ON LERAY'S PROBLEM FOR ALMOST PERIODIC FLOWS

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ABSTRACT. We prove existence and uniqueness for fully-developed (Poiseuille-type) flows in semi-infinite cylinders, in the setting of (time) almost-periodic functions. In the case of Stepanov almost-periodic functions the proof is based on a detailed variational analysis of a linear "inverse" problem, while in the Besicovitch setting the proof follows by a precise analysis in wave-numbers.

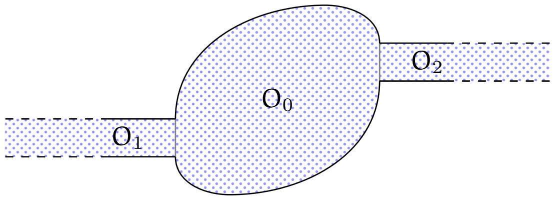
Next, we use our results to construct a unique almost periodic solution to the so called "Leray's problem" concerning 3D fluid motion in two semi-infinite cylinders connected by a bounded reservoir. In the case of Stepanov functions we need a natural restriction on the size of the flux, while for Besicovitch solutions certain limitations on the generalized Fourier coefficients are requested.

CONTENTS

1. INTRODUCTION

We consider the motion of a viscous fluid in semi-infinite cylindrical pipes, with an assigned (time) almost-periodic flux. The results are aimed to find solutions of the so-called "Leray's problem." Moreover, this work can be considered as a intermediate step towards the analysis of (deterministic) statistical solutions for the flow of Poiseuille-type, which is the object of our current and ongoing research. We recall that Leray's problem (which seems to have been proposed by Leray to Ladyžhenskaya [17, 19]) is that of determining a motion in a region with cylindrical exits, subject to a given flux, and tending to the Poiseuille solution in each exit. More precisely, let be given a connected open domain $O \subset \mathbf{R}^3$ made of a "reservoir", a bounded and smooth open set O_0, with two cylindrical exits O_1 and O_2. These two semi-infinite exits (pipes)

Date: December 9, 2010.
2000 *Mathematics Subject Classification.* 35Q30, 76D03, 35B15.
Key words and phrases. Almost periodic flux, channel flow, Leray's problem.
This work has been partially supported by the GNAMPA project *Modelli aleatorii e computazionali per l'analisi della turbolenza generata da pareti ruvide*.



are described in coordinate systems directed along the axis as

$$O_i = D_i \times \mathbf{R}^+,$$

where the smooth cross sections D_i, $i = 1, 2$, may be possibly of different shape and measure. We denote by $z \in \mathbf{R}^+$ the axial coordinate in both cylinders. Pioneering results in the stationary case are those of Ladyžhenskaya [18] and Amick [2]. See also the review in Finn [7]. The extensive literature on the stationary problem is recalled for instance in [11, 12] for the linearized and full Navier-Stokes problem, respectively. More recently the problem of motion in pipes has also been addressed in the time-evolution case, see Ladyžhenskaya and Solonnikov [20] and also the review in Solonnikov [26]. In the last decade Beirão da Veiga [3] and Pileckas [24] gave new contributions to the study of the time-dependent problem with assigned flux, and the special role of the pressure has been also emphasized by Galdi and coworkers [14, 13].

In [3] Leray's problem has been considered in the context of time periodic flows, especially in view of application to the study of blood flow and we recall that the role of blood flow in mathematical research has been put in evidence by Quarteroni [25]. We also stress that the (non-trivial) explicit solution introduced by Womersley is periodic, in some sense generalizes the Poiseuille flow, and has been discovered in the study of physiological flows. Since the heart is pumping with a flux which is not periodic, but a superposition of possibly non-rational frequencies, this suggests also to study the problem in the setting of almost periodic functions. This work has been originated by the inspiring results in [3] and especially from Remark 3 therein: The independence of the various constant on the period of the flux let the author suggest about the possible extension to almost periodic solutions. The problem nevertheless requires a precise functional setting in order to detect the largest class of almost periodic functions to be employed. Moreover, it seems that the very-nice proof based on Fourier series in [3] cannot be directly applied to the new setting and in addition new difficulties in treating the nonlinearities arise when almost periodic functions are employed. This leads us to propose two different approaches in two different functional settings. We finally remark that, in addition to early results of Foias [8], the approach *via* almost-periodic functions finds wide applications in fluid mechanics (see for instance the recent paper by Gérard-Varet and Masmoudi [15]).

1.1. **Setting of the problem.** The problem we wish to solve is to find a (time) almost-periodic solution of the Navier-Stokes equations

$$\begin{cases} \partial_t u - \nu\Delta u + (u \cdot \nabla)\, u + \nabla p = 0, & x \in O, \quad t \in \mathbf{R}, \\ \operatorname{div} u = 0, & x \in O, \quad t \in \mathbf{R}, \\ u = 0 & \text{on } \partial O, \quad t \in \mathbf{R}, \end{cases} \tag{1.1}$$

such that u converges in both pipes as $|z| \to \infty$ (in a sense we shall make clear later) to the solution of the Poiseuille-type problem. For clarity we recall (cf.[3, § 2]) that by solution of the Poiseuille-type problem (of fully-developed flow) we mean a solution of the Navier-Stokes equations such that, in a reference frame with z directed along the axis of the pipe and $x := (x_1, x_2)$ belonging to the orthogonal plane, is of the form

$$u(t, x, z) = (0, 0, w(t, x)) \qquad \text{and} \qquad p(t, x, z) = \pi(t, x, z) + p_0(t).$$

Here $p_0(t)$ is an arbitrary function and in addition the flux condition is satisfied

$$\int_D w(t,x)\,dx = f(t),$$

for some given function f, where D is the section of the domain.

The Poiseuille-type *ansatz* implies that Navier-Stokes equations reduce in the semi-infinite pipes O_i, $i = 1, 2$, to the following equations

$$\begin{cases} \partial_t w^i - \nu\Delta_x w^i + \partial_z p^i = 0, & (x,z) \in O_i,\ t \in \mathbf{R}, \\ \partial_{x_1} p^i = \partial_{x_2} p^i = 0 & (x,z) \in O_i,\ t \in \mathbf{R}, \\ w^i(t,x) = 0, & x \in \partial O_i,\ t \in \mathbf{R}, \end{cases}$$

where Δ_x denotes the Laplacian with respect to the variables x_1 and x_2. From the first equation it follows that $\partial_z p^i$ is independent of z. From the second equation, we also obtain that p^i is independent of x, hence $p^i(t,x,z) = -\pi^i(t)\,z + p_0^i(t)$. Since the term $p_0^i(t)$ does not affect the velocity field, we may assume that the pressure has the form $p^i(t,z) = -\pi(t)^i\,z$. Moreover, the dependence of w^i on the space variables x_1 and x_2 allows us to consider a problem reduced to the cross section D_i of O_i, and the flux condition is $\int_{D_i} w^i(t,x)\,dx = f(t)$. This implies that we have to study in each pipe the following problem (called in the sequel the "basic flow"): Find $(w^i(t,x), \pi^i(t))$ such that

$$\begin{cases} \partial_t w^i(t,x) - \nu\Delta w^i(t,x) = \pi^i(t), & x \in D_i,\ t \in \mathbf{R}, \\ w^i(t,x) = 0 & x \in \partial D_i,\ t \in \mathbf{R}, \\ \int_{D_i} w^i(t,x)\,dx = f(t) & t \in \mathbf{R}, \end{cases} \tag{1.2}$$

showing (under suitable assumptions) that if f is almost periodic, then the couple (w,π) is almost-periodic, too.

We observe that, contrary to the stationary problem where the same approach gives the well-known Poiseuille solutions, the solution of the time-dependent motion is more complex for the determination of the non-constant pressure (Observe that the classical Poiseuille solution is that obtained for circular pipes, but nevertheless in more general domains the same approach gives corresponding results). In our problem for the basic flow we have two scalar unknowns and two equations, but contrary to the classical problems in fluid mechanics one cannot get rid of the quantity π by means of projection operators. The problem we have to solve can be considered as an *inverse* problem. Moreover the problem cannot be treated with the standard variational tools in a direct way. We can write a single equation (the "elimination" of π^i is obtained by taking the mean value over D_i) obtaining

$$\begin{cases} \partial_t w^i - \nu\Delta w^i + \nu \int_D \Delta w^i = f'(t), & x \in D_i,\ t \in \mathbf{R}, \\ w^i(t,x) = 0, & x \in \partial D_i,\ t \in \mathbf{R}, \end{cases}$$

and the latter equation makes easy to understand why some knowledge also of the derivative of the flux will be needed in order to solve (2.2). Moreover the usual energy-type estimates obtained by testing with w^i, or with $-\Delta w^i$, and with w^i_t are not-conclusive when applied to this problem. In particular, the lack of coercivity prevents from a direct application of the standard techniques employed for parabolic problems, see [3, Sec. 3]. This particular issue has been addressed in two different ways by Beirão da Veiga [3] (periodic case) and Pileckas [24] (given smooth flux).

Even if we generalize to the almost-periodic setting the periodic results obtained in [3], in the first part of the paper we will mainly follow and suitably adapt the approach of [24]. In particular we give special emphasis to the solution of (1.2) since this represents one of the main technical difficulties. The nonlinear problem is then treated by means of perturbation arguments in a more or less standard way. We want also to point out that in the huge literature on almost periodic solutions we find particularly inspiring (for the choice of Stepanov functions as suitable for our problem) the paper by Marcati and Valli [22] concerning compressible fluids.

In the second part of the paper we consider the problem in the larger class of Besicovitch almost periodic solutions with an approach which is more in the spirit of Fourier analysis. We give a different proof of the existence of the basic flow which also covers the $H^1(\mathbf{R})$ case and provides an alternative proof of [3, Thm. 1], when restricted to a time-periodic flux. The fully nonlinear case needs, besides the natural assumption of large viscosity, an additional assumption of regularity on the flux (see (3.12)) which accounts of the technical difficulties of this case, due essentially to the non–local (in time) quantities that are used, see Section 3.4.7.

Remark 1.1. For its variational formulation and the use of energy estimates the problem seems to be naturally set in Hilbert spaces and this is not well fitting with the classical continuous (Bohr) spaces of almost periodic functions. A suitable choice of the spaces represents then a fundamental starting point. We are presenting two different proofs in two different settings, since they are substantial different and the assumptions we make on the flux are of very different nature. In the first part we deal with Stepanov a.p. functions, and the setting is much similar to the classical variational one for evolution partial differential equations. In the second part we deal with Besicovitch a.p. functions and the proof use analysis in wave-numbers. We also point out that while the linear problem can be also treated in a unified way, for the nonlinear one the differences in the functional setting imply special assumptions on the size of the flux and on the Fourier coefficients, respectively.

Plan of the paper: In Section 2 we consider the problem under the condition of a Stepanov almost periodic flux. After recalling the main definition we give a complete solution of Leray's problem, with the *natural* (in space dimension three) restriction of a large viscosity. As by-product of our results, we also prove existence in the case of $H^1(\mathbf{R})$ fluxes. In Section 3 we consider the problem in the larger class of Besicovitch almost periodic solutions and we prove existence for the basic flow, together with existence for the nonlinear problem under suitable restrictions on the flux.

2. Leray's problem in the framework of Stepanov a.p. functions

Here we introduce a functional setting in which it is possible to extend the result of [3] to almost periodic solutions.

2.1. **Functional setting.** The problem of almost periodic solutions of partial differential equations has been studied extensively in the last century, starting with the work of Bohr, Muckenhoupt, Bochner, and Favard and many others. See the review in Amerio and Prouse [1], Besicovitch [5], Corduneanu [6], and Levitan and Zhikov [21].

In the sequel we will use the standard Lebesgue L^p and Sobolev spaces $H^s = W^{s,2}$. For simplicity we also denote by $\|\,.\,\|$ the L^2-norm. We will use the symbol C to denote a generic constant, possibly different from line to line, depending on the domain and not on the viscosity ν or on the flux f. Next, given a Banach space $(X, \|\,.\,\|_X)$ we denote by $\mathsf{UAP}(\mathbf{R}; X)$ the space of almost periodic functions in the sense of Bohr-Bochner. We recall that a function $f \in C^0(\mathbf{R}; X)$ is almost periodic if and only if the set of its translates is relatively compact in the $C^0(\mathbf{R}; X)$-topology (observe that if $C^0_b(\mathbf{R}; X)$ denotes the space of continuous bounded functions, then $\mathsf{UAP}(\mathbf{R}; X) \subset C^0_b(\mathbf{R}; X)$). In the context of weak and strong solutions to partial differential equations it is probably better to work with a more general notion of almost-periodicity, given for functions $f \in L^p_{loc}(\mathbf{R}; X)$, which is suited to deal with distributional solutions.

Definition 2.1 (Stepanov p-almost periodicity)**.** We say that the function $f: \mathbf{R} \to X$ is Stepanov p-almost periodic (denoted by $f \in \mathcal{S}^p(\mathbf{R}; X)$) if $f \in L^p_{loc}(\mathbf{R}; X)$ and if the set of its translates is relatively compact in the $L^p_{uloc}(\mathbf{R}; X)$ topology defined by the norm

$$\|f\|_{L^p_{uloc}(\mathbf{R},X)} := \sup_{t\in\mathbf{R}} \left[\int_t^{t+1} \|f(s)\|_X^p \, ds\right]^{1/p}.$$

When $p = 2$ we say simply that the function $f: \mathbf{R} \to X$ is Stepanov almost periodic.

We will give the main result by using fluxes belonging to this class, together with their first derivative. However, in the second part of the paper we will consider also a wider class of almost periodic functions: functions almost periodic in the sense of Besicovitch. Further generalities (not needed in this section) on almost periodic functions are given in Section 3.1.

A first main result that we will prove concerns the existence of the "basic flow" problem in this framework.

Theorem 2.2. *Let be given a smooth, connected, and bounded open set* $D \subset \mathbf{R}^2$ *and let be given* f *such that* $f, f' \in \mathcal{S}^2(\mathbf{R})$*. Then, there exists a unique solution* (w, π) *of* (2.2) *such that*

$$\begin{aligned} &\Delta w, w_t \in \mathcal{S}^2(\mathbf{R}; L^2(D)), \\ &\nabla w \in \mathcal{S}^2(\mathbf{R}; L^2(D)) \cap C^0_b(\mathbf{R}; L^2(D)), \\ &\pi \in \mathcal{S}^2(\mathbf{R}), \end{aligned}$$

and

$$\begin{aligned} \sup_{t\in\mathbf{R}} \Big[\nu\|\nabla w(t)\|^2 + \int_t^{t+1} \left(\nu^2\|\Delta w(s)\|^2 + \|w_t(s)\|^2 + |\pi(s)|^2\right) ds\Big] \\ \leqslant C\big(\nu^2 + 1 + \frac{1}{\nu}\big)\|f\|^2_{H^1_{uloc}(\mathbf{R})}, \end{aligned} \tag{2.1}$$

Remark 2.3. The result concerning the linear problem for the basic flow holds true in any space dimension.

This allows to obtain in a rather standard way the following result for the Navier-Stokes equations.

Theorem 2.4. *Let* O *as in the introduction and let be given* f *such that* $f, f' \in \mathcal{S}^2(\mathbf{R})$*. There exists* $\nu_0 = \nu_0(f, O) \geqslant 0$ *such that if* $\nu > \nu_0$ *there exists a unique solution* u *of* (1.1) *such that*

$$u \in \mathcal{S}^2(\mathbf{R}; H^s(O)) \quad \text{for all } s < 2$$

and u *converges to a Poiseuille-type solution* w^i *in each pipe, as* $|z| \to +\infty$*.*

Remark 2.5. The restriction on the viscosity is not surprising and is common to several results concerning the three-dimensional Navier-Stokes equations. This is also observed in [12, Ch. XI] since the existence of a flux carrier that can be absorbed by the dissipation for any positive viscosity is generally not known for cylindrical domains. This imposes (also in the stationary case) limitations on the size of the flux, in terms of the viscosity.

We observe that also in the time-periodic case [3] largeness (in terms of data of the problem) of the viscosity is required. Nevertheless, the results in [3] concern weak solutions and uniqueness is not stated. On the other hand in [24] there is no restriction on the viscosity, since special "*two-dimensional-like*" solutions are considered.

