Some Properties of Finitely Presented Groups with Topological Viewpoints

Behrooz Mashayekhy [*] **and Hanieh Mirebrahimi**
Department of Pure Mathematics,
Center of Excellence in Analysis on Algebraic Structures,
Ferdowsi University of Mashhad,
P. O. Box 1159-91775, Mashhad, Iran

Abstract

In this paper, using some properties of fundamental groups and covering spaces of connected polyhedra and CW-complexes, we present topological proof for some famous theorems about finitely presented groups.

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[*]Correspondence: mashaf@math.um.ac.ir

1 Introduction

There are some famous results about subgroups of free groups, free products and finitely presented groups with complicated group theoretical proofs. For example, a famous corollary of the Reidemeister-Schreier rewriting process[3] tells us that every subgroup of a finitely presented group with finite index is also finitely presented. In this paper, using some well-known relationship between covering spaces of connected polyhedra (simplicial complexes) and their fundamental groups, we intend to prove some results for finitely presented groups with a topological approach.

2 Notation and Preliminaries

We suppose that the reader is familiar with some well-known notion such as free groups, free products and presentation in group theory and simplicial complexes (polyhedra), covering spaces, and fundamental groups in algebraic topology.

Definition 2.1 Let T be a connected simplicial complex, then T is called a tree if $dimT \leq 1$ and which contains no circuits. Let K be a connected simplicial complex with a maximal tree T in K. Define a group $G_{K,T}$ with the following presentation:

$$G_{K,T} = \langle\ (p,q) \in K \mid (p,q) \in T,\ (p,q)(q,r) = (p,r)\ if\ \{p,q,r\}\ is\ a\ simplex\ in\ K\ \rangle.$$

The following are some facts in algebraic topology which we need in the proof of main results.

Theorem 2.2 ([5]). Let K be a connected polyhedron with a base point p. Then its fundamental group $\pi_1(K,p)$ is isomorphic to $G_{K,T}$, where T is a maximal tree in K (note that we identify the simplicial complex K with its underlying set the polyhedron $|K|$).

Corollary 2.3. If K is a graph i.e. a connected 1-complex, then $\pi_1(K,p)$ is a free group of rank $|\{(p,q) \in K\backslash T \mid T \text{ is a maximal tree in } K\}|$.

Theorem 2.4 ([5]). A group G is finitely presented if and only if there exists a finite connected polyhedron X with $G \cong \pi_1(X,p)$.

Theorem 2.5 ([5]). For any group G, there exists a CW-complex $K(G)$ with

$$\pi_1(K(G)) \cong G \text{ and } \pi_n(K(G)) = 1 \text{ for all } n \geq 2.$$

The space $K(G)$ is called Eilenberg-MacLane space of G.

Remark 2.6 ([5]). With respect to the way of constructing the Eilenberg-MacLane space, generators and relators of the group G are in one to one corresponding to 1-cells and 2-cells in $K(G)$.

Corollary 2.7. A group G is finitely presented if and only if the number of 1-cells and 2-cells in it's Eilenberg-MacLane space $K(G)$ is finite.

Theorem 2.8 ([1]). For any group G and its Eilenberg-MacLane space, K say, we have

$$H_2(K) \cong M(G),$$

where $M(G)$ is the Schur multiplier of G.

Lemma 2.9 ([5]). Let $(\tilde{X},p)$ be a covering space of X, $x_0 \in X$, and $Y = p^{-1}(x_0)$ be the fiber over x_0. Then $|Y| = [\pi_1(X,x_0) : p_*(\pi_1(\tilde{X},x_0))]$.

Definition 2.10. A space X is called semilocally 1-connected if for every $x \in X$ there exists an open neighborhood U of x so that every closed path at x in U is nullhomotopic in X.

Note that any CW-complex, particularly any Eilenberg-MacLane space, is semilocally 1-connected space.

Theorem 2.11 ([5]). If X is connected, locally path connected, and semilocally 1-connected and $G \leq \pi_1(X,x_0)$, then there exists a constructed covering space of X, $(\tilde{X}_G,p)$ such that

$$p_*(\pi_1(\tilde{X}_G,\tilde{x_0})) = G.$$

Theorem 2.12 ([5]). If X is a connected CW-complex and $\tilde{X}$ is a covering space of X, then $\tilde{X}$ is also a CW-complex with $dim\tilde{X} = dimX$. Moreover, if X has m k-cells, and $\tilde{X}$ is n-sheeted, then the number of k-cells in $\tilde{X}$ is exactly equal to mn.

