,|L|}\bigg(\ \Big[(\overrightarrow{\mathrm{SL}} + \overrightarrow{\mathrm{EL}}) \bullet (\overrightarrow{\mathrm{SL}} - \overrightarrow{\mathrm{EL}}) + \mathrm{ES}^2 + 2\,\overrightarrow{\mathrm{ES}} \bullet \overrightarrow{\mathrm{SL}} - 2\,(\overrightarrow{\mathrm{ES}} \bullet n_{\mathrm{SN}})\,(\overrightarrow{\mathrm{SL}} \bullet n_{\mathrm{SN}})\Big]\ \eta \, + \\
&\qquad + \ \bigg[\ \frac{1}{36}\,\overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{AE}} + \overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{AN}}) \ + \ (\overrightarrow{\mathrm{EL}} + \overrightarrow{\mathrm{EM}})\,(\overrightarrow{\mathrm{EL}} - \overrightarrow{\mathrm{EM}}) \, + \\
&\qquad + \ \frac{1}{6}\,\overrightarrow{\mathrm{SA}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{SN}}) \ + \ 2\,\overrightarrow{\mathrm{ES}} \bullet \overrightarrow{\mathrm{LM}} \ - \ 2\,(\overrightarrow{\mathrm{ES}} \bullet n_{\mathrm{SN}})\,(\overrightarrow{\mathrm{LM}} \bullet n_{\mathrm{SN}})\bigg]\ \alpha \, + \\
&\qquad + \ \bigg[\ \frac{1}{36}\,\overrightarrow{\mathrm{ND}} \bullet (\overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{DS}} + \overrightarrow{\mathrm{NE}} + \overrightarrow{\mathrm{DE}}) \ + \ (\overrightarrow{\mathrm{EL}} + \overrightarrow{\mathrm{ER}})\,(\overrightarrow{\mathrm{EL}} - \overrightarrow{\mathrm{ER}}) \, + \\
&\qquad + \ \frac{1}{6}\,\overrightarrow{\mathrm{ND}} \bullet (\overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SN}}) \ + \ 2\,\overrightarrow{\mathrm{ES}} \bullet \overrightarrow{\mathrm{LR}} \ - \ 2\,(\overrightarrow{\mathrm{ES}} \bullet n_{\mathrm{SN}})\,(\overrightarrow{\mathrm{LR}} \bullet n_{\mathrm{SN}})\ \bigg]\ \delta\bigg) \\
&= \frac{1}{4\,|L|}\bigg(\ \Big[\overrightarrow{\mathrm{SE}} \bullet (\overrightarrow{\mathrm{SL}} + \overrightarrow{\mathrm{EL}} + \overrightarrow{\mathrm{SE}} - 2\,\overrightarrow{\mathrm{SL}}) \ + \ 2\,(\overrightarrow{\mathrm{SE}} \bullet n_{\mathrm{SN}})\,(\overrightarrow{\mathrm{SL}} \bullet n_{\mathrm{SN}})\Big]\ \eta \, +
\end{aligned}
$$

$$+ \Big[\frac{1}{36} \overrightarrow{\rm SA} \bullet \big(\overrightarrow{\rm SE} + \overrightarrow{\rm AS} + \overrightarrow{\rm SE} + \overrightarrow{\rm SN} + \overrightarrow{\rm AS} + \overrightarrow{\rm SN} + 6 \, (\overrightarrow{\rm SE} + \overrightarrow{\rm SA} + \overrightarrow{\rm SN}) \big)$$

$$+ \overrightarrow{\rm LM} \bullet (\overrightarrow{\rm LE} + \overrightarrow{\rm ME} + 2 \, \overrightarrow{\rm ES}) \, + \, 2 \, (\overrightarrow{\rm SE} \bullet n_{\rm SN}) \, (\overrightarrow{\rm LM} \bullet n_{\rm SN}) \Big] \, \alpha +$$

$$+ \Big[\frac{1}{36} \overrightarrow{\rm ND} \bullet \big(\overrightarrow{\rm NS} + \overrightarrow{\rm DS} + \overrightarrow{\rm NS} + \overrightarrow{\rm SE} + \overrightarrow{\rm DS} + \overrightarrow{\rm SE} + 6 \, (\overrightarrow{\rm SD} + \overrightarrow{\rm SE} + \overrightarrow{\rm SN}) \big)$$