2.2. **Construction of the solution of the "basic flow".** In this section we give a detailed analysis of the existence of an almost periodic basic flow and a complete proof of Theorem 2.2. The problem is the following: given f, $f \in \mathcal{S}^2(\mathbf{R})$ find a Stepanov almost periodic solution of

$$\begin{cases} \partial_t w - \nu \Delta w = \pi, & x \in D,\ t \in \mathbf{R}, \\ w(t,x) = 0, & x \in \partial D,\ t \in \mathbf{R}, \\ \int_D w(t,x)\, dx = f(t) & t \in \mathbf{R}, \end{cases} \tag{2.2}$$

Remark 2.6. It is easy to check that one can analyze the slightly more general problem where (2.2) is replaced by

$$\begin{cases} \partial_t w + \nu A w = \pi e, & t \in \mathbf{R}, \\ \langle w(t), e\rangle_H = f(t) & t \in \mathbf{R}. \end{cases}$$

with an unbounded, linear, and with compact inverse operator A on the Hilbert space H with domain $D(A)$ and $e \in H$ with $e \notin D(A)$ is given. Under suitable assumptions on A, the same procedure that we will employ can be used, see also [4]. The same remark holds also for the results of Section 3.

We start by solving the following initial-boundary value problem in the unknowns (w, π),

$$\begin{cases} \partial_t w - \nu \Delta w = \pi, & x \in D,\ t \in]0,T], \\ w(t,x) = 0, & x \in \partial D,\ t \in]0,T], \\ \int_D w(t,x)\, dx = f(t), & t \in [0,T], \\ w(0,x) = w_0(x), & x \in D. \end{cases} \tag{2.3}$$

We follow essentially the same approach of [24], with additional care on the analysis of the initial datum and on the dependence of the solution on the various parameters of the problem. In the sequel we will employ a spectral (spatial) approximation using the $L^2(D)$-orthonormal eigenfunctions $(e_k)_{k\in\mathbf{N}}$ of the Laplace operator,

$$\begin{cases} -\Delta e_k = \lambda_k e_k & x \in D, \\ e_k = 0 & x \in \partial D. \end{cases}$$

Define

$$\beta_j := (\mathbb{1}, e_j), \qquad j \in \mathbf{N},$$

where $(\cdot,\cdot)$ denotes the $L^2(D)$ scalar product and $\mathbb{1}$ is the function defined on D such that $\mathbb{1}(x) = 1$ a. e.. Without loss of generality from now on we assume that $|D|$, the Lebesgue measure of D, is equal to one. Clearly,

$$\mathbb{1} = \sum_{k=1}^{\infty} \beta_j e_j(x) \qquad \text{and} \qquad \sum_{k=1}^{\infty} \beta_j^2 = |D| = 1.$$

A special role is played by the pipe's flux carrier, i. e., by the function Φ (which belongs to $H^2(D) \cap H_0^1(D)$ under the smoothness assumptions on D) defined as solution of the following Poisson problem

$$\begin{cases} -\Delta \Phi = \mathbb{1} & x \in D, \\ \Phi = 0 & x \in \partial D. \end{cases}$$

We define the following quantities

$$\chi_0^2 := \int_D \Phi\, dx = \int_D |\nabla \Phi|^2 dx > 0 \qquad \text{and} \qquad \eta_0^2 := \int_D |\Phi|^2 dx > 0,$$

which clearly depend only on D. Observe that the function Φ is enough in the stationary case to construct Poiseuille-type flows, since in that case the problem for the flux and that for the pressure completely decouple. On the other hand, in the time-dependent case the situation if more complex, since both unknown depend also on the time.

To work with in our problem with a general viscosity we need the scaled version of the flux carrier $\varphi := \nu^{-1}\Phi$ which solves

$$\begin{cases} -\nu\Delta\varphi = \mathbb{1} & x \in D, \\ \varphi = 0 & x \in \partial D, \end{cases} \tag{2.4}$$

and such that

$$\int_D \varphi\, dx = \nu \int_D |\nabla\varphi|^2 dx = \frac{\chi_0^2}{\nu} > 0 \qquad \text{and} \qquad \int_D |\varphi|^2 dx = \frac{\eta_0^2}{\nu^2}.$$

We start our analysis by proving the following result.

Proposition 2.7. *Given* $f \in H^1(0,T)$, *assume that*[1] $w_0(x) = \frac{\nu\,\varphi(x)}{\chi_0^2} f(0)$. *Then, there exists a unique solution* (w, π) *of* (2.3) *such that*

$$w \in C(0,T; H_0^1(D)) \cap H^1(0,T; L^2(D)) \cap L^2(0,T; H^2(D)),$$
$$\pi \in L^2(0,T),$$

satisfying the following estimate

$$\begin{aligned} \nu\|\nabla w(t)\|^2 + \nu^2 \int_0^t \|\Delta w(s)\|^2 + \int_0^t \|w_t(s)\|^2\, ds + \int_0^t |\pi(s)|^2\, ds \\ \leqslant C \int_0^t \left((1+\nu^2)|f(s)|^2 + (1+\nu)|f'(s)|^2\right) ds, \end{aligned} \tag{2.5}$$

with a constant C *depending only on* D *(and in particular independent of* T*).*

Proof. We start by constructing, with the Faedo-Galerkin method, a global unique approximate solution in $V_m = \text{Span}\langle e_1, \dots, e_m\rangle$. The first step is to approximate the initial condition. Let φ be the function introduced in (2.4) and write $\varphi = \sum_{k=1}^{\infty} \varphi_k e_k$, where the series converges in $H^2(D)$ and $\varphi_k = (\varphi, e_k)$. Hence, the projection of φ over V_m is given by

$$P_m\varphi := \sum_{k=1}^{m} \varphi_k e_k.$$

In order to satisfy the flux condition also at time $t = 0$ we set

$$w^m(0,x) := \frac{f(0) P_m\varphi(x)}{\int_D P_m\varphi(x)\, dx}.$$

Observe that, for large enough $m \in \mathbf{N}$, the approximate initial datum is well-defined. In fact, $P_m\varphi \to \varphi$ in $L^2(D)$ and since $|D| < +\infty$ then $P_m\varphi \to \varphi$ in $L^1(D)$. Since $\int_D P_m\varphi \to \int_D \varphi = \nu^{-1}\chi_0^2 > 0$, there exists $m_0 \in \mathbf{N}$ such that $\int_D P_m\varphi \neq 0$ for all $m \geqslant m_0$. Moreover, $w^m(0,x) \to w_0(x)$ in $H^2(D)$, as $m \to +\infty$.

We write Galerkin approximate functions

$$w^m(t,x) = \sum_{k=1}^{m} c_k^m(t)\, e_k(x),$$

[1] The initial condition here is chosen in such a way that the compatibility conditions on the flux at time $t = 0$ are satisfied.

and we look for a couple (w^m, π^m) such that

$$\frac{d}{dt}(w^m, e_j) + \nu(\nabla w^m, \nabla e_j) = \pi^m(\mathbb{1}, e_j) \qquad \text{for } j = 1, \dots, m, \tag{2.6}$$

and $\pi^m : (0, T) \to (0, T)$ chosen so that the flux condition

$$\int_D w^m(t, x)\, dx = f(t) \qquad \forall\, t \in (0, T), \tag{2.7}$$

is satisfied. The equality is meaningful since f is a. e. equal to a continuous function. In terms of Galerkin coefficients $(c_j^m)_{1\leqslant j\leqslant m}$ we have for the initial condition that

$$c_j^m(0) := f(0)\frac{\varphi_j}{\sum_{j=0}^m \varphi_k \beta_k},$$

while the system of ordinary differential equations reads as

$$\frac{d}{dt}c_j^m(t) + \nu\lambda_j c_j^m(t) = \pi^m(t)\beta_j \qquad \text{for } j = 1, \dots, m,$$

and the solution can be written as follows

$$c_j^m(t) = c_j^m(0)\, e^{-\nu\lambda_j t} + \beta_j \int_0^t \pi^m(s)\, e^{-\nu\lambda_j (t-s)}\, ds \qquad \text{for } j = 1, \dots, m.$$

To find the equation satisfied by π^m we multiply the latter equality by β_j and sum over $j = 1, \dots, m$ to get

$$f(t) = \sum_{j=0}^m \beta_j c_j^m(0)\, e^{-\nu\lambda_j t} + \sum_{j=0}^m \beta_j^2 \int_0^t \pi^m(s)\, e^{-\nu\lambda_j (t-s)}\, ds.$$

Finally, to obtain an integral equation for π^m, we differentiate with respect to time deducing

$$f'(t) = -\nu\sum_{j=0}^m \lambda_j\beta_j c_j^m(0)\, e^{-\nu\lambda_j t} + \sum_{j=0}^m \Big(\beta_j^2\pi^m(t) - \nu\lambda_j \int_0^t \beta_j^2 \pi^m(s)\, e^{-\nu\lambda_j (t-s)}\, ds\Big),$$

and this yields the following *Volterra integral equation of the second type*

$$\pi^m(t) - \nu\int_0^t \sum_{j=1}^m \frac{\lambda_j\beta_j^2}{|\beta|^2}\, e^{\nu\lambda_j(t-s)}\, \pi^m(s)\, ds = \frac{1}{|\beta|^2} f'(t) + f(0)\frac{\sum_{j=1}^m \beta_j\varphi_j\, e^{-\nu\lambda_j t}}{\sum_{k=1}^m \beta_k\varphi_k}, \tag{2.8}$$

where $|\beta|^2 := \sum_{k=1}^m \beta_m^2$. For any fixed $m \in \mathbf{N}$ the kernel of the integral equation (2.8) is bounded for all $0 \leqslant s \leqslant t$. This is enough to infer that if $f \in H^1(0, T)$, then there exists a unique $\pi^m \in L^2(0, T)$ satisfying the integral equation (2.8) and such that

$$\|\pi^m\|_{L^2(0,T)} \leqslant C_m(\nu)\|f\|_{H^1(0,T)},$$

for a constant $C_m(\nu)$ possibly depending on m and also on ν. Especially the dependence on m is crucial, since we consider the problem at fixed viscosity, while we need uniform estimates in $m \in \mathbf{N}$ to employ the Galerkin method. In particular, the uniform estimate does not follow directly since the series defining the kernel for $s = t$, that is $\big(\sum_{k=1}^m \beta_k^2\big)^{-1} \sum_{k=1}^m \lambda_k\beta_k^2$ does not converge for $m \to +\infty$, see [24] for further details.

We need to find the *a priori* estimate in a different way, but observe that, once we have constructed π^m, we can use it as a *given external force* in the equation for the velocity (2.6). By using

w^m as test function, we obtain (with the Schwarz inequality and by using (2.7)) that

$$\begin{aligned}\frac{1}{2}\|w^m\|^2 + \nu\int_0^t \|\nabla w^m\|^2 &= \frac{1}{2}\|w_0^m\|^2 + \int_0^t \pi^m(s)(\mathbb{1}, w^m(s))\,ds\\ &= \frac{1}{2}\|w_0^m\|^2 + \int_0^t \pi^m(s)f(s)\,ds\\ &\leqslant \|w_0\|^2 + \frac{1}{2\epsilon}\int_0^t |f(s)|^2 + \frac{\epsilon}{2}\int_0^t |\pi^m(s)|^2\,ds.\end{aligned}$$

Observe also that $w_0^m \to w_0$ in $L^2(D)$ and that $\|w_0^m\|^2 \leqslant 2\|w_0\|^2$ for large enough m and in addition

$$\|\nabla w_0^m\|^2 \leqslant 2\|\nabla w_0\|^2 \leqslant \frac{2\nu^2}{\chi_0^2}\|f\|^2_{H^1(0,T)}\|\nabla\varphi\|^2 \leqslant 2\|f\|^2_{H^1(0,T)}.$$

This gives the first a-priori estimate showing that, for all $m \geqslant m_0$ there exists a unique solution $w^m \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$. Since the bounds we obtain on the solution are not independent of m, they cannot be used to make directly the Galerkin method to work. We can prove even more regularity on w^m by standard estimates. In fact, by using as test function w_t^m in the system satisfied by w^m we get

$$\|w_t^m(t)\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla w^m(t)\|^2 = \pi^m(t)(\mathbb{1}, w_t^m(t)) = \pi^m(t)f'(t).$$

Hence, an application of the Schwarz inequality gives

$$\frac{\nu}{2}\|\nabla w^m(t)\|^2 + \int_0^t \|w_t^m\|^2\,ds \leqslant \nu\|\nabla w_0\|^2 + \frac{\epsilon}{2}\int_0^t |\pi^m(s)|^2\,ds + \frac{1}{2\epsilon}\int_0^t |f'(s)|^2\,ds. \tag{2.9}$$

By using $-\Delta w^m$ as test function in the equation satisfied by w^m we also obtain

$$\frac{1}{2}\|\nabla w^m(t)\|^2 + \frac{\nu}{2}\int_0^t \|\Delta w^m(s)\|^2\,ds \leqslant \|\nabla w_0\|^2 + \frac{1}{2\nu}\int_0^t |\pi^m(s)|^2\,ds. \tag{2.10}$$

These estimates are enough to construct, for each fixed $m \in \mathbf{N}$, a unique solution (w^m, π^m), which is smooth, say $H^1(0,T;L^2(D)) \cap L^2(0,T;H^2(D))$. Again the presence of π^m in the right-hand side prevents from uniformity in m.

This issue is solved by using a special test function and a couple of nested a-priori estimates in the following lemma.

Lemma 2.8. *There exists $M_0 \in \mathbf{N}$ (larger or equal than m_0) such that for all $m > M_0$ it holds for all $t \in [0,T]$*

$$\begin{aligned}\nu\|\nabla w^m(t)\|^2 + \nu^2\int_0^t \|\Delta w^m(s)\|^2 + \int_0^t \|w_t^m(s)\|^2\,ds + \int_0^t |\pi^m(s)|^2\,ds\\ \leqslant C\int_0^t \big((1+\nu^2)|f(s)|^2 + (1+\nu)|f'(s)|^2\big)\,ds,\end{aligned} \tag{2.11}$$

with a constant C depending only on D (hence independent of m and of T).

Proof. We observe that the function $P_m\varphi$, where φ is defined in (2.4), turns out to be a legitimate test function for the Galerkin system. With integration by parts, we obtain

$$(w_t^m, P_m\varphi) - \nu(w^m, \Delta P_m\varphi) = \pi^m(t)(\mathbb{1}, \varphi + (P_m\varphi - \varphi)).$$

By adding to both sides of the previous equality the quantity $-\nu(w^m, \Delta\varphi) = (w^m, \mathbb{1}) = f(t)$ we obtain

$$(w_t^m, P_m\varphi) + f = \pi^m(\mathbb{1}, \varphi) + \pi^m(\mathbb{1}, P_m\varphi - \varphi) + \nu(w^m, \Delta(P_m\varphi - \varphi)).$$

The last term from the right-hand side vanishes since $w^m \in V_m$, while $\Delta(P_m\varphi - \varphi) \in V_m^\perp$. Hence squaring the latter equality (remember that $(\mathbb{1}, \varphi) = \nu^{-1}\chi_0^2$) we obtain

$$\frac{\chi_0^4}{\nu^2}|\pi^m(t)|^2 \leqslant 3\big(\|w_t^m(t)\|^2\|P_m\varphi\|^2 + |f(t)|^2 + |\pi^m(t)|^2\|P_m\varphi - \varphi\|^2\big).$$

Next, since P_m is a projection operator it follows that there exists $M_0 \in \mathbf{N}$ such that $\|P_m\varphi - \varphi\|^2 < \chi_0^4/(6\nu^2)$ for all $m \geqslant M_0$. Consequently, we can absorb in the left-hand side the term involving π^m from the right-hand side. Consequently, after integration over $(0, T)$, we get

$$\int_0^t |\pi^m(s)|^2\,ds \leqslant C\int_0^t \big(\|w_t^m(s)\|^2 + \nu^2|f(s)|^2\big)\,ds \qquad \forall\, m \geqslant M_0, \tag{2.12}$$

with a constant C depending only on D (*via* χ_0 and η_0). Consider again (2.9) and use now (2.12) with $\epsilon = 1/C$. It is possible to absorb the term involving π from the right-hand side, obtaining

$$\begin{aligned}
&\frac{\nu}{2}\|\nabla w^m(t)\|^2 + \frac{1}{2}\int_0^t \|w_t^m(s)\|^2\,ds\\
&\leqslant \nu\|\nabla w_0\|^2 + C\int_0^t \nu^2|f(s)|^2 + |f'(s)|^2\,ds\\
&\leqslant C\int_0^t (1+\nu^2)|f(s)|^2 + (1+\nu)|f'(s)|^2\,ds
\end{aligned}$$

This shows a uniform bound on $\|w_t^m\|_{L^2(0,T)}$. Hence, coming back again to (2.12) we also obtain that

$$\int_0^t |\pi^m(s)|^2\,ds \leqslant C\int_0^t (1+\nu^2)|f(s)|^2 + (1+\nu)|f'(s)|^2\,ds$$

Next, by using the bound obtained in (2.10) we also get

$$\nu^2\int_0^t \|\Delta w^m(s)\|^2 \leqslant C\int_0^t (1+\nu^2)|f(s)|^2 + (1+\nu)|f'(s)|^2\,ds$$

By collecting all the estimates and with Young's inequality we obtain (2.11). □

With the above lemma we can conclude the proof of the existence result, since uniform bounds imply that there exists a couple (w, π) and a sub-sequence $\{m_k\}_{k\in\mathbf{N}}$ such that

$$\begin{aligned}
&w^{m_k} \rightharpoonup w && \text{in } H^1(0,T;L^2(D)) \cap L^2(0,T;H^2(D)),\\
&w^{m_k} \overset{*}{\rightharpoonup} w && \text{in } L^\infty(0,T;H_0^1(D)),\\
&\pi^{m_k} \rightharpoonup \pi && \text{in } L^2(0,T).
\end{aligned}$$

The problem is linear and this implies that (w, π) is a distributional solution of (2.3) with the requested regularity. Uniqueness follows again from linearity of the problem. □

Since the estimates on the norm of the solution are independent of T (they depend just on D and on ν) the same argument shows that if $f \in H^1(0, +\infty)$, one can study the problem with arbitrary $T > 0$ and then a unique solution (with the same regularity) exists in $[0, +\infty)$. More generally we have also the following result on the whole real line, which is obtained by letting the initial time to go to $-\infty$.