Theorem 2.13 ([4]). For any two groups G_1 and G_2 with their Eilenberg-MacLane spaces K_1 and K_2, respectively, the topological wedge space $K_1 \vee K_2$ is an Eilenberg-MacLane space corresponding to the free product $G_1 * G_2$.

Theorem 2.14 ([4]). For any two groups G_1 and G_2 with their Eilenberg-MacLane spaces K_1 and K_2, respectively, the topological product space $K_1 \times K_2$ is an Eilenberg-MacLane space corresponding to the direct product $G_1 \times G_2$.

3 Main Results

The following theorem is a consequence of the Reidemeister-Schreier rewriting process [3, Prop. 4.2].

Theorem 3.1. *Every subgroup of a finitely presented group with finite index is also finitely presented.*

Proof. Let G be a finitely presented group and $H \leq G$ with finite index. By Theorem 2.4, there exists a finite connected polyhedron X with $G \cong \pi_1(X)$. Since X is connected, locally path connected and semilocally 1-connected, there exists a covering space $\tilde{X}_H$ so that $\pi_1(\tilde{X}_H) \cong H$, by Theorem 2.11. Since $[G : H] \leq \infty$, $\tilde{X}_H$ is a finite sheeted covering space of X and so by Theorem 2.12, $\tilde{X}_H$ is a finite polyhedron. Now, by Theorem 2.4, $\pi_1(\tilde{X}_H) \cong H$ is finitely presented. $\Box$

Theorem 3.2. *If G is a finitely presented group, then its Schur multiplier $M(G)$ is finitely presented.*

Proof. First, note that the Schur multiplier of any group G is isomorphic to the second homology group of its corresponding Eilenberg-MacLane space [1], K say. Now using the fact that the number of i-cells, for any $i \in \mathbf{N}$, in the Eilenberg-MacLane space of any finitely presented group G is finite, any homology group of K and in particular, the Schur multiplier of G is finitely presented. □

Corollary 3.3. *Any covering group of a finite group is also a finitely presented group.*

Proof. Using the definition of covering group $\widetilde{G}$ considered as an extension of the Schur multiplier of G by the group G itself, this note is straightforward result of two recent theorems. □

Theorem 3.4. *The number of finitely presented groups is countable.*

Proof. First, recall that there exists a bijection between all finitely generated groups and special 2-simplicial complexes [6]. Hence to prove the result, it is sufficient to show the number of such spaces is countable. Note that each polyhedron corresponding to a finitely presented group G, with a presentation $G =< x_1, \cdots, x_n \mid r_1, \cdots, r_n >$, is obtained by attaching r 2-cells to an n-rose via some particular maps.

Suppose K is an n-rose lying on the plane $\mathbf{R}^2$ and $\{K^n_\lambda\}_{\lambda \in \Lambda}$ be the family of all polyhedra obtained by attaching finitely many 2-cells to K, in several ways.

Now by Whitney Theorem [7] which states that any n-simplicial complex can be embedded in $\mathbf{R}^{2n+1}$, we can consider all the constructed complexes as above in the Euclidean space $\mathbf{R}^5$ and then using the axiom of choice and the denseness of $\mathbf{Q}^5$ in $\mathbf{R}^5$, we can consider the rational points $x_\lambda \in \mathbf{Q}^5$ belonging to one and only one K^n_λ.

Finally, we conclude that all finitely presented groups with n generators in their presentations are in one to one corresponding to a subset of rational points in $\mathbf{R}^5$ and so we are done. □

Theorem 3.5. *The free product of two finitely presented groups is finitely presented.*

Proof. Suppose that G_1 and G_2 are finitely presented groups with Eilenberg-MacLane spaces K_1 and K_2, respectively. Using Theorem 2.13, $K_1 \vee K_2$ is an

Eilenberg-MacLane space corresponding to $G_1 * G_2$. Also, by the definition, clearly the number of i-cells in wedge space of two spaces having finitely many i-cells, is also finite and so by Theorem 2.7, the result satisfied. □

Theorem 3.6. *The product of two finitely presented groups is finitely presented.*

Proof. By the hypothesis of the previous proof, we only note Theorem 2.14, and the fact that the number of i-cells in product of two spaces having finitely many i-cells, is also finite. Hence similar to the above proof, we complete the proof. □

Theorem 3.7. *The free amalgamated product of two finitely presented groups G_1 and G_2 over a finitely presented subgroup H is also finitely presented.*

Proof. First, we consider an Eilenberg-Maclane space corresponding to the presentation of H, X say, and note that we can extend the algebraic presentation of H to the presentations for G_1 and G_2.