$$+ \overrightarrow{\rm LR} \bullet (\overrightarrow{\rm LE} + \overrightarrow{\rm RE} + 2 \, \overrightarrow{\rm ES}) \, + \, 2 \, (\overrightarrow{\rm SE} \bullet n_{\rm SN}) \, (\overrightarrow{\rm LR} \bullet n_{\rm SN}) \Big] \, \delta \Big)$$

$$= \frac{1}{4 \, |L|} \Big(\big[2 \, (\overrightarrow{\rm SE} \bullet n_{\rm SN}) \, (\overrightarrow{\rm SL} \bullet n_{\rm SN}) \big] \, \eta +$$

$$+ \big[\frac{1}{9} \overrightarrow{\rm SA} \bullet (2 \, \overrightarrow{\rm SE} + \overrightarrow{\rm SA} + 2 \, \overrightarrow{\rm SN}) + \frac{1}{3} \overrightarrow{\rm SA} \bullet (\overrightarrow{\rm LS} + \overrightarrow{\rm MS}) + 2 \, (\overrightarrow{\rm SE} \bullet n_{\rm SN}) \, (\overrightarrow{\rm LM} \bullet n_{\rm SN}) \big] \, \alpha +$$

$$+ \big[\frac{1}{9} \overrightarrow{\rm ND} \bullet (2 \, \overrightarrow{\rm SE} + \overrightarrow{\rm SN} + \overrightarrow{\rm SD}) + \frac{1}{3} \overrightarrow{\rm ND} \bullet (\overrightarrow{\rm LS} + \overrightarrow{\rm RS}) + 2 \, (\overrightarrow{\rm SE} \bullet n_{\rm SN}) \, (\overrightarrow{\rm LR} \bullet n_{\rm SN}) \big] \, \delta \Big)$$

$$= \frac{1}{2 \, |L|} (\overrightarrow{\rm SE} \bullet n_{\rm SN}) \Big[(\overrightarrow{\rm SL} \bullet n_{\rm SN}) \, \eta \, + \, (\overrightarrow{\rm LM} \bullet n_{\rm SN}) \, \alpha \, + \, (\overrightarrow{\rm LR} \bullet n_{\rm SN}) \, \delta \Big]$$

$$= \frac{1}{|\overrightarrow{\rm NS}|} \Big[(\overrightarrow{\rm SL} \bullet n_{\rm SN}) \, \eta \, + \, (\overrightarrow{\rm LM} \bullet n_{\rm SN}) \, \alpha \, + \, (\overrightarrow{\rm LR} \bullet n_{\rm SN}) \, \delta \Big]$$

that establishes the relation (3.24).

• The relation (3.25) is a consequence of the previous relation (3.24) with the following modifications : $\rm E \longleftrightarrow W$, $\rm N \longleftrightarrow S$, $\rm D \longleftrightarrow B$, $\rm A \longleftrightarrow C$, $L \longleftrightarrow K$, $M \longleftrightarrow Q$, $R \longleftrightarrow P$, $\eta \longleftrightarrow -\eta$, $\alpha \longleftrightarrow \gamma$, $\delta \longleftrightarrow \beta$ and $n_{\rm SN} \longleftrightarrow -n_{\rm SN}$. □

Proof of Theorem 2.

• We first eliminate the variable η_1 between the relations (3.22) and (3.23) ; we obtain

$$(3.26) \qquad \big(\overrightarrow{\rm KL} \, \eta \, + \, \overrightarrow{\rm LM} \, \alpha \, + \, \overrightarrow{\rm KP} \, \beta \, + \, \overrightarrow{\rm KQ} \, \gamma \, + \, \overrightarrow{\rm LR} \, \delta \big) \bullet \overrightarrow{\rm SN} \, = \, 0 \, .$$

• We write secondly the orthonormality relation (1.32) between the vector function $\varphi_{\rm SN}$ and its dual $\varphi^\star_{\rm SN}$, with the help of (3.24) and (3.25) :

$$1 \, = \, (\varphi^\star_{\rm SN} \, , \, \varphi_{\rm SN} \,) \, = \, \int_K \varphi^\star_{\rm SN} \bullet \varphi_{\rm SN} \, {\rm d}x \, + \, \int_L \varphi^\star_{\rm SN} \bullet \varphi_{\rm SN} \, {\rm d}x$$