Corollary 2.9. *Let be given* $f \in H^1(\mathbf{R})$. *Then, there exists a unique solution* (w, π) *of* (2.3) *defined for all* $t \in \mathbf{R}$ *such that*

$$\begin{aligned}
&w \in C_o(\mathbf{R}; H_0^1(D)) \cap H^1(\mathbf{R}; L^2(D)) \cap L^2(\mathbf{R}; H^2(D)),\\
&\pi \in L^2(\mathbf{R}),
\end{aligned}$$

with the same bounds as before (here C_o denotes the subspace of continuous functions vanishing at infinity).

For our purposes of studying almost periodic solutions it is important to show that one has a global solution, with uniformly bounded gradients, also if the force f is not in $H^1(0,+\infty)$, but just in $H^1_{\text{uloc}}(0,+\infty)$, where

$$\|f\|_{H^1_{\text{uloc}}(0,+\infty)} := \sup_{t\geqslant 0}\Big(\int_t^{t+1}|f(s)|^2+|f'(s)|^2\,ds\Big)^{\frac{1}{2}}.$$

In particular, the following result will be crucial for the rest of the paper.

Proposition 2.10. *Let be given $f\in H^1_{uloc}(0,+\infty)$, then the unique solution of (2.3) exists for all positive time and it satisfies*

$$w\in C_b(0,+\infty;H^1_0(D))\cap H^1_{uloc}(0,+\infty;L^2(D))\cap L^2_{uloc}(0,+\infty;H^2(D)),$$
$$\pi\in L^2_{uloc}(0,+\infty),$$

with the estimate (2.19) in terms of the data.

Proof. Generally this result is straightforward in presence of a standard parabolic problem. Since here we deal essentially with an inverse problem, we give a detailed proof, which is nevertheless obtained adapting the usual techniques typical of almost periodic solutions, see e. g. Amerio and Prouse [1]. Observe that the estimate (2.11) does not give a direct control of $\sup_{0<t<T}\|\nabla w(t)\|$ since the bound depends on $\|f\|_{H^1(0,T)}$ and consequently the H^1-norm of w may become unbounded when $T\to+\infty$, if $f\notin H^1(\mathbf{R})$. We first prove that $\|\nabla w\|\in L^\infty(0,+\infty)$.

First observe that Proposition 2.7 imply that there exists a unique solution

$$w\in C(0,+\infty;H^1_0(D))\cap H^1_{\text{loc}}(0,+\infty;L^2(D))\cap L^2_{\text{loc}}(0,+\infty;H^2(D)),$$
$$\pi\in L^2_{\text{loc}}(0,+\infty),$$

hence the following calculations will be completely justified. By using the function φ from (2.4) as test function we obtain with integration by parts the following identity

$$(w_t,\varphi)-\nu(\Delta w,\varphi)=(w_t,\varphi)-(w,\nu\Delta\varphi)=\pi(\mathbb{1},\varphi).$$

Hence, by recalling the definition of φ and the flux condition we get

$$\frac{\chi_0^2}{\nu}\pi(t)=(w_t(t,x),\varphi(x))+f(t)$$

and consequently taking the square and integrating over the $[\xi,\tau]$ (for all couples $\xi\leqslant\tau$) we obtain that there exists C depending only on D such that

$$\int_\xi^\tau|\pi(s)|^2\,ds\leqslant C\int_\xi^\tau\|w_t(s)\|^2+\nu^2|f(s)|^2\,ds. \tag{2.13}$$

Next, we test the equation satisfied by (w,π) by w and $\nu^{-1}w_t$, and by recalling that $\int_D w_t\,dx=\big(\int_D w\,dx\big)'=f'(t)$ we obtain (for any $\epsilon,\eta>0$) the following differential inequalities, a.e. $t\in(0,+\infty)$

$$\frac{1}{2}\|w\|^2+\nu\|\nabla w\|^2\leqslant\frac{\epsilon}{2}|\pi|^2+\frac{1}{2\epsilon}|f|^2,$$
$$\frac{1}{2}\|\nabla w\|^2+\frac{1}{\nu}\|w_t\|^2\leqslant\frac{\eta}{2\nu}|\pi|^2+\frac{1}{2\eta\nu}|f'|^2.$$

By choosing $\epsilon = (2\nu C)^{-1}$ and $\eta = (2C)^{-1}$, where C is the constant appearing in (2.13) and by integrating over an arbitrary interval $[\xi, \tau]$ we get

$$\begin{aligned}(2.14)\qquad &\frac{1}{2}\|w(\tau)\|^2_{H^1} + \int_\xi^\tau \left(\nu\|\nabla w(s)\|^2 + \frac{1}{\nu}\|w_t(s)\|^2\right) ds \\ &\leqslant \frac{1}{2}\|w(\xi)\|^2_{H^1} + \int_\xi^\tau \nu(2+C)|f(s)|^2 + \frac{C}{\nu}|f'(s)|^2\, ds.\end{aligned}$$

Hence, by using the Poincaré inequality and dropping the non-negative term $\nu^{-1}\|w_t\|^2$ we finally obtain that there exist $c_1, c_2 > 0$ and depending only on D such that

$$(2.15)\qquad \|w(\tau)\|^2_{H^1} + c_1\nu\int_\xi^\tau \|w(s)\|^2_{H^1}\, ds \leqslant \frac{1}{2}\|w(\xi)\|^2_{H^1} + c_2\int_\xi^\tau \nu|f(s)|^2 + \frac{1}{\nu}|f'(s)|^2\, ds.$$

Suppose now that for a given $\bar{t} \in [0, +\infty[$ it holds that

$$(2.16)\qquad \|w(\bar{t})\|^2_{H^1} \leqslant \|w(\bar{t}+1)\|^2_{H^1}$$

and rewrite (2.15) in the interval $[\bar{t}, \bar{t}+1]$ as follows:

$$(2.17)\quad \|w(\bar{t}+1)\|^2_{H^1} - \|w(\bar{t})\|^2_{H^1} + c_1\nu\int_{\bar{t}}^{\bar{t}+1} \|w(s)\|^2_{H^1}\, ds \leqslant c_2\int_{\bar{t}}^{\bar{t}+1} \nu|f(s)|^2 + \frac{1}{\nu}|f'(s)|^2\, ds.$$

Since by hypothesis (2.16) the first term is non-negative we obtain in particular that

$$(2.18)\qquad c_1\nu\int_{\bar{t}}^{\bar{t}+1} \|w(s)\|^2_{H^1}\, ds \leqslant c_2\left(\nu + \frac{1}{\nu}\right)\|f\|^2_{H^1_{uloc}}.$$

Next, by using the estimate (2.18) and the same argument we get, for each couple $\tau \leqslant \xi$, with $\tau, \xi \in [\bar{t}, \bar{t}+1]$, that

$$\begin{aligned}\left|\|w(\tau)\|^2_{H^1} - \|w(\xi)\|^2_{H^1}\right| &\leqslant c_1\nu\int_\xi^\tau \|w^m(s)\|^2_{H^1}\, ds + c_2(\nu + \frac{1}{\nu})\|f\|^2_{H^1_{uloc}} \\ &\leqslant 2c_2(\nu + \frac{1}{\nu})\|f\|^2_{H^1_{uloc}}.\end{aligned}$$

Since $\|w\|^2_{H^1}$ is a continuous function we can fix $\xi \in [\bar{t}, \bar{t}+1]$ such that

$$\|w(\xi)\|^2_{H^1} = \min_{\bar{t}\leqslant s\leqslant \bar{t}+1} \|w(s)\|^2_{H^1}.$$

We have then, using (2.16) and the definition of ξ,

$$\begin{aligned}\|w(\bar{t})\|^2_{H^1} &\leqslant \|w(\bar{t}+1)\|^2_{H^1} \\ &\leqslant \left|\|w(\bar{t}+1)\|^2_{H^1} - \|w(\xi)\|^2_{H^1}\right| + \|w(\xi)\|^2_{H^1} \\ &= \left|\|w(\bar{t}+1)\|^2_{H^1} - \|w(\xi)\|^2_{H^1}\right| + \int_{\bar{t}}^{\bar{t}+1} \|w(\xi)\|^2_{H^1}\, ds \\ &\leqslant \left|\|w(\bar{t}+1)\|^2_{H^1} - \|w(\xi)\|^2_{H^1}\right| + \int_{\bar{t}}^{\bar{t}+1} \|w(s)\|^2_{H^1}\, ds.\end{aligned}$$

Finally, from (2.17) and (2.18) we obtain

$$\|w(\bar{t})\|^2_{H^1} \leqslant C\left(\nu + 1 + \frac{1}{\nu^2}\right)\|f\|^2_{H^1_{uloc}}.$$

Hence, for each $\bar{t} \in (0, +\infty)$ such that (2.16) holds true we have that $\|w(\bar{t})\|^2_{H^1}$ is bounded uniformly. On the contrary, if

$$\|w(\bar{t})\|^2_{H^1} > \|w(\bar{t}+1)\|^2_{H^1},$$

we can repeat the same argument on the "window" $[\overline{t}-1,\overline{t}]$. In this way it is clear that if we are not able to find an interval $[\overline{t}-n-1,\overline{t}-n]$ with $n<\overline{t}-1$ such that

$$\|w(\overline{t}-n-1)\|_{H^1}^2 \leqslant \|w(\overline{t}-n)\|_{H^1}^2$$

we obtain, for some $\overline{n}\in\mathbf{N}$ such that $0\leqslant\overline{t}-n<1$, the inequalities

$$\|w(\overline{t}-n)\|_{H^1}^2 > \cdots > \|w(\overline{t})\|_{H^1}^2,$$

hence that

$$\|w(\overline{t})\|_{H^1} \leqslant \sup_{0\leqslant t\leqslant 1}\|w(t)\|_{H^1}^2.$$

The latter is bounded simply by the local existence result. This finally shows that, since $\|w_0\|_{H^1}$ depends itself on the norm of f in $H^1_{\text{uloc}}(\mathbf{R})$, for all $t\in\mathbf{R}^+$,

$$\|w(t)\|_{H^1}^2 + c_1\nu\int_t^{t+1}\|w(s)\|_{H^1}^2\,ds \leqslant C\left(\nu+1+\frac{1}{\nu^2}\right)\|f\|_{H^1_{\text{uloc}}}^2.$$

Next, by going back to (2.14) we obtain an estimate on $\|w_t\|_{L^2(t,t+1)}^2$, which implies by (2.13) the corresponding estimate for $|\pi|_{L^2(t,t+1)}^2$. Finally, by comparison we get also an estimate for $\|\Delta w\|_{L^2(t,t+1)}^2$ and by collecting all inequalities we finally get

$$\sup_{t\geqslant 0}\left[\nu\|\nabla w(t)\|^2 + \int_t^{t+1}\left(\nu^2\|\Delta w(s)\|^2 + \|w_t(s)\|^2 + |\pi(s)|^2\right)ds\right] \leqslant C\big(\nu^2+1+\frac{1}{\nu}\big)\|f\|_{H^1_{\text{uloc}}(0,+\infty)}^2, \tag{2.19}$$

ending the proof of the Proposition. □

We can finally construct a solution over the whole real line and we have the following result.

Proposition 2.11. *Let be given* $f\in H^1_{uloc}(\mathbf{R})$, *with*

$$\|f\|_{H^1_{uloc}(\mathbf{R})} := \sup_{t\in\mathbf{R}}\left[\int_t^{t+1}|f(s)|^2+|f'(s)|^2\,ds\right]^{1/2},$$

then the unique solution of (2.2) *exists for all times and it satisfies*

$$w\in C_b(\mathbf{R};H^1)\cap H^1_{uloc}(\mathbf{R};L^2)\cap L^2_{uloc}(\mathbf{R};H^2),$$
$$\pi\in L^2_{uloc}(\mathbf{R}),$$

satisfying the estimate (2.1).

Proof. We start by observing that since $f\in H^1_{\text{uloc}}(\mathbf{R})$, then it follows that $f\in C_b(\mathbf{R})$ (again by a.e. identification). It follows directly that $f\in C(\mathbf{R})$, but the control of the maximum of $|f|$ is obtained as follows. We claim that $\sup_{x\in\mathbf{R}}|f(x)|\leqslant 2\|f\|_{H^1_{\text{uloc}}}$. In fact, for each couple of points $x,y\in\mathbf{R}$ such that $|x-y|\leqslant 1$ it follows that

$$|f(x)-f(y)| = \left|\int_x^y f'(s)\,ds\right| \leqslant \|f\|_{H^1_{\text{uloc}}}.$$

Suppose now *per absurdum* that there exists $x_0\in\mathbf{R}$ such that $|f(x_0)|>2\|f\|_{H^1_{\text{uloc}}}$. The previous inequality implies that

$$|f(x)| > \|f\|_{H^1_{\text{uloc}}} \qquad \text{for all } x\in[x_0,x_0+1],$$

hence the contradiction

$$\int_{x_0}^{x_0+1}|f(s)|^2\,ds > \|f\|_{H^1_{\text{uloc}}}^2.$$

This proves that the bound on $|f(x)|$ is true for all $x \in \mathbf{R}$.