Also, by joining some 1-cells and attaching 2-cells via the relations, similar to the method of [5, Theorem 7.45] and [6, Note 6.44], we extend the space X to Eilenberg-Maclane spaces X_1 and X_2 corresponding to the presentations of G_1 and G_2, respectively. Note that the construction is considered so that X is a deformation retract of the space $X_1 \cap X_2$.

Now using van-Kampen theorem, the fundamental group $\pi_1(X_1 \cup X_2)$ is the free amalgamated product of two groups $\pi_1(X_1) \cong G_1$ and $\pi_1(X_2) \cong G_2$ over the subgroup $\pi_1(Y) \cong \pi_1(X_1 \cap X_2) \cong H$ [5].

Hence by uniqueness of the free amalgamated product up to isomorphism, we conclude that

$$G \cong \pi_1(X_1 \cup X_2).$$

On the other hand, by the assumption of being finitely presented for the groups H, G_1 and G_2 we conclude the spaces X, X_1, X_2 and so the space $X_1 \cup X_2$ have finitely many cells, which implies the group $\pi_1(X_1 \cup X_2)$ to be finitely presented. □

Finally, by the definition of two new concepts, the Schur multiplier of a pair and the Schur multiplier of a triple of groups [2], we conclude the following results. Note that for a pair of groups (G, N), the natural epimorphism $G \to G/N$ induces functorially the continuous map $f : K(G) \to K(G/N)$. Suppose that $M(f)$ is the mapping cylinder of f containing $K(G)$ as a subspace and is also homotopically equivalent to the space $K(G/N)$. We take $K(G, N)$ to be the mapping cone of the cofibration $K(G) \hookrightarrow M(f)$. The Schur multiplier of the pair (G, N) is considered as the third homology group of the cofiber space $K(G, N)$.

In addition, we can extend the above notes to a topological argument for the Schur multiplier of a triple of groups. If we consider the space X as the cofibration of the natural sequence $K(G, N) \to K(G/M, MN/M)$, which is noted by $K(G, M, N)$ [2, Sec. 6], then the Schur multiplier of the triple (G, M, N) is defined to be the fourth homology group of the cofiber space $K(G, M, N)$.

Theorem 3.8. *The Schur multiplier of a pair of finitely presented groups is finitely presented.*

Proof. We remark that a mapping cone space obtained from two spaces having finitely many cells, have also finitely many cells, which holds the result. □

Using a similar argument, we establish the following theorem:

Theorem 3.9. *The Schur multiplier of a triple of finitely presented groups is finitely presented.*

References

[1] E.B. Curtis; Simplicial homotopy theory, *Advances in Math.*, **6**, (1971) 107-209.

[2] G. Ellis; The Schur multiplier of a pair of groups, *Applied Categorical Structures*, **6**, (1998) 355-371.

[3] R.C. Lyndon, P.E. Schupp; *Combinatorial Group Theory*, Springer-Verlag, New York, 1977.

[4] B. Mashayekhy, H. Mirebrahimi; Some properties of the Schur multiplier with algebraic topological approach, *International Journal of Mathematics, Game Theory and Algebra*, **17:3**, (2008) 125-129.

[5] J.J. Rotman; *An Introduction to Algebraic Topology*, Graduate Texts in Mathematics, Vol. **119**, Springer-Verlag, New York, 1988.

[6] P. Scott, T. Wall; Topological methods in group theory, Homological group theory (Proc. Sympos., Durham, 1977), pp. 137-203, *London Math Soc. Lecture Note Ser.*, 36, Cambridge Univ. Press, Cambridge-New york, 1979.

[7] H. Whitney; *Geometric Integration Theory*, Princeton University Press, 1957.