$$= \frac{1}{2 \, |K|} \int_K \varphi^\star_{\rm SN} \bullet (x - {\rm W}) \, {\rm d}x \, + \, \frac{1}{2 \, |L|} \int_L \varphi^\star_{\rm SN} \bullet ({\rm E} - x) \, {\rm d}x$$

$$= \frac{1}{\rm NS} \big((\overrightarrow{\rm SL} + \overrightarrow{\rm KN}) \, \eta \, + \, \overrightarrow{\rm LM} \, \alpha \, + \, \overrightarrow{\rm KP} \, \beta \, + \, \overrightarrow{\rm KQ} \, \gamma \, + \, \overrightarrow{\rm LR} \, \delta \big) \bullet n_{\rm SN}$$

$$= \frac{1}{\mathrm{NS}} \big((\overrightarrow{\mathrm{SL}} + \overrightarrow{\mathrm{KN}} + \overrightarrow{\mathrm{NS}})\, \eta \;+\; \overrightarrow{\mathrm{LM}}\, \alpha \;+\; \overrightarrow{\mathrm{KP}}\, \beta \;+\; \overrightarrow{\mathrm{KQ}}\, \gamma \;+\; \overrightarrow{\mathrm{LR}}\, \delta \big) \bullet n_{\mathrm{SN}}$$

$$= \frac{1}{\mathrm{NS}} \big(\overrightarrow{\mathrm{KL}}\, \eta \;+\; \overrightarrow{\mathrm{LM}}\, \alpha \;+\; \overrightarrow{\mathrm{KP}}\, \beta \;+\; \overrightarrow{\mathrm{KQ}}\, \gamma \;+\; \overrightarrow{\mathrm{LR}}\, \delta \big) \bullet n_{\mathrm{SN}} \,.$$

The relation (3.5) is a direct consequence of the above expression and of the previous relation (3.26).

• Thirdly, we eliminate the variable η_2 between the relations (3.17) and (3.20). The coefficient of the variable η is equal to

$$(\rho_L^2 + \mathrm{SL}^2) - (\rho_K^2 + \mathrm{SK}^2) \;=\; \frac{1}{36} \big((\mathrm{EN}^2 + \mathrm{ES}^2) - (\mathrm{NW}^2 + \mathrm{SW}^2) \big) \;+\; \mathrm{SL}^2 - \mathrm{SK}^2$$

$$= \frac{1}{36} \big((\overrightarrow{\mathrm{EN}} + \overrightarrow{\mathrm{NW}}) \bullet (\overrightarrow{\mathrm{EN}} + \overrightarrow{\mathrm{WN}}) + (\overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{SW}}) \bullet (\overrightarrow{\mathrm{ES}} + \overrightarrow{\mathrm{WS}}) \big) \;+\; (\overrightarrow{\mathrm{SL}} + \overrightarrow{\mathrm{SK}}) \bullet (\overrightarrow{\mathrm{SL}} + \overrightarrow{\mathrm{KS}})$$

$$= \frac{1}{36}\, \overrightarrow{\mathrm{WE}} \bullet (\overrightarrow{\mathrm{NE}} + \overrightarrow{\mathrm{NW}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SW}}) \;+\; \frac{1}{3}\, \overrightarrow{\mathrm{WE}} \bullet \left(\frac{1}{3} (\overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SW}} + \overrightarrow{\mathrm{SN}}) \right)$$

$$= \frac{1}{12}\, \overrightarrow{\mathrm{KL}} \bullet (\overrightarrow{\mathrm{NE}} + \overrightarrow{\mathrm{NW}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SW}} \;+\; 8\, \overrightarrow{\mathrm{SN}} \;+\; 4\, \overrightarrow{\mathrm{SE}} \;+\; 4\, \overrightarrow{\mathrm{SW}})$$

$$= \frac{1}{12}\, \overrightarrow{\mathrm{KL}} \bullet (\overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{NS}} + \overrightarrow{\mathrm{SW}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SW}} \;+\; 8\, \overrightarrow{\mathrm{SN}} \;+\; 4\, \overrightarrow{\mathrm{SE}} \;+\; 4\, \overrightarrow{\mathrm{SW}})$$

$$= \frac{1}{2}\, \overrightarrow{\mathrm{KL}} \bullet (\overrightarrow{\mathrm{SN}} \;+\; \overrightarrow{\mathrm{SE}} \;+\; \overrightarrow{\mathrm{SW}}) \,.$$