The proof of Proposition 2.11 is then obtained by using the previous results from Proposition 2.10 to solve the following family of problems parametrized by $n \in \mathbf{N}$,

$$\begin{cases} \partial_t w_n - \nu\Delta w_n = \pi_n, & x \in D,\ t > -n, \\ w_n(t,x) = 0, & x \in \partial D,\ t > -n, \\ \int_D w_n(t,x)\,dx = f(t) & t > -n, \\ w_n(-n,x) = \frac{\nu\varphi(x)}{\chi_0^2} f(-n), & x \in D,\ t = -n. \end{cases}$$

The same arguments as before imply that

$$\begin{aligned} &w_n \in L^2_{\text{uloc}}(-n,+\infty;H^2(D)) \cap H^1_{\text{uloc}}(-n,+\infty;L^2(D)), \\ &\nabla w_n \in L^\infty(-n,+\infty;L^2(D)), \\ &\pi_n \in L^2_{\text{uloc}}(-n,+\infty), \end{aligned}$$

with bounds independent of $n \in \mathbf{N}$ (the dependence on ν is the same as in (2.5)). By defining the extended functions $\widetilde{w}_n(t,x)$ and $\widetilde{\pi}_n(t,x)$ on the whole real line as

$$\widetilde{w}_n(t,x) := \begin{cases} w_n(t,x) & t \geqslant -n, \\ \frac{\nu\varphi(x)}{\chi_0^2} f(-n) & t \leqslant -n, \end{cases} \quad \text{and} \quad \widetilde{\pi}_n(t) := \begin{cases} \pi_n(t) & t \geqslant -n, \\ \frac{\nu f(-n)}{\chi_0^2} & t < -n, \end{cases}$$

we can extract a sub-sequence (relabeled) as $(\widetilde{w}_n(t,x), \widetilde{\pi}_n(t,x))$ such that

$$\begin{aligned} &\widetilde{w}_n \rightharpoonup w && \text{in } L^2_{\text{uloc}}(\mathbf{R};H^2(D)) \cap H^1_{\text{uloc}}(\mathbf{R};L^2(D)), \\ &\nabla\widetilde{w}_n \overset{*}{\rightharpoonup} \nabla w && \text{in } L^\infty(\mathbf{R};L^2(D)), \\ &\widetilde{\pi}_n \rightharpoonup \pi && \text{in } L^2_{\text{uloc}}(\mathbf{R}). \end{aligned}$$

It is easy to see that (w,π) is a distributional solution to (2.3). Since the problem is linear this is the unique solution. □

We finally prove the result in Stepanov space of almost periodic functions.

Proof of Theorem 2.2. Since the function f is almost periodic, for each sequence $\{r_m\} \subset \mathbf{R}$ we can find a sub-sequence $\{r_{m_k}\} \subset \mathbf{R}$ and a function $\widehat{f}$ such that

$$\sup_{t\in\mathbf{R}} \int_t^{t+1} \left(|f(\tau + r_{m_k}) - \widehat{f}(\tau)|^2 + |f'(\tau + r_{m_k}) - \widehat{f}'(\tau)|^2\right) d\tau \overset{k\to+\infty}{\longrightarrow} 0.$$

By using a standard argument by contradiction (see e. g. Foias and Zaidman [10] and Foias and Prodi [9]) we assume *per absurdum* that w and π are not almost periodic, hence that there exist a sequence $\{h_m\} \subset \mathbf{R}$, a function $\widehat{f} : \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{t\in\mathbf{R}} \int_t^{t+1} \left(|f(\tau + h_m) - \widehat{f}(\tau)|^2 + |f'(\tau + h_m) - \widehat{f}'(\tau)|^2\right) d\tau \overset{m\to+\infty}{\longrightarrow} 0,$$

a constant $\delta_0 > 0$, and three sequences $\{t_p\}$, $\{h_{m_p}\}$, $\{h_{n_p}\}$ such that for all $p \in \mathbf{N}$,

$$\sup_{t\in\mathbf{R}} \int_{t_p}^{t_p+1} \left(\nu^2\|\Delta w(\tau + h_{m_p}) - \Delta w(\tau + h_{n_p})\|^2 + |\pi(\tau + h_{m_p}) - \pi(\tau + h_{n_p})|^2\right) d\tau \geqslant \delta_0.$$

In addition, there are (by eventually relabeling the sequences) two real functions $\widehat{f}_1$ and $\widehat{f}_2$ such that

$$\sup_{t\in\mathbf{R}}\int_t^{t+1}\left(|f(\tau+t_p+h_{m_p})-\widehat{f}_1(\tau)|^2+|f'(\tau+t_p+h_{m_p})-\widehat{f}_1'(\tau)|^2\right)d\tau \overset{p\to+\infty}{\longrightarrow} 0,$$
$$\sup_{t\in\mathbf{R}}\int_t^{t+1}\left(|f(\tau+t_p+h_{n_p})-\widehat{f}_2(\tau)|^2+|f'(\tau+t_p+h_{n_p})-\widehat{f}_2'(\tau)|^2\right)d\tau \overset{p\to+\infty}{\longrightarrow} 0.$$

It is clear that $\widehat{f}=\widehat{f}_1=\widehat{f}_2$. We consider now the problem (2.3) with the two fluxes

$$F_{1p}(t):=f(t+t_p+h_{m_p}),$$
$$F_{2p}(t):=f(t+t_p+h_{n_p}),$$

and the corresponding solutions

$$w_{1p}(t,x):=w(t+t_p+h_{m_p},x),\qquad \pi_{1p}(t,x):=\pi(t+t_p+h_{m_p}),$$
$$w_{2p}(t,x):=w(t+t_p+h_{n_p},x),\qquad \pi_{2p}(t,x):=\pi(t+t_p+h_{n_p}).$$

Observe that $F_{1p}(t)\to f(t)$ and $F_{2p}(t)\to f(t)$ in $H^1_{uloc}(\mathbf{R})$ as $p\to\infty$. Hence, by passing to the limit as $p\to+\infty$ we construct two solutions (w_1,π_1) and (w_2,π_2) corresponding to the same flux f. In particular,

$$w_{1p}\to w_1 \qquad\text{and}\qquad w_{2p}\to w_2,$$

in $C_b(\mathbf{R};H^1_0(D))$, but also in the topologies given in the statement of Proposition 2.11. Hence, we have in particular

$$\delta_0\leqslant\int_{t_{p_s}}^{t_{p_s}+1}\nu^2\|\Delta w(\tau+h_{m_{p_s}})-\Delta w(\tau+h_{n_{p_s}})\|^2+|\pi(\tau+h_{m_{p_s}})-\pi(\tau+h_{n_{p_s}})|^2\,d\tau=$$
$$=\int_0^1\nu^2\|\Delta w_{1p_s}(\tau)-\Delta w_{2p_s}(\tau)\|^2+|\pi_{1p_s}(\tau)-\pi_{2p_s}(\tau)|^2\,d\tau\longrightarrow 0,$$

because the problem is linear and the two solutions w_1 and w_2 corresponding to the same flux, coincide. This proves the almost periodicity of w and the same argument applied to w_t ends the proof of the result. □

2.3. **The full nonlinear problem in $S^2(\mathbf{R})$.** We now finally consider the Navier-Stokes equations and we look for solutions of the following problem

$$\begin{cases}\partial_t u-\nu\Delta u+(u\cdot\nabla)u+\nabla p=0, & x\in O,\ t\in\mathbf{R},\\ \nabla\cdot u=0, & x\in O,\ t\in\mathbf{R},\\ u=0, & x\in\partial O,\ t\in\mathbf{R},\end{cases}$$

with

$$\sup_{t\in\mathbf{R}}\|u(t)-w_i(t)\|_{H^1_0(O_i)}\leqslant c_0,$$

for $i=1,2$. Observe that this constraint implies (see [3, § 7] and [11, VI.I]) that

$$\lim_{z\to+\infty}\|u(t)-w_i(t)\|_{H^{1/2}(D_i))}=0,$$

uniformly in t.

As a preliminary remark we recall that it is well-known (see for example [11, VI], Amick [2]) that due to the particular shape of the unbounded domain O the Poincaré inequality holds true

$$\exists\, c_P>0\qquad \|u\|_{L^2(O)}\leqslant c_P\|\nabla u\|_{L^2(O)}\qquad \forall u\in H^1_0(O) \tag{2.20}$$

and, if the boundary is smooth enough (say of class $C^{1,1}$), then the following estimate for the Stokes operator holds true

$$\|u\|_{H^2(O)} \leqslant c\|P\Delta u\|_{L^2(O)}, \qquad u \in H^2(O) \cap H^1_0(O).$$

We now take advantage of the results on the linear case of the previous section, but first we need to "glue" the basic flows constructed in the two pipes. In this part of the paper we do not claim any originality and we use a classical approach. We denote by $z \in \mathbf{R}^+$ the axial coordinate in both cylinders and we define the "truncated pipes"

$$O_i^r := \{(x, z) \in O_i : \ z < r\} \qquad i = 1, 2,$$

and we also define the truncated domain

$$O^r := O_0 \cup O_1^r \cup O_2^r.$$

We first define a field (which is as smooth as w_i) defined on O and such that is equal to w_i in the sets $O_i \backslash O_i^1$. This extension is obtained by freezing the time variable and by gluing together the functions w_i by cut-off functions $\phi_i(z) \in C^\infty(\mathbf{R}^3)$ depending just on the axial coordinate z, and such that

$$\phi_i(z) = \begin{cases} 1 & x \in O_i \backslash O_i^1, \\ 0 & x \in O_i^{1/2}. \end{cases}$$

Next observe that,

$$V_0 := \sum_{i=1}^{2} \phi_i\, w_i \in H^1_{\text{uloc}}(\mathbf{R}; L^2(O)) \cap L^2_{\text{uloc}}(\mathbf{R}; H^2(O)) \tag{2.21}$$

since the mapping $(w_1, w_2) \mapsto V_0$ is bi-linear and the extension does not involve the time-variable.

Remark 2.12. We also observe that if we are in a different functional framework the same procedure can be applied because the properties of the extension with respect to the time variable are the same of those of the basic flows (w^i, π^i).

This is not exactly the required extension, since the function V_0 is not divergence-free. To this end one has to use the Bogovskiĭ formula to solve the *linear* problem in the bounded domain O^1: Find $\mathcal{B}\, V_0$ such that

$$\begin{cases} \nabla \cdot (\mathcal{B}\, V_0) = -\nabla \cdot V_0, & x \in O^1 \\ \mathcal{B}\, V_0 = 0, & x \in \partial O^1. \end{cases}$$

Since $\nabla \cdot V_0 \in H^1_0(O^1)$ and the compatibility condition $\int_{O^1} \nabla \cdot V_0 = 0$ is satisfied the problem has a solution $\mathcal{B}\, V_0 \in H^2(O^1)$. Denoting again by $\mathcal{B}\, V_0$ the null extension of O^1, we finally set

$$w := V_0 + \mathcal{B} V_0. \tag{2.22}$$

The regularity of the solution of the divergence equation and the previous argument shows that if $w_i \in H^1_{\text{uloc}}(\mathbf{R}; L^2(O_i)) \cap L^2_{\text{uloc}}(\mathbf{R}; H^2(O_i))$, then

$$w = w_i \qquad \text{in } O_i \backslash O_i^1$$

and

$$\|w\|_{H^1(\mathbf{R}; L^2(O)) \cap L^2(\mathbf{R}; H^2(O))} \leqslant c \sum_{i=1}^{2} \|w_i\|_{H^1_{\text{uloc}}(\mathbf{R}; L^2(O_i)) \cap L^2_{\text{uloc}}(\mathbf{R}; H^2(O_i))},$$

for some $c = c(O_0, O_1, O_2)$. We have finally the following result.

Lemma 2.13. *Let (w_i, π_i) be solutions of the basic flow with the regularity of Theorem 2.2. Then the extended function w is Stepanov almost periodic and*

$$\|w\|_{L^\infty(\mathbf{R};H^1_0(O)\cap H^1_{uloc}(\mathbf{R};L^2(O))\cap L^2_{uloc}(\mathbf{R};H^2(O))}$$
$$\leqslant c\sum_{i=1}^{2}\|w_i\|_{L^\infty(\mathbf{R};H^1_0(O)\cap H^1_{uloc}(\mathbf{R};L^2(O_i))\cap L^2_{uloc}(\mathbf{R};H^2(O_i))}.$$

With this result, we look now for solutions in the form

$$u = U + w, \qquad p = P + \pi.$$

We have to find $U \in L^\infty_{uloc}(\mathbf{R}; L^2(O)) \cap L^2_{uloc}(\mathbf{R}; H^1_0(O))$ solving

$$\begin{cases} \partial_t U - \nu\Delta U + (U\cdot\nabla)\,U + (U\cdot\nabla)\,w + (w\cdot\nabla)\,U + \nabla P = F, \\ \nabla\cdot U = 0, \\ U = 0, & \text{on } \partial O, \\ U(0,x) = -w(0,x). \end{cases} \tag{2.23}$$

where

$$F(t,x,z) := -\big(\partial_t w(t,x) - \nu\Delta w(t,x) + (w(t,x)\cdot\nabla)\,w(t,x)\big) + \sum_{i=1}^{2}\frac{\partial}{\partial z}(z\,\phi_i(z)\pi_i(t)). \tag{2.24}$$

From the results of the previous section we can infer that $F \in S^2(\mathbf{R}; L^2(O))$, hence we can now use standard results to show existence of an almost periodic solution U. The support of F is contained in the bounded subset $\mathcal{O}^1$, hence we can use the standard variational techniques to show existence of a unique solution, provided that the viscosity is large enough. In particular the only property that we need to check is that $(w(t,x)\cdot\nabla)\,w(t,x)$ is almost periodic. This follows since if we take the $L^2(O)$-norm of F, this is equal to the $L^2(O_1)$-norm. All other terms in the summation are clearly relatively compact in $L^2_{uloc}(\mathbf{R})$, the nonlinear one can be estimated as

$$\int_t^{t+1}\left|\int_{O_1} w(t,x)\cdot\nabla)\,w(t,x)\right|^2 \leqslant \sup_t\|\nabla w(t)\|^2_{L^2(O_1)}\int_t^{t+1}\|\Delta w(s)\|^2_{L^2(O_1)}\,ds.$$

This proves that $F \in \mathcal{S}^2(\mathbf{R}; L^2(O))$ and, by using this expression and from the estimate (2.1) on (w^i, π^i), we obtain that

$$\|F\|^2_{L^2_{uloc}} \leqslant C\Big(\nu^2 + 1 + \frac{1}{\nu^4}\Big)\|f\|^2_{H^1_{uloc}}.$$

In particular, as in the previous section, we need to show that there exists a solution $U \in L^2_{uloc}(0,+\infty; H^2(O))$, and this will follow by using that the right-hand side is in $L^2_{uloc}(0,+\infty; L^2(O))$. The existence of a local solution $L^2_{loc}(0,+\infty; H^1_0(O))$ follows by standard arguments. In particular, one has to perform a truncation in the space variables (which is possible since F is of compact support, see again [3]) and the usual a-priori estimates. In addition to these rather standard results, we now prove that the solution is strong (provided that the viscosity is large enough, which is nevertheless needed also for weak solutions) and that the solution is uniformly bounded in $H^1(O)$ and $L^2_{uloc}(\mathbf{R}; H^2(O))$. We present just the a priori estimates, which can be justified by the usual Galerkin method and truncation of the domain.

Proof of Theorem 2.4. We multiply (2.23) by $-P\Delta U$ and we carefully treat the various terms. First note that

$$\begin{aligned}\int_{O_i}(P\Delta U)\cdot(U\cdot\nabla)\,U &= \sum_k\int_k^{k+1}\int_{D_i}(U\cdot\nabla)\,U\,P\Delta U\,dx\,dz\\ &\leqslant C\|U\|_{L^4((k,k+1)\times D_i)}\|\nabla U\|_{L^4((k,k+1)\times D_i)}\|\Delta U\|_{L^2((k,k+1)\times D_i)}\\ &\leqslant C\|U\|_{H^1(O_i)}\|\nabla U\|_{H^1(O_i)}\|\Delta U\|_{L^2(O_i)},\end{aligned}$$

since the constant of the Sobolev embedding $H^1\subset L^4$ are uniformly bounded in each of the strips $(k,k+1)\times D_i$, see [12, Lemma 2.1]. Moreover the standard regularity theory for the Stokes operator shows also that

$$\int_{O_i}(P\Delta U)\cdot(U\cdot\nabla)\,U\leqslant C\|U\|_{H^1(O_i)}\|P\Delta U\|^2_{L^2(O_i)}.$$

The term $\int_{O_i}(U\cdot\nabla)\,w$ is then estimated as follows, by using the same splitting into slices of width 1 in the z-direction and the Sobolev embedding $H^2\subset L^\infty$,

$$\int_{O_i}(U\cdot\nabla)\,w\,P\Delta U\leqslant C\|w\|_{L^2(O_i)}\|P\Delta U\|^2_{L^2(O_i)},$$

The other term is estimated in the same way, and the contribution from the domain O_0 is handled with standard tools. Hence adding together the three terms, we finally we arrive at the differential inequality

$$\frac{d}{dt}\|\nabla U\|^2+\big(\nu-C_1(\|\nabla U\|+\|\nabla w\|)\big)\|P\Delta U\|^2\leqslant C_2\frac{\|F\|^2}{\nu}$$

We fix the viscosity large enough, so that

$$\nu-C_1(\|\nabla U(0)\|-\|\nabla w(0)\|)=\nu-2C_1\|\nabla w(0)\|>0.$$

and this is possible since $\|\nabla w(0)\|=\frac{\|\nabla\varphi\|}{\chi_0^2}|f(0)|\leqslant\frac{C}{\nu}\|f\|_{H^1_{\text{uloc}}}$. Since $\|F(t)\|\in L^2_{\text{uloc}}(\mathbf{R})$ this is enough to show existence of a local unique solution, in a time interval $[0,\delta[$ for some positive δ.

The next step is to show that, under the same assumptions, the solution is global. This is rather standard, since if we have the uniform bound on $\|\nabla w\|^2$ coming from Proposition 2.11,

$$\sup_{t\in\mathbf{R}}\|\nabla w(t)\|^2\leqslant C\big(\nu+1+\frac{1}{\nu^2}\big)\|f\|^2_{H^1_{\text{uloc}}(\mathbf{R})},$$

we can fix $\nu_0=\nu_0(f,O)$ large enough such that

$$\text{if }\ \nu>\nu_0\qquad\text{then}\qquad \nu-C_1\|\nabla w(t)\|\geqslant\frac{\nu}{2},\qquad t\in\mathbf{R}.$$

For such ν we are reduced to solve (possibly redefining the constants)

$$\frac{d}{dt}\|\nabla U\|^2+c_1\Big(\frac{\nu}{2}-\|\nabla U\|\Big)\|\nabla U\|^2\leqslant\frac{C_2}{\nu}\|F\|^2$$

Hence, if we define $Z(t)$ as the solution of the following Cauchy problem

$$\begin{cases}Z'+c_1\big(\frac{\nu}{2}-\sqrt{Z}\big)Z=\frac{C}{\nu}\|F\|^2\\ Z(0)=\|\nabla U(0)\|^2\end{cases}\tag{2.25}$$

we have that $\|\nabla U(t)\|^2\leqslant Z(t)$. We employ now a fixed point argument in the space of function which are continuous and bounded over $[0,+\infty]$ to show that Z is well defined and bounded in the same interval. Let be given $\overline{z}\in C_b(0,+\infty)$ such that

$$0\leqslant\overline{z}\leqslant\frac{\nu^2}{4},$$

Solve now the problem

$$\begin{cases} z' + c_1\left(\frac{\nu}{2} - \sqrt{\overline{z}}\right)z = \frac{c_2}{\nu}\|F\|^2. \\ z(0) = Z(0) \end{cases}$$

Since $\frac{\nu}{2} - C_1\sqrt{\overline{Z}} \geqslant \frac{\nu}{4}$ and both $z(0)$ and $\frac{c_2}{\nu}\|F\|^2$ are non-negative, it follows that $z(t) \geqslant 0$. Hence z satisfies

$$z' + \frac{c_1\nu}{4}z \leqslant \frac{c_2}{\nu}\|F\|^2. \tag{2.26}$$

The same argument employed in the proof of Proposition 2.10 shows that

$$0 \leqslant z(t) \leqslant z(0) + \frac{5c_2}{\nu}\|F\|^2_{L^2_{\mathrm{uloc}}} \leqslant C\Big(\nu + 1 + \frac{1}{\nu^5}\Big)\|f\|^2_{H^1_{\mathrm{uloc}}}.$$

Then, if ν is large enough such that

$$C\Big(\nu + 1 + \frac{1}{\nu^5}\Big)\|f\|^2_{H^1_{\mathrm{uloc}}} \leqslant \frac{\nu^2}{4}.$$

Consider now the map $\Phi : \overline{z} \mapsto z$ and given $\tau > 0$ define

$$K_\tau := \left\{\overline{z} \in C^0([0,\tau]) :\ 0 \leqslant \overline{z} \leqslant \frac{\nu^2}{4}\right\}.$$

Clearly we have $\Phi(K_\tau) \subseteq K_\tau$ and the map is relatively compact by Ascoli-Arzelà theorem since z is Lipschitz continuous as solution of (2.26). Hence Φ as a unique fixed point which is a solution to (2.25). Since $\tau > 0$ is arbitrary this proves that Z exists on the whole interval $[0, +\infty)$. A standard comparison argument shows that $\|\nabla U(t)\|^2 \leqslant Z(t) \leqslant \frac{\nu^2}{4}$ for all positive t.

This estimate implies, by using standard argument well established for the Navier-Stokes equations, that there exists a solution

$$U \in C_b(0, +\infty; H^1_0(O)) \cap L^2_{\mathrm{uloc}}(0, +\infty; H^2(O)) \cap H^1_{\mathrm{uloc}}(0, +\infty; L^2(O)),$$

and due to the regularity proved the solution is unique.

Next, one can construct a global solution, by solving the following family of problems in $[-n, \infty) \times O$,

$$\begin{cases} \partial_t U_n - \nu\Delta U_n + (U_n\cdot\nabla)U_n + (U_n\cdot\nabla)w + (w\cdot\nabla)\,U_n + \nabla P_n = F, \\ \nabla\cdot U_n = 0, \\ U_n = 0, \qquad \text{on } \partial O, \\ U_n(-n,x) = -w(-n,x) \qquad \text{in } O. \end{cases}$$

Again by prolongation we define a velocity on the whole real line by

$$\tilde{U}_n(t,x) = \begin{cases} U_n(t,x) & t \geqslant -n, \\ U(-n,x) & t < -n. \end{cases}$$

We can show that $\tilde{U}_n(t,x)$ converges to a solution U such that

$$U \in C_b(\mathbf{R}; H^1_0(O)) \cap L^2_{\mathrm{uloc}}(\mathbf{R}; H^2(O)) \cap H^1_{\mathrm{uloc}}(\mathbf{R}; L^2(O)).$$

To end the proof we need to show that if the external force is almost periodic, then U is almost periodic too. For this result we need a result of "asymptotic equivalence," which is obtained as follows. Let us suppose the we have two solutions of (2.23) on the interval $[t_0, +\infty)$ corresponding to different initial data, but to the same external force. Then the difference $\widetilde{U} := U_1 - U_2$ satisfies

$$\begin{cases} \partial_t \widetilde{U} - \nu\Delta\widetilde{U} + (U_1 \cdot \nabla)\, U_1 - (U_2 \cdot \nabla)\, U_2 + (\widetilde{U} \cdot \nabla)\, w + (w \cdot \nabla)\, \widetilde{U} + \nabla\widetilde{P} = 0, \\ \nabla \cdot \widetilde{U} = 0, \\ \widetilde{U} = 0, \qquad \text{on } \partial O, \\ \widetilde{U}(t_0, x) = \widetilde{U}_0. \end{cases}$$

By multiplication by $\widetilde{U}$ and usual integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{U}\|^2 + \nu\|\nabla\widetilde{U}\|^2 \leqslant \int_O (\widetilde{U} \cdot \nabla)\,(w + U_2)\widetilde{U}\, dx.$$

By using the same techniques employed before to handle unbounded domains we show that

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{U}\|^2 + \big(\nu - C(\|\nabla w\| + \|\nabla U_2\|)\big)\|\nabla\widetilde{U}\|^2 \leqslant 0.$$

The uniform bounds on w and U_2 in $H^1_0(O)$ imply that for large enough viscosity one has

$$\frac{d}{dt}\|\widetilde{U}\|^2 + \nu C_P\|\widetilde{U}\|^2 \leqslant 0.$$

This finally shows that

$$\|\widetilde{U}\|^2 \leqslant \|\widetilde{U}(t_0)\|^2\, e^{-\nu C_P (t - t_0)}, \qquad t \geqslant t_0.$$

The same argument employed in the proof of Theorem 2.2 can be employed and, for each sequence $\{r_m\} \subset \mathbf{R}$ we can find a sub-sequence $\{r_{m_k}\} \subset \mathbf{R}$ and a function $\tilde{F}$ such that

$$\sup_{t\in\mathbf{R}} \int_t^{t+1} \|F(\tau + r_{m_k}) - \tilde{F}(\tau)\|^2\, d\tau \overset{k\to+\infty}{\longrightarrow} 0.$$

Assume by contradiction that U is not almost periodic, hence that there exist a sequence $\{h_m\} \subset \mathbf{R}$ and a function $\tilde{f} : \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{t\in\mathbf{R}} \int_t^{t+1} \|F(\tau + h_m) - \tilde{F}(\tau)\|^2\, d\tau \overset{m\to+\infty}{\longrightarrow} 0,$$

and a constant $\delta_0 > 0$ and three sequences $\{t_p\}, \{h_{m_p}\}$ and $\{h_{n_p}\}$ such that for all $p \in \mathbf{N}$,

$$\sup_{t\in\mathbf{R}} \int_{t_p}^{t_p+1} \|U(\tau + h_{m_p}) - U(\tau + h_{n_p})\|^2\, d\tau \geqslant \delta_0.$$

There exists two real functions $\tilde{F}_1$ and $\tilde{F}_2$ such that

$$\sup_{t\in\mathbf{R}} \int_t^{t+1} \|F(\tau + t_p + h_{m_p}) - \tilde{F}_1(\tau)\|^2\, d\tau \overset{k\to+\infty}{\longrightarrow} 0.$$

$$\sup_{t\in\mathbf{R}} \int_t^{t+1} \|F(\tau + t_p + h_{n_p}) - \tilde{F}_2(\tau)\|^2\, d\tau \overset{k\to+\infty}{\longrightarrow} 0.$$

It follows in a standard way that $\tilde{F} = \tilde{F}_1 = \tilde{F}_2$ and we consider now the problem (2.23) with the two forces

$$F_{1p}(t, x) := F(t + t_p + h_{m_p}, x) \quad \text{and} \quad F_{2p}(t, x) := F(t + t_p + h_{n_p}, x),$$

and the corresponding solutions

$$U_{1p}(t, x) := U(t + t_p + h_{m_p}, x) \quad \text{and} \quad U_{2p}(t, x) := U(t + t_p + h_{n_p}, x).$$

By passing to the limit as $p \to +\infty$ we construct two solutions U_1 and U_2 corresponding to the same force $\tilde{F}$. In particular

$$w_{1p} \to w_1 \text{ in } C_{loc}(\mathbf{R}; L^2), \qquad w_{2p} \to w_2 \text{ in } C_{loc}(\mathbf{R}; L^2).$$

In addition we have

$$\delta_0 \leqslant \int_{t_{p_s}}^{t_{p_s}+1} \|U(\tau + h_{m_{p_s}}) - U(\tau + h_{n_{p_s}})\|^2 \, d\tau = \\ = \int_0^1 \|U_{1p_s}(\tau) - U_{2p_s}(\tau)\|^2 \, d\tau \longrightarrow \int_0^1 \|U_1(\tau) - U_2(\tau)\|^2 \, d\tau.$$

On the other hand the asymptotic equivalence implies that $U_1 = U_2$, since for all $t \in \mathbf{R}$,

$$\|U_1(t) - U_2(t)\|^2 \leqslant \|U_1(t_0) - U_2(t_0)\|^2 \, e^{-\nu C_P (t-t_0)} \qquad t \geqslant t_0,$$

and letting $t_0 \to -\infty$ we obtain a contradiction.

To conclude the proof we show that the function $U \in \mathcal{S}^2(\mathbf{R}; H^s(O))$ for all $0 \leqslant s < 2$. In fact, take a sequence $\{r_n\}$ we can find a sub-sequence $\{r_{m_k}\} \subset \mathbf{R}$ and a function $\tilde{U}$ such that the sequence $\{U(\tau + r_{m_k})\}$ satisfies

$$\sup_{t \in \mathbf{R}} \int_t^{t+1} \|U(\tau + r_{m_k}) - \tilde{U}(\tau)\|^2 \, d\tau \overset{k \to +\infty}{\longrightarrow} 0.$$

Hence the sequence $\{U(\tau + r_{m_k})\}$ is a Cauchy sequence in $L^2_{uloc}(\mathbf{R}; L^2(O))$, that is for every $\epsilon > 0$ there is $N \in \mathbf{N}$ such that

$$\sup_{t \in \mathbf{R}} \int_t^{t+1} \|U(\tau + r_{m_p}) - U(\tau + r_{m_s})\|^2 \, d\tau \leqslant \epsilon \qquad p, s \geqslant N.$$

Since $U \in H^1_{uloc}(\mathbf{R}; L^2(O)) \cap L^2_{uloc}(\mathbf{R}; H^2(O))$, by classical interpolation it follows that

$$U_* \in C(\mathbf{R}; H^\sigma(0, 1; H^{2-2\sigma}(O))), \qquad 0 \leqslant \sigma \leqslant 1.$$

Hence we obtain that

$$\sup_{t \in \mathbf{R}} \int_t^{t+1} \|U(\tau + r_{m_p}) - U(\tau + r_{m_s})\|^2_{H^s(O)} \, d\tau \leqslant \\ \leqslant \sup_{t \in \mathbf{R}} \int_t^{t+1} \|U(\tau + r_{m_p}) - U(\tau + r_{m_s})\|^{2-s}_{L^2(O)} \|U(\tau + r_{m_p}) - U(\tau + r_{m_s})\|^s_{H^2(O)} \, d\tau,$$

and by using Hölder inequality,

$$\int_t^{t+1} \|U(\tau + r_{m_p}) - U(\tau + r_{m_s})\|^2_{H^s(O)} \, d\tau \leqslant \left[\int_t^{t+1} \|U(\tau + r_{m_p}) - U(\tau + r_{m_s})\|^2_{L^2(O)} \right]^{\frac{2-s}{2}} \cdot \\ \left[\int_t^{t+1} \|U(\tau + r_{m_p}) - U(\tau + r_{m_s})\|^2_{H^2(O)} \, d\tau \right]^{\frac{s}{2}}.$$

Since

$$\sup_{t \in \mathbf{R}} \int_t^{t+1} \|U(\tau + r_{m_p})\|^2_{H^2(O)} + \|U(\tau + r_{m_s})\|^2_{H^2(O)} \, d\tau \leqslant C,$$

it follows that the sequence $\{U(\tau + r_{m_k})\}$ is a Cauchy sequence in $L^2_{uloc}(\mathbf{R}; H^2(O))$ as well, ending the proof. □

Remark 2.14. The result with $f \in H^1(\mathbf{R})$ follows in the same, even simpler, way.

3. Leray's problem in the framework of Besicovitch a. p. functions

In this section we discuss the same problem in a more general setting, and first we recall some definitions on almost periodic solutions.

3.1. Generalities on almost periodic functions.

In the literature there are different definitions of *almost periodic* functions and we need now to explain the precise setting we are using. We refer mainly to [5, ch. I] for further details and references. Let $\mathrm{Trig}(\mathbf{R})$ be the set of all trigonometric polynomials, that is, $u \in \mathrm{Trig}(\mathbf{R})$ if there exist $n \in \mathbf{N}$, $\xi_1, \ldots, \xi_n \in \mathbf{R}$ and $u_1, \ldots, u_n \in \mathbf{C}$ such that

$$u(x) = \sum_{k=1}^{n} u_k \, e^{i\,\xi_k\,x}, \qquad x \in \mathbf{R}.$$

Next, a set $A \subseteq \mathbf{R}$ is *relatively dense* if there exists $L > 0$ such that each interval of length L contains an element of the set A.

Definition 3.1 (Bohr). A *uniformly almost periodic function* ($\mathsf{UAP}(\mathbf{R})$) is a continuous function $f : \mathbf{R} \to \mathbf{R}$ such that there is a *relatively-dense* set of ε-almost-periods. That is for all $\varepsilon > 0$, there exist translations $T_\varepsilon > 0$ of the variable t such that

$$|f(t + T_\varepsilon) - f(t)| \leqslant \varepsilon.$$

It is easy to see that all trigonometric polynomials are almost periodic according to the previous definition. Let $\mathsf{UAP}(\mathbf{R})$ be the set of all uniformly almost periodic functions. Then $\mathsf{UAP}(\mathbf{R})$ coincides with the closure of $\mathrm{Trig}(\mathbf{R})$ with respect to the *sup-norm* $\|\cdot\|_{L^\infty}$. Alternatively, as recalled in Section 2, a function $f \in \mathsf{UAP}(\mathbf{R})$ if the set $\{f(\tau + \cdot) : \tau \in \mathbf{R}\}$ of translates of f is relatively compact in $C(\mathbf{R})$. A more general notion of almost periodicity was introduced by Stepanov in 1925. To this end for $p \geqslant 1$ and $r > 0$, define the norm

$$\|f\|_{\mathcal{S}^p,r} := \sup_{t\in\mathbf{R}} \Big(\frac{1}{r} \int_t^{t+r} |f(s)|^p \, ds \Big)^{\frac{1}{p}}.$$

Then, the space $\mathcal{S}^p(\mathbf{R})$ is the closure of $\mathrm{Trig}(\mathbf{R})$ with respect to the norm $\|\cdot\|_{\mathcal{S}^p,r}$ above. Notice also that while the norm depends on r, the topology is independent of the value of r, hence we re-obtain the definition used in Section 2 (Cf. [5]).