We deduce from (3.17), (3.20) and the previous calculus :

$$\overrightarrow{\mathrm{KL}} \bullet (\overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SW}})\, \eta \;+\; \overrightarrow{\mathrm{LM}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}} + \overrightarrow{\mathrm{SN}})\, \alpha \;+\; \overrightarrow{\mathrm{KP}} \bullet (\overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{SB}} + \overrightarrow{\mathrm{SW}})\, \beta \,+$$
$$+\; \overrightarrow{\mathrm{KQ}} \bullet (\overrightarrow{\mathrm{SN}} + \overrightarrow{\mathrm{SW}} + \overrightarrow{\mathrm{SC}})\, \gamma \;+\; \overrightarrow{\mathrm{LR}} \bullet (\overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SN}})\, \delta \;=\; 0$$

and taking into consideration the relation (3.26) :

$$\overrightarrow{\mathrm{KL}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SW}})\, \eta \;+\; \overrightarrow{\mathrm{LM}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}})\, \alpha \;+\; \overrightarrow{\mathrm{KP}} \bullet (\overrightarrow{\mathrm{SB}} + \overrightarrow{\mathrm{SW}})\, \beta \,+$$
$$+\; \overrightarrow{\mathrm{KQ}} \bullet (\overrightarrow{\mathrm{SW}} + \overrightarrow{\mathrm{SC}})\, \gamma \;+\; \overrightarrow{\mathrm{LR}} \bullet (\overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SE}})\, \delta \;=\; 0 \,.$$

We eliminate the variable η between the previous relation and the relation (3.5) after multiplying it by $-(\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SW}})$. We get :

$$\overrightarrow{\mathrm{LM}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SA}} - \overrightarrow{\mathrm{SE}} - \overrightarrow{\mathrm{SW}})\, \alpha \;+\; \overrightarrow{\mathrm{KP}} \bullet (\overrightarrow{\mathrm{SB}} + \overrightarrow{\mathrm{SW}} - \overrightarrow{\mathrm{SE}} - \overrightarrow{\mathrm{SW}})\, \beta \,+$$
$$+\; \overrightarrow{\mathrm{KQ}} \bullet (\overrightarrow{\mathrm{SW}} + \overrightarrow{\mathrm{SC}} - \overrightarrow{\mathrm{SE}} - \overrightarrow{\mathrm{SW}})\, \gamma \;+\; \overrightarrow{\mathrm{LR}} \bullet (\overrightarrow{\mathrm{SD}} + \overrightarrow{\mathrm{SE}} - \overrightarrow{\mathrm{SE}} - \overrightarrow{\mathrm{SW}})\, \delta \;=$$
$$=\; -\,|\mathrm{NS}|\; \mathrm{n}_{\mathrm{SN}} \bullet (\overrightarrow{\mathrm{SE}} + \overrightarrow{\mathrm{SW}})$$

id est

$$\overrightarrow{\mathrm{LM}} \bullet \overrightarrow{\mathrm{WA}}\, \alpha \;+\; \overrightarrow{\mathrm{KP}} \bullet \overrightarrow{\mathrm{EB}}\, \beta \;+\; \overrightarrow{\mathrm{KQ}} \bullet \overrightarrow{\mathrm{EC}}\, \gamma \;+\; \overrightarrow{\mathrm{LR}} \bullet \overrightarrow{\mathrm{WD}}\, \delta \;=$$
$$=\; -3\; |\mathrm{NS}|\; \mathrm{n}_{\mathrm{SN}} \bullet (\overrightarrow{\mathrm{SL}} + \overrightarrow{\mathrm{SK}})$$

and the relation (3.6) is a direct consequence of the above relation. The Theorem 2 is established. □

4) Perspectives

• We have proposed to formulate the finite volume method for the Poisson equation in two space dimensions with the help of Petrov-Galerkin mixed finite elements. The unknown is constant in each triangle and the momentum is discretized with the Raviart-Thomas vectorial finite elements of lower degree. The conservation law is integrated in each triangle and our stencil for the discrete gradient operator is composed by six triangles in the vicinity of each edge of the mesh. The question of the determination of such a scheme conducts to a two-parameter family for a possible choice of a so-called "dual Raviart-Thomas basis function" for the finite volume scheme. We have also developed a sufficient hypothesis to prove the stability and the optimal convergence of the associated finite volume scheme. The next step of this research is to construct explicitly an interpolation vector valued function in the particular case where $\Omega = \mathbb{R}^2$ in order to determine free coefficients and to establish the stability property.

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