The definition was later extended by Weyl in 1927 by considering the closure $\mathcal{W}^p(\mathbf{R})$ of trigonometric polynomials with respect to the semi-norm

$$\|f\|_{\mathcal{W}^p} := \lim_{r\to\infty} \|f\|_{\mathcal{S}^p,r}.$$

Finally, Besicovitch [5] defined the space $\mathcal{B}^p(\mathbf{R})$ as the closure of $\mathrm{Trig}(\mathbf{R})$ with respect to the *semi-norm*

$$\|f\|_{\mathcal{B}^p} := \limsup_{R\to+\infty} \Big(\frac{1}{2R} \int_{-R}^{R} |f(s)|^p \, ds \Big)^{\frac{1}{p}}.$$

Notice that one can have $\|f\|_{\mathcal{B}^p} = 0$ even though $f \not\equiv 0$. For example this happens if f is in $L^q(\mathbf{R})$ with $q > p$ or $f \in L^\infty(\mathbf{R})$ and $|f(x)| \to 0$ as $|x| \to \infty$. One has the following strict inclusions

$$\mathsf{UAP}(\mathbf{R}) \subset \mathcal{S}^p(\mathbf{R}) \subset \mathcal{W}^p(\mathbf{R}) \subset \mathcal{B}^p(\mathbf{R}), \quad \text{for any } p \in]1, +\infty[,$$

(with obvious inclusions with different values of p). It turns out that the spaces $\mathcal{B}^p(\mathbf{R})$ of Besicovitch almost periodic functions are among the "largest possible" compatible with the treatment of partial differential equations as we shall see in Proposition 3.2. Let us focus on the case $p = 2$, since $\mathcal{B}^2(\mathbf{R})$ has an Hilbert structure. Given $f \in L^1_{\mathrm{loc}}(\mathbf{R})$, define

$$\overline{\mathcal{M}}(f) := \limsup_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} f(t) \, dt.$$

The *mean operator* $\mathcal{M}(f)$ is defined as the above quantity when the limit exists. Given $f \in \mathcal{B}^1(\mathbf{R})$, the *(generalized) Fourier coefficients* of f are defined as follows

$$a_\lambda(f) := \mathcal{M}(f(t)\, e^{i\lambda t})$$

and the set

$$\sigma(f) := \{\lambda \in \mathbf{R} : a_\lambda(f) \neq 0\},$$

the *spectrum* of f, is at most countable.

Define the equivalence relation $\stackrel{ap}{\sim}$ as $f \stackrel{ap}{\sim} g$ if $a_\lambda(f-g) = 0$ for all $\lambda \in \mathbf{R}$. As stated above, $|\cdot|_{\mathcal{B}^p}$ is not a norm, as it can be zero on nonzero functions. It turns out that the quotient space $\mathcal{B}^p/\stackrel{ap}{\sim}$ is a Banach space (see [5]). If $f, g \in \mathcal{B}^2(\mathbf{R})$ then the mean $\mathcal{M}(fg)$ is well-defined and $\mathcal{M}(f\, g) = \mathcal{M}(f_1\, g_1)$ if $f \stackrel{ap}{\sim} f_1$ and $g \stackrel{ap}{\sim} g_1$. The space $\mathcal{B}^2/\stackrel{ap}{\sim}$ is an Hilbert space when endowed with the scalar product $\langle f, g\rangle_{\mathcal{B}^2} = \mathcal{M}(fg)$ and we have the fundamental result due to Besicovitch.

Proposition 3.2. *The exponential functions $t \mapsto e^{i\lambda t}$ are an orthonormal Hilbert basis for $\mathcal{B}^2(\mathbf{R})$. In different words, any $f \in \mathcal{B}^2(\mathbf{R})$ can be represented by its generalized Fourier series*

$$f(t) \stackrel{ap}{\sim} \sum_{\lambda\in\sigma(f)} a_\lambda(f)\, e^{i\lambda t}, \tag{3.1}$$

and $\sum_\lambda |a_\lambda(f)|^2 < \infty$.

Conversely, if one has a generalized series as above with square summable coefficients (as a generalized series), then there is a function $f \in \mathcal{B}^2(\mathbf{R})$ having the series as its own generalized Fourier series.

Hence, we can identify a function by means of its generalized series as follows

$$\mathcal{B}^2(\mathbf{R}) := \Big\{ f \in L^2_{loc}(\mathbf{R}) :\ \|f\|^2_{\mathcal{B}^2} := \sum_{\lambda\in\sigma(f)} |a_\lambda(f)|^2 < +\infty \Big\} \tag{3.2}$$

and the identification $f(t) \stackrel{ap}{\sim} \sum_{\lambda\in\sigma(f)} a_\lambda(f)\, e^{i\lambda t}$ holds in the sense of convergence in $\mathcal{B}^2(\mathbf{R})$.

We turn to the pipe problem (2.2) in the *almost periodic* case. Since the problem is linear, it is reasonable to find a solution in terms of Fourier transform (or series). It is clear that, once the problem is solved in the Fourier space, we are given with coefficients $a(\xi) \in \mathbf{C}$ for $\xi \in \mathbf{R}$ and we are left with the problem of reconstructing the solution by inverse Fourier transforming. Since the convergence for classical Fourier series is robust in L^2 for ℓ^2 coefficients, likewise in the context of almost periodic functions we consider the (correct) space $\mathcal{B}^2$, for which the analogous of the Riesz-Fischer theorems holds true.

In the following we shall also need spaces of the type Sobolev-Besicovitch, which are defined in the following way.

Definition 3.3. Given a real $s > 0$, a function $f \in \mathcal{B}^2(\mathbf{R})$ belongs to $\mathcal{B}^{s,2}(\mathbf{R})$ if

$$\|f\|^2_{\mathcal{B}^{s,2}} := \sum_{\lambda\in\sigma(f)} (1+|\lambda|^2)^s |a_\lambda(f)|^2 < +\infty.$$

In particular, if $f \in \mathcal{B}^{1,2}(\mathbf{R})$, then the Fourier series for the (formal) derivative f' of f is convergent and defines an element of $\mathcal{B}^2(\mathbf{R})$.

3.2. **Construction of the "basic flow" in the Besicovitch setting.** In this section we solve problem (2.2) in the unknowns (w, π), with the Besicovitch meaning. As a by-product of the method we obtain also a new proof of the existence of the basic flow in the periodic case.

Theorem 3.4. *Given* $f \in \mathcal{B}^{1,2}(\mathbf{R})$ *there are* $w \in \mathcal{B}^2(\mathbf{R}; H^2(D))$*, with* $\partial_t w \in \mathcal{B}^2(\mathbf{R}; L^2(D))$*, and* $\pi \in \mathcal{B}^2(\mathbf{R})$ *such that*

$$\partial_t w - \nu \Delta w \overset{ap}{\sim} \pi \qquad \text{and} \qquad \int_D w(t,x)\,dx \overset{ap}{\sim} f(t).$$

Moreover, w *and* π *are unique up to identification as almost periodic functions. Finally, there exists* $c > 0$ *such that*

$$\|\Delta w\|_{\mathcal{B}^2(L^2)} \leqslant c\|f\|_{\mathcal{B}^2} + \frac{c}{\nu}\|f'\|_{\mathcal{B}^2},$$
$$\|\pi\|_{\mathcal{B}^2} + \|\partial_t w\|_{\mathcal{B}^2(L^2(D))} \leqslant c\nu\|f\|_{\mathcal{B}^2} + c\|f'\|_{\mathcal{B}^2}.$$

In order to prove the theorem, we restate the problem by spectral analysis in terms of Fourier transform with respect to the time variable t (with conjugate variable ξ). Once Fourier transformed, problem (2.2) reads as follows: Find $(\widehat{w}, \widehat{\pi})$ such that

$$\begin{cases} i\,\xi\,\widehat{w}(\xi,x) - \nu\Delta\widehat{w}(\xi,x) = \widehat{\pi}(\xi), & x \in D,\ \xi \in \mathbf{R}, \\ \widehat{w}(\xi,x) = 0, & x \in \partial D,\ \xi \in \mathbf{R}, \\ \int_D \widehat{w}(\xi,x)\,dx = \widehat{f}(\xi) & \xi \in \mathbf{R}. \end{cases} \tag{3.3}$$

Clearly, the same result follows by a decomposition in Fourier series. The first equation yields $\widehat{w}(\xi,x) = \widehat{\pi}(\xi) W_\xi(x)$, where

$$W_\xi(x) := \big((i\,\xi\,\mathrm{Id} - \nu\Delta)^{-1}\mathbb{1}\big)(x)$$

is defined to be the solution to the linear, stationary, and complex system

$$\begin{cases} i\,\xi\,W_\xi - \nu\Delta W_\xi = 1, & x \in D, \\ W_\xi(x) = 0, & x \in \partial D, \end{cases} \tag{3.4}$$

parametrized by $\xi \in \mathbf{R}$. Set

$$a_\xi := \int_D W_\xi(x)\,dx,$$

then by Fourier transforming the flux condition in (2.2) we get $\widehat{f}(\xi) = a_\xi \widehat{\pi}(\xi)$ and in conclusion the solution to (2.2) (or, more precisely, to (3.3)), is given by

$$\widehat{\pi}(\xi) = \frac{1}{a_\xi}\widehat{f}(\xi) \qquad \text{and} \qquad \widehat{w}(\xi,x) = \frac{1}{a_\xi}\widehat{f}(\xi) W_\xi(x). \tag{3.5}$$

The problem reduces to analyse the behavior of the two terms a_ξ and W_ξ with respect to $\xi \in \mathbf{R}$. The main properties are summarised in the following lemma.

Lemma 3.5. *For every* $\xi \in \mathbf{R}$ *it holds*

1. $a_\xi = \nu \int_D |\nabla W_\xi(x)|^2\,dx - i\xi \int_D |W_\xi(x)|^2\,dx$,
2. $\nu \int_D \Delta W_\xi(x)\,dx = i\xi a_\xi - 1$,
3. $\nu^2 \int_D |\Delta W_\xi(x)|^2\,dx + \xi^2 \int_D |W_\xi(x)|^2\,dx = 1$.

Proof. To prove the first property, take the complex conjugate of the equation satisfied by W_ξ, multiply by W_ξ and integrate by parts, obtaining

$$\begin{aligned}\int_D W_\xi(x)\,dx &= \int_D (-i\xi|W_\xi|^2 - \nu W_\xi \Delta\overline{W_\xi})\,dx \\ &= \nu \int_D |\nabla W_\xi(x)|^2\,dx - i\xi \int_D |W_\xi(x)|^2\,dx.\end{aligned}$$

For the second property, just integrate the equation for W_ξ on D. In order to prove the third, take again the complex conjugate of the equation for W_ξ, but this time multiply by ΔW_ξ. Next, integrate by parts and use the first two properties to get

$$\begin{aligned} i\xi a_\xi - 1 &= \nu \int_D \Delta W_\xi \, dx = -\nu^2 \int_D |\Delta W_\xi|^2 \, dx + i\nu\xi \int_D |\nabla W_\xi|^2 \, dx \\ &= -\nu^2 \int_D |\Delta W_\xi|^2 \, dx + i\xi a_\xi - \xi^2 \int_D |W_\xi|^2 \, dx, \end{aligned}$$

which proves the equality. □

Next, we need to understand the growth/decay of a_ξ and $W_\xi(x)$ with respect to ξ, in order to show that the formal expression (3.5) defines a solution (in a suitable sense).

Lemma 3.6. *The map* $\xi \mapsto W_\xi$ *is continuous on* $\mathbf{R}$ *with values in* $H^2(D) \cap H^1_0(D)$. *Moreover, as* $|\xi| \to +\infty$ *we have* $W_\xi \to 0$ *in* $H^2(D)$ *and also*

$$\lim_{|\xi|\to+\infty} \xi \|\nabla W_\xi\|^2_{L^2} = 0, \qquad \textit{and} \qquad \lim_{|\xi|\to+\infty} \xi^2 \|W_\xi\|^2_{L^2} = 1. \tag{3.6}$$

Finally, the map $\xi \to a_\xi$ *is continuous on* $\mathbf{R}$ *with values in* $\mathbf{R}$ *and*

$$\lim_{|\xi|\to+\infty} a_\xi = -\frac{1}{\xi}.$$

Proof. Fix $\xi, \xi_0 \in \mathbf{R}$, with $\xi \neq 0$, and set $V := W_{\xi_0} - W_\xi$. By symmetry we can assume that $\xi > 0$ and start with the case $\xi_0 > 0$. The new function V solves $i\xi V - \nu\Delta V = i(\xi - \xi_0) W_{\xi_0}$. Multiply by $\Delta\overline{V}$ and integrate by parts to get

$$i\xi \|\nabla V\|^2_{L^2} + \nu \|\Delta V\|^2_{L^2} = i(\xi - \xi_0) \int_D \nabla W_{\xi_0} \cdot \nabla\overline{V} \, dx.$$

The imaginary part of the above formula yields

$$\xi \|\nabla V\|^2_{L^2} = (\xi - \xi_0) \Re\Big(\int_D \nabla W_{\xi_0} \cdot \nabla\overline{V} \, dx\Big) \leqslant \frac{\xi}{2} \|\nabla V\|^2_{L^2} + \frac{(\xi - \xi_0)^2}{2\xi} \|\nabla W_{\xi_0}\|^2_{L^2}$$

and so $\|\nabla V\|_{L^2} \leqslant \frac{|\xi - \xi_0|}{|\xi|} \|\nabla W_{\xi_0}\|_{L^2}$. On the other hand the real part yields

$$\begin{aligned} \nu \|\Delta V\|^2_{L^2} = (\xi_0 - \xi) \Im\Big(\int_D \nabla W_{\xi_0} \cdot \nabla\overline{V} \, dx\Big) \leqslant \\ \leqslant |\xi - \xi_0| \, \|\nabla W_{\xi_0}\|_{L^2} \|\nabla V\|_{L^2} \leqslant \frac{(\xi - \xi_0)^2}{|\xi|} \|\nabla W_{\xi_0}\|^2_{L^2} \end{aligned}$$

and as $\xi \to \xi_0$, continuity follows.

In the case $\xi_0 = 0$ the proof is slightly different, since one can prove directly taking the real part that

$$\nu \|\Delta V\|^2_{L^2} \leqslant |\xi| \|\nabla W_{\xi_0}\|^2_{L^2}.$$

Since $V \in H^1_0(D)$, this proves that as $\xi \to 0$, then R V tends to zero in $H^2(D)$, but now with the order of $|\xi|^{\frac{1}{2}}$.

Next, we consider the limit at ∞. By the previous lemma we know that $\nu \|\Delta W_\xi\|_{L^2} \leqslant 1$, so there is a sequence W_{ξ_n} converging weakly in $H^2(D)$ to some $W_\infty \in H^2(D)$. Indeed $W_\infty = 0$, since by the second property of Lemma 3.5 ξa_ξ converges to a finite limit (and in particular implies (3.6)) and so $a_\xi \to 0$, in particular $W_\infty = 0$. Moreover, from this it follows that the whole function $W_\xi \rightharpoonup 0$ weakly in $H^2(D)$. Finally, from the third property of Lemma 3.5 $\|\Delta W_\xi\|_{L^2}$ converges strongly to 0.

The statement on a_ξ follows in the same way, by using again Lemma 3.5. □

Remark 3.7. With a little more effort one can show that $W_\xi \to -\frac{i}{\xi}$ point-wise as $|\xi| \to +\infty$, but we are not going to use this property in the sequel.

Proof of Theorem 3.4. We use the identification (3.2) of an almost periodic function with its Fourier series and of the $\mathcal{B}^2$ semi-norm with the sum of squares of Fourier coefficients (namely, Parseval's identity). Consider $f \overset{\text{ap}}{\sim} \sum_\xi \widehat{f}(\xi)\, e^{i\xi t}$, then we only need to prove suitable bounds for the quantities $|a_\xi|$, $\|\Delta W_\xi\|_{L^2}$, and $\|W_\xi\|_{L^2}$.

Indeed, solving problem (3.3) for the Fourier components yields

$$w(t,x) \overset{\text{ap}}{\sim} \sum_\xi \frac{W_\xi(x)}{a_\xi}\widehat{f}(\xi)\, e^{i\xi t} \qquad \text{and} \qquad \pi(t) \overset{\text{ap}}{\sim} \sum_\xi \frac{\widehat{f}(\xi)}{a_\xi}\, e^{i\xi t}.$$

In order to capture the dependence of constants from ν, we observe that $W_\xi = \frac{1}{\nu}\widetilde{W}_{\xi/\nu}$, where $\widetilde{W}_\xi$ is the solution to (3.4) corresponding to $\nu = 1$. Set

$$n_0(\xi) = \|\widetilde{W}_\xi\|^2_{L^2}, \qquad n_1(\xi) = \|\nabla\widetilde{W}_\xi\|^2_{L^2}, \qquad n_2(\xi) = \|\Delta\widetilde{W}_\xi\|^2_{L^2}.$$

Indeed,

$$\begin{aligned}
\|\Delta w\|^2_{\mathcal{B}^2(\mathbf{R};L^2)} &= \sum_\xi \|\Delta\widehat{w}\|^2_{L^2} = \sum_\xi \frac{1}{|a_\xi|^2}|\widehat{f}(\xi)|^2\|\Delta W_\xi\|^2_{L^2}\\
&= \sum_\xi |\widehat{f}(\xi)|^2 \frac{\nu^2 n_2(\frac{\xi}{\nu})}{\nu^2 n_1(\frac{\xi}{\nu})^2 + \xi^2 n_0(\frac{\xi}{\nu})^2}\\
&\leqslant c_1\|f\|^2_{\mathcal{B}^2(\mathbf{R})} + \frac{c_2}{\nu^2}\|f'\|^2_{\mathcal{B}^2(\mathbf{R})},
\end{aligned}$$

since for $|\xi| \leqslant \nu$,

$$\frac{\nu^2 n_2(\frac{\xi}{\nu})}{\nu^2 n_1(\frac{\xi}{\nu})^2 + \xi^2 n_0(\frac{\xi}{\nu})^2} \leqslant \frac{n_2(\frac{\xi}{\nu})}{n_1(\frac{\xi}{\nu})^2} \leqslant \max_{|\sigma|\leqslant 1}\frac{n_2(\sigma)}{n_1(\sigma)^2} = c_1,$$

while for $|\xi| \geqslant \nu$,

$$\frac{\nu^2 n_2(\frac{\xi}{\nu})}{\nu^2 n_1(\frac{\xi}{\nu})^2 + \xi^2 n_0(\frac{\xi}{\nu})^2} \leqslant \frac{1}{\nu^2}\xi^2 \frac{n_2(\frac{\xi}{\nu})}{\big[(\frac{\xi}{\nu})^2 n_0(\frac{\xi}{\nu})\big]^2} \leqslant \frac{1}{\nu^2}\xi^2 \sup_{|\sigma|\geqslant 1}\frac{n_2(\sigma)}{\big(\sigma^2 n_0(\sigma)\big)^2} = \frac{c_2}{\nu^2}\xi^2.$$

Similarly,

$$\begin{aligned}
\|\partial_t w\|^2_{\mathcal{B}^2(L^2)} &= \sum_\xi \xi^2\|\widehat{w}(\xi)\|^2_{L^2} = \sum_\xi \frac{\xi^2}{|a_\xi|^2}|\widehat{f}(\xi)|^2\|W_\xi\|^2_{L^2}\\
&= \sum_\xi |\widehat{f}(\xi)|^2 \frac{\nu^2\xi^2 n_0(\frac{\xi}{\nu})}{\nu^2 n_1(\frac{\xi}{\nu})^2 + \xi^2 n_0(\frac{\xi}{\nu})^2}\\
&\leqslant c_3\nu^2\|f\|^2_{\mathcal{B}^2} + c_4\|f'\|^2_{\mathcal{B}^2},
\end{aligned}$$

since for $|\xi| \leqslant \nu$,

$$\frac{\nu^2\xi^2 n_0(\frac{\xi}{\nu})}{\nu^2 n_1(\frac{\xi}{\nu})^2 + \xi^2 n_0(\frac{\xi}{\nu})^2} \leqslant \frac{\nu^2}{n_0(\frac{\xi}{\nu})} \leqslant \nu^2\max_{|\sigma|\leqslant 1}\frac{1}{n_0(\sigma)} = c_3\nu^2,$$

while for $|\xi| \geqslant \nu$,

$$\frac{\nu^2\xi^2 n_0(\frac{\xi}{\nu})}{\nu^2 n_1(\frac{\xi}{\nu})^2 + \xi^2 n_0(\frac{\xi}{\nu})^2} \leqslant \xi^2\frac{1}{(\frac{\xi}{\nu})^2 n_0(\frac{\xi}{\nu})} \leqslant \xi^2\frac{1}{\inf_{|\sigma|\geqslant 1}\sigma^2 n_0(\sigma)} = c_4\xi^2.$$

Finally, with similar computations,

$$\|\pi\|_{\mathcal{B}^2} = \sum_\xi |\widehat{\pi}(\xi)|^2 = \sum_\xi \frac{1}{|a_\xi|^2}|\widehat{f}(\xi)|^2 = \sum_\xi |\widehat{f}(\xi)|^2 \frac{\nu^4}{\nu^2 n_1(\frac{\xi}{\nu})^2 + \xi^2 n_0(\frac{\xi}{\nu})^2}$$
$$\leqslant c_5\nu^2\|f\|^2_{\mathcal{B}^2} + c_4\|f'\|^2_{\mathcal{B}^2},$$

where $c_5 = \max_{|\sigma|\leqslant 1} \frac{1}{n_1(\sigma)^2}$. The quantities $c_1, \dots, c_5$ are easily seen to be finite by the previous lemma. □

Remark 3.8. The computations in the proof of Theorem 3.4 provide an alternate proof to Theorem 1 in [3], as well as to Corollary 2.9 (once (generalized) Fourier transform are replaced by Fourier series). Moreover, the following estimates hold,

$$\int_{\mathbf{R}} \|\Delta w\|^2_{L^2}\,dt \leqslant c\|f\|^2_{L^2(\mathbf{R})} + \frac{c}{\nu^2}\|f'\|^2_{L^2(\mathbf{R})},$$
$$\int_{\mathbf{R}} \|\partial_t w\|^2_{L^2}\,dt + \int_{\mathbf{R}} |\pi(t)|^2\,dt \leqslant c\nu^2\|f\|^2_{L^2(\mathbf{R})} + c\|f'\|^2_{L^2(\mathbf{R})},$$
$$\sup_{t\in\mathbf{R}} \|\nabla w(t)\|^2_{L^2} \leqslant c\big(\nu\|f\|^2_{L^2(\mathbf{R})} + \frac{1}{\nu}\|f'\|^2_{L^2(\mathbf{R})}\big).$$

Indeed, in the proof above we have shown that

$$\begin{aligned} \frac{\|\Delta W_\xi\|_{L^2}}{|a_\xi|} &\leqslant c\max\{1, \nu^{-1}|\xi|\} \\ \frac{|\xi|\,\|W_\xi\|_{L^2}}{|a_\xi|} &\leqslant c\max\{|\xi|, \nu\} \\ \frac{1}{|a_\xi|} &\leqslant c\max\{|\xi|, \nu\}. \end{aligned} \tag{3.7}$$

Hence, by Parseval's identity, we get

$$\int_{\mathbf{R}} \|\Delta w\|^2_{L^2} = \int_{\mathbf{R}} \frac{1}{|a_\xi|^2}|\widehat{f}(\xi)|^2\|\Delta W_\xi\|^2_{L^2} \leqslant c\|f\|^2_{L^2} + \frac{c}{\nu^2}\|f'\|^2_{L^2},$$

and the other inequalities are obtained similarly. Finally, the inequality for ∇w follows by integration by parts and the identity $\frac{d}{dt}\|\nabla w\|^2_{L^2} = -2\int_D w_t\,\Delta w\,dx$.

3.3. On the meaning of the solution.

We need to spend a few words about the notion of solution we constructed. We observe that a given $f \in \mathcal{B}^{1,2}(\mathbf{R})$ is clearly identified in the sense of $\mathcal{B}^{1,2}(\mathbf{R})$, hence by means of its generalized Fourier series. This implies, for instance that if w is a solution in the sense of Besicovitch spaces, then $w + \overline{w}$ is also a solution, for any $\overline{w} \in L^2(\mathbf{R}; H^2) \cap H^1(\mathbf{R}; L^2)$. This poses some restrictions to the interpretation of the result. One would like to have some embedding in the space of continuous functions in order to have a more precise identification of the solution. A larger spaces in which we are able to solve the equation is balanced by a weaker notion of solution.

In general one cannot expect the validity of the usual Sobolev embeddings in $\mathcal{B}^{s,p}(\mathbf{R})$ as is explained for instance in Pankov [23] and especially the identification with $UAP(\mathbf{R})$ functions is not a trivial fact. Classical counterexamples can be found in the references cited, while the following general embedding result is proved for instance in [16].

Proposition 3.9. *Let $\Xi \subset \mathbf{R}$ be countable and assume there is $\beta > 0$ such that the generalized sum satisfies*

$$\sum_{\xi\in\Xi} \frac{1}{|\xi|^\gamma} \quad \begin{cases} < +\infty & \text{for } \gamma > \beta \\ = +\infty & \text{for } \gamma < \beta. \end{cases}$$

If $\beta < 2s$, then for every $f \in \mathcal{B}^{s,2}(\mathbf{R})$ such that $\sigma(f) \subset \Xi$, we have $f \in C^{r,\alpha}(R) \cap \mathsf{UAP}(\mathbf{R})$ for all $\alpha \in [0, s - r - \beta/2)$, where $r = \lceil s - \frac{\beta}{2} \rceil$ (with corresponding inequality for the norms).

Remark 3.10. To simplify the notation from now on we denote by f_ξ the (generalized) Fourier coefficient $\widehat{f}(\xi)$ that is (more precisely) written as $a_\xi(f)$ in (3.1).

To understand this result, let us observe that if $c_\xi \in \ell^1(\mathbf{C})$, then the series $\sum_\xi c_\xi\, e^{i\xi t}$ converges uniformly and can be identified with a continuous almost periodic function $f \overset{ap}{\sim} \sum_\xi c_\xi\, e^{i\xi t}$.
Moreover, for classical Fourier series, i. e. $\sigma(f) = \Xi \subseteq \mathbf{Z}$, the β-condition is satisfied for $\beta = 1$ and this shows that if $(c_j)_{j\in\mathbf{Z}}, (jc_j)_{j\in\mathbf{Z}} \in \ell^2(\mathbf{C})$, then

$$\Big|\sum_{j\in\mathbf{Z}} c_j\, e^{ijt}\Big|^2 \leqslant \Big|\sum_{j\in\mathbf{Z}} |c_j|\Big|^2 \leqslant \Big(\sum_{j\in\mathbf{Z}} j^2|c_j|^2\Big)\Big(\sum_{j\in\mathbf{Z}\setminus 0} \frac{1}{j^2}\Big) < +\infty.$$

This is the guideline to understand the result for $\mathcal{B}^{s,2}(\mathbf{R})$, since one has – roughly speaking – to show an inequality similar to

$$\Big|\sum_{\xi\in\sigma(f)} c_\xi\, e^{i\xi t}\Big|^2 \leqslant \Big|\sum_{\xi\in\sigma(f)} |c_\xi|\Big|^2 \leqslant \Big(\sum_{\xi\in\sigma(f)} |\xi|^{2s}|c_\xi|^2\Big)\Big(\sum_{\xi\in\sigma(f)} \frac{1}{|\xi|^{2s}}\Big) < +\infty.$$

For instance Proposition 3.9 implies the following result.

Corollary 3.11. *Let be given $f \in \mathcal{B}^{1,2}(\mathbf{R})$, with $f \overset{ap}{\sim} \sum_\xi f_\xi\, e^{i\xi t}$ such that*

$$\sum_{\xi\in\sigma(f)} \frac{1}{|\xi|^2} < +\infty.$$

Then, there are $w \in \mathcal{B}^2(\mathbf{R}; H^2(D)) \cap \mathsf{UAP}(\mathbf{R}; H^1_0(D))$, with $\partial_t w \in \mathcal{B}^2(\mathbf{R}; L^2(D))$, and $\pi \in \mathcal{B}^2(\mathbf{R})$ such that (2.2) is satisfied in the sense of Besicovitch.

This makes also possible to consider the flux as

$$f(t) = f_1(t) + f_2(t),$$

with $f_1 \in \mathcal{B}^{1,2}(\mathbf{R})$ and $f_2 \in H^1(\mathbf{R})$, so that $\|f_2\|_{\mathcal{B}^{1,2}} = 0$. One can construct the solutions $w_1 \in \mathcal{B}^{1,2}(\mathbf{R}; L^2(D))$ and $w_2 \in W^{1,2}(\mathbf{R}; L^2(D))$ corresponding to f_1 and f_2 respectively, and add together.

This is not completely satisfactory, since we still do not have a precise identification on the pressure. To this end one would like to have a solution in the classical $\mathsf{UAP}(\mathbf{R})$ space for example. This can be achieved by assuming stronger conditions on f (rather than on its spectrum), as shown by the following result.

Proposition 3.12. *Let f be given, with $f \overset{ap}{\sim} \sum_\xi f_\xi\, e^{i\xi t}$, such that*

$$\sum_\xi (1 + |\xi|)|f_\xi| < +\infty.$$

Then, there exists a unique solution (w, π) to (2.2) such that

$$\begin{aligned} &\sum_\xi \|\Delta w_\xi\|_{L^2} \leqslant c \sum_\xi |f_\xi| + \frac{c}{\nu} \sum_\xi |\xi|\,|f_\xi|, \\ &\sum_\xi |\pi_\xi| + \sum_\xi |\xi|\,\|w_\xi\|_{L^2} \leqslant c\nu \sum_\xi |f_\xi| + c \sum_\xi |\xi|\,|f_\xi|, \end{aligned} \tag{3.8}$$

where w_ξ and π_ξ are the (generalized) Fourier coefficients of w and π, respectively. In particular, $w \in \mathsf{UAP}(\mathbf{R}; H^2)$, $\partial_t w \in \mathsf{UAP}(\mathbf{R}; H)$ and $\pi \in \mathsf{UAP}(\mathbf{R})$.

Proof. First we notice that since $\sum_\xi |f_\xi| < +\infty$, then

$$\sum_\xi |f_\xi|^2 \leqslant \Big(\sum_\xi |f_\xi|\Big)^2,$$

showing that $f \in \mathcal{B}^2(\mathbf{R})$. The same argument shows also that $f' \in \mathcal{B}^2(\mathbf{R})$, and so Theorem 3.4 ensures the existence of a unique solution. Since $w_\xi = a_\xi^{-1} W_\xi f_\xi$ and $\pi_\xi = a_\xi^{-1} f_\xi$, the estimates (3.8) follow immediately from (3.7). In order to show that w and π are Bohr-almost periodic, we consider a truncation

$$f_n = \sum_{\xi \in \sigma_n(f)} f_\xi e^{i\xi t},$$

where $(\sigma_n(f))_{n\in\mathbf{N}}$ is an increasing sequence of finite subset of $\sigma(f)$ such that $\bigcup_n \sigma_n(f) = \sigma(f)$. For each $n \in \mathbf{N}$ we can consider (2.2) with flux given by the trigonometric polynomial f_n and the estimates (3.8) imply uniform convergence of the corresponding solutions (w_n, π_n) towards UAP functions with the requested properties. □

3.4. **The nonlinear case.** In this last section we consider the non-linear problem. Assume preliminarily (we shall assume stronger assumptions on f later) that $f \in \mathcal{B}^{1,2}(\mathbf{R})$ and denote by w_1, w_2 the basic flows in the two pipes O_1, O_2 respectively, provided by Theorem 3.4. Let V_0 be the flow defined as in (2.21), it is clear that V_0 is also almost periodic and keeps the same regularity properties of w_1 and w_2, namely

$$V_0 \in \mathcal{B}^2(\mathbf{R}; H^2(O) \cap H^1_0(O)) \cap \mathcal{B}^{1,2}(\mathbf{R}; L^2(O)),$$

as well as the flow w defined in (2.22). Indeed, both flows are obtained by applying only linear operators in the space variable to w_1 and w_2.

Consider the full nonlinear Leray's problem in the (Besicovitch) almost periodic setting, namely to find a solution (u, p) to the problem

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)\, u + \nabla p \overset{ap}{\sim} 0, \\ \nabla\cdot u \overset{ap}{\sim} 0, \end{cases} \tag{3.9}$$

such that

$$\|u - w_i\|_{\mathcal{B}^2(\mathbf{R};H^1(O_i))} \leqslant c_0, \qquad i = 1, 2,$$

and this implies that

$$\lim_{z\to+\infty} \|u - w_i\|_{\mathcal{B}^2(\mathbf{R};H^{1/2}(D_i))} = 0. \tag{3.10}$$

If w is the flow defined in (2.22), consider the solution $u = U + w$ as a perturbation of w. Consequently,

$$\begin{cases} \partial_t U - \nu\Delta U + (U\cdot\nabla)\, U + (U\cdot\nabla)\, w + (w\cdot\nabla)\, U + \nabla P \overset{ap}{\sim} F, \\ \nabla\cdot U \overset{ap}{\sim} 0, \end{cases} \tag{3.11}$$

where F is defined in (2.24). The main theorem of the section is the following.

Theorem 3.13. *Assume that the flux* f *satisfies*

$$\phi_* := \sum_\xi (1+|\xi|)|f_\xi| < \infty. \tag{3.12}$$

Then, there exists $\nu_0 > 0$, with $\nu_0 = \nu_0(f, O)$, such that for every $\nu \geqslant \nu_0$ problem (3.9) admits a solution

$$u \in \mathcal{B}^2(\mathbf{R}; H^1_0(O)) \cap \mathcal{B}^{\frac12,2}(\mathbf{R}; L^2(O)),$$

which satisfies (3.10).

The rest of the section is devoted to the proof of this result.

3.4.1. *Spectrum and module.* Before turning to the analysis of problem (3.11), we recall that, since $f \in \mathcal{B}^2(\mathbf{R})$, its generalized Fourier series is well-defined and its spectrum $\sigma(f)$ is the set of modes $\xi \in \mathbf{R}$ corresponding to non-zero coefficients f_ξ in the Fourier expansion of f.

Since f is real, it follows that $\overline{f_\xi} = f_{-\xi}$ and so the spectrum is symmetric, namely $-\sigma(f) = \sigma(f)$.

Definition 3.14. The set $\mu(f)$ is the $\mathbf{Z}$-*module* of the spectrum of f, namely the smallest subset of $\mathbf{R}$ which contains $\sigma(f)$ and is closed for the sum (that is, if $\xi, \eta \in \mu(f)$, then $a\,\xi + b\,\eta \in \mu(f)$, for all $a, b \in \mathbf{Z}$).

It is clear that $\mu(f)$ is also symmetric and, since $\sigma(f)$ is at most countable, $\mu(f)$ is at most countable too. Moreover, it is easy to see that the spectra of w and F, by linearity, are contained in the spectrum of f. Indeed, by construction, the terms V_0 and w, defined in (2.21) and in (2.22), respectively have spectrum contained in $\sigma(f)$.

In the following, with the purpose of approximations, we shall need to consider finite dimensional truncations. To this aim, we fix an increasing sequence $(\mu_N(f))_{N\in\mathbf{N}}$ of subsets of $\mu(f)$ converging to $\mu(f)$, that is $\mu_N(f) \subset \mu_{N+1}(f)$ and $\mu(f) = \bigcup_N \mu_N(f)$, and such that $\mu_N(f) = -\mu_N(f)$.

3.4.2. *Reduction to a system in Fourier variables.* A remarkable feature of the nonlinearity we are going to analyse is that if u_1, u_2 are Besicovitch almost-periodic, then $(u_1 \cdot \nabla)\,u_2$ is also in the same class. This result on product of almost periodic functions is not true in general, but as we will see in our case it holds since the spectrum of the nonlinearity is contained in the module generated by $\sigma(u_1)$ and $\sigma(u_2)$. Having this in mind, we recast problem (3.11) in Fourier variables,

$$i\xi U_\xi - \nu\Delta U_\xi + \sum_{\eta+\theta=\xi} \big[(U_\eta \cdot \nabla)\,U_\theta + (U_\eta \cdot \nabla)\,w_\theta + (w_\eta \cdot \nabla)\,U_\theta\big] + \nabla P_\xi = f_\xi, \tag{3.13}$$

for $\xi \in \mu(f)$, with $\nabla \cdot U_\xi = 0$, and the sum in the formula above is extended over all $\eta, \theta \in \mu(f)$.

We shall use the following strategy to prove Theorem 3.13. We linearise the nonlinearity (by introducing an auxiliary field $\widetilde{U}$) and solve the new linearised problem (in two steps, first for a finite number of modes, then for all modes). The assumption on the viscosity allows to have a uniquely defined map that gives a solution to the linearised problem for each field $\widetilde{U}$. The same assumption ensures that this map is a contraction and its fixed point is the solution to problem (3.9).

3.4.3. *Preliminary tools.* We prove two preliminary tools for the analysis of the problem.

We first consider the fields w and F defined respectively as in (2.22) and (2.24) and prove the following estimates in terms of ϕ_*.

Lemma 3.15. *Let be given* $f \in \mathcal{B}^{1,2}(\mathbf{R})$, *assume that* (3.12) *holds. Then, there is* $c > 0$ *(independent of* ν*) such that*

$$\|\Delta w\|_{\mathcal{B}^2(\mathbf{R};L^2)} \leqslant c\big(1+\tfrac{1}{\nu}\big)\|f\|_{\mathcal{B}^{1,2}(\mathbf{R})},$$
$$\|\pi\|_{\mathcal{B}^2(\mathbf{R})} + \|\partial_t w\|_{\mathcal{B}^2(\mathbf{R};L^2)}, \leqslant c(1+\nu)\|f\|_{\mathcal{B}^{1,2}(\mathbf{R})}$$
$$\sum_\xi \|\Delta w_\xi\|_{L^2} \leqslant c\big(1+\tfrac{1}{\nu}\big)\phi_\star,$$
$$\sum_\xi \big(|\pi_\xi| + |\xi|\|w_\xi\|_{L^2}\big) \leqslant c(1+\nu)\phi_\star,$$
$$\|F\|_{\mathcal{B}^2(\mathbf{R};L^2)} \leqslant c\big((1+\nu) + (1+\tfrac{1}{\nu})^2\phi_\star\big)\|f\|_{\mathcal{B}^{1,2}(\mathbf{R})},$$
$$\sum_\xi \|f_\xi\|_{L^2} \leqslant c\big((1+\nu) + (1+\tfrac{1}{\nu})^2\phi_\star\big)\phi_\star.$$

Proof. The inequalities for w, π are a straightforward consequence of Theorem 3.4, Proposition 3.12 and the definition (2.22). To prove the inequalities for F, we only need to consider the term $(w\cdot\nabla)\,w$ (the estimate of the other terms in F follow from the estimates for w and π). For $\xi \in \mu(f)$, writing the explicit expression for $[(w\cdot\nabla)\,w]_\xi = \sum_{\eta+\theta=\xi}(w_\eta\cdot\nabla)\,w_\theta$, we obtain that

$$\|[(w\cdot\nabla)\,w]_\xi\|_{L^2} \leqslant c \sum_{\eta+\theta=\xi} \|\nabla w_\eta\|_{L^2}\|\Delta w_\theta\|_{L^2},$$

hence

$$\begin{aligned}\|(w\cdot\nabla)\,w\|^2_{\mathcal{B}^2(\mathbf{R};L^2)} &\leqslant \sum_\xi\Big(\sum_{\eta+\theta=\xi}\|\nabla w_\eta\|_{L^2}\|\Delta w_\theta\|_{L^2}\Big)^2\\ &\leqslant \Big(\sum_\xi \|\nabla w_\xi\|_{L^2}\Big)^2\|\Delta w\|^2_{\mathcal{B}^2(\mathbf{R};L^2)},\end{aligned}$$

and also

$$\begin{aligned}\sum_\xi \|[(w\cdot\nabla)\,w]_\xi\|_{L^2} &\leqslant \sum_\xi\sum_{\eta+\theta=\xi}\|\nabla w_\eta\|_{L^2}\|\Delta w_\theta\|_{L^2}\\ &\leqslant \Big(\sum_\xi\|\nabla w_\xi\|_{L^2}\Big)\Big(\sum_\xi\|\Delta w_\xi\|_{L^2}\Big),\end{aligned}$$

which complete the proof. □

The second result is the extension to our setting of the usual cancellation property of the nonlinear convective term, when energy estimates are derived.

Lemma 3.16. *Let be given* $X = \sum X_\xi\, e^{i\xi t}$ *and* $X = \sum X_\xi\, e^{i\xi t}$ *both belonging to* $\mathcal{B}^2(\mathbf{R}; H^1_0)$, *with* Y *divergence-free. Assume that the spectra of* X *and* Y *are contained in* $\mu(f)$ *and fix an integer* $N \geqslant 1$. *Then*

$$\sum_\xi^{(N)} \sum_{\eta+\theta=\xi}^{(N)} \int_O \overline{X_\xi}\cdot(Y_\eta\cdot\nabla)\,X_\theta\,dx = 0,$$

where the superscript on the sum above means that the sum is extended only over modes in $\mu_N(f)$.

Moreover, the same holds true for $N=\infty$ *if at least one between* X *and* Y *is in* $\mathcal{B}^\star(\mathbf{R}; H^1_0)$, *with*

$$\mathcal{B}^\star(\mathbf{R}; H^1_0) := \Big\{ f \in L^1_{loc}(\mathbf{R}) : \ \|f\|_{\mathcal{B}^\star(\mathbf{R};H^1_0)} := \sum_{\xi\in\sigma(f)} (1+|\xi|)\|f_\xi\|_{H^1} < +\infty \Big\}.$$

Proof. Let us denote by ⓝ the sum in the statement of the lemma, then by a change of summation index

$$ⓝ = \sum_{\xi+\eta+\theta=0}^{(N)} \int_O \overline{X_\xi} \cdot (Y_\eta \cdot \nabla) X_\theta \, dx = -\sum_{\xi+\eta+\theta=0}^{(N)} \int_O \overline{X_\theta} \cdot (Y_\eta \cdot \nabla) X_\xi \, dx, = -ⓝ,$$

since $\overline{X_\xi} = X_{-\xi}$, and the claim is true. To show that the same holds for $N = \infty$, it is sufficient to prove that the following sum extended is bounded uniformly in N. Indeed, if Z is another field,

$$\sum_{\xi}^{(N)} \sum_{\eta+\theta=\xi}^{(N)} \int_O \overline{X_\xi} \cdot (Y_\eta \cdot \nabla) Z_\theta \, dx \leqslant \sum_{\xi}^{(N)} \sum_{\eta+\theta=\xi}^{(N)} \|\nabla X_\xi\|_{L^2} \|\nabla Y_\eta\|_{L^2} \|\nabla Z_\theta\|_{L^2}$$
$$\leqslant \|\nabla X\|_{\mathcal{B}^2(H^1_0)} \|\nabla Y\|_{\mathcal{B}^2(H^1_0)} \|Z\|_{\mathcal{B}^\star(H^1_0)},$$

by Young's inequality for convolutions. □

We observe that in the cancellation property above it is fundamental that we deal with the complex conjugate of X and with the fact that the spectrum is symmetric, since flux and solution are both real–valued.

The proof of Theorem 3.13 is split into three preliminary steps.

3.4.4. *First step: existence for the finite modes approximation.* Given $\widetilde{U} \in \mathcal{B}^2(\mathbf{R}; H^1_0)$ with $\nabla \cdot \widetilde{U} = 0$ and an integer $N \geqslant 1$, we seek for a solution to the following problem,

$$i\xi U_\xi - \nu \Delta U_\xi + \sum_{\eta+\theta=\xi}^{(N)} \big[(\widetilde{U}_\eta \cdot \nabla) U_\theta + (\widetilde{U}_\eta \cdot \nabla) w_\theta + (w_\eta \cdot \nabla) U_\theta\big] + \nabla P_\xi = f_\xi, \tag{3.14}$$

for $\xi \in \mu_N(f)$, where again the superscript on the above sum means that the sum is extended only over modes in $\mu_N(f)$.

Proposition 3.17. *Let* $f \in \mathcal{B}^{1,2}(\mathbf{R})$ *and* $\widetilde{U} \in \mathcal{B}^{1,2}(\mathbf{R}; H^1_0)$ *and set*

$$\phi^N_\star := \sum_{\xi}^{(N)} (1 + |\xi|)|f_\xi| \qquad \textit{and} \qquad \psi^N_\star := \sum_{\xi}^{(N)} \|\nabla \widetilde{U}_\xi\|_{L^2}.$$

Then, there is at least one solution $U^{(N)}$ *to problem* (3.14)*. Moreover, there are non-negative* $c_1(\cdot, \dots)$ *and* $c_2(\cdot, \dots, \dots)$ *increasing functions of their arguments and (depending only on the domain* O*) such that*

$$\|U^{(N)}\|_{B^{\frac12,2}(\mathbf{R};L^2)} + \|U^{(N)}\|_{B^2(\mathbf{R};H^1)} \leqslant c(\nu, \|f\|_{\mathcal{B}^{1,2}(\mathbf{R})})\psi^N_\star + c(\nu, \phi^N_\star, \|f\|_{\mathcal{B}^{1,2}(\mathbf{R})}). \tag{3.15}$$

If additionally

$$\nu > \psi^N_\star + c(1 + \tfrac{1}{\nu})\phi^N_\star,$$

where c *is the constant of Lemma 3.15, then the solution is unique and*

$$\begin{aligned}\big(\nu - \psi^N_\star - c(1 + \tfrac{1}{\nu})\phi^N_\star\big)\Big(\sum_{\xi}^{(N)} \|\nabla U^{(N)}_\xi\|_{L^2}\Big)\\ \leqslant c(1 + \nu)\phi^N_\star + c(1 + \tfrac{1}{\nu})^2(\phi^N_\star)^2 + c(1 + \tfrac{1}{\nu})\phi^N_\star\psi^N_\star.\end{aligned} \tag{3.16}$$

Proof. The proof can be carried on with the standard technique of Fujita (cf. Theorem 1.4 of [27, Ch. 2]) for the case of existence of solutions for the steady Navier-Stokes equations in unbounded domains (but we have the additional advantage of the Poincaré inequality (2.20)). We use Galerkin approximations $U^n = \sum_{k=1}^n u^k e_k$ (not necessarily made with eigenfunctions) and we consider the projection of (3.14) on the finite dimensional Galerkin space as a problem on $\mathbf{R}^{2nN}$ (the real and imaginary parts of each u^k count as two variables) with the scalar product induced by the one of $H^1_0(O)^{\otimes 2N}$.

We show existence of a solution of the finite dimensional problem by means of Lemma 1.4 of [27, Chapter 2] (which in turns is a consequence of Brouwer's fixed point theorem). Let $P(U^n)$ be given component-wise by the projection of (3.14) so that if $P(U^n) = 0$ then U^n is the solution to the Galerkin projected problem. It is sufficient to show that $P(U^n)\cdot U_n > 0$ on $|U^n| = C_0$ for some $C_0 > 0$, where product and norm are those we have given on $\mathbf{R}^{2nN}$. This is immediate by Lemma 3.16 since

$$\begin{aligned} P(U^n)\cdot U^n &\geqslant \nu|U^n|^2 - \sum_{\xi}^{(N)}\sum_{\eta+\theta=\xi}^{(N)} |U^n_\xi|\|\nabla\widetilde{U}_\eta\|_{L^2}\|\nabla w_\theta\|_{L^2} - c\|F\|_{\mathcal{B}^2(L^2)}|U^n| \\ &\geqslant \nu|U^n|^2 - \psi^N_\star\|\nabla w\|_{\mathcal{B}^2(L^2)}|U^n| - c\|F\|_{\mathcal{B}^2(L^2)}|U^n|, \end{aligned}$$

which is strictly positive if we choose ν such that

$$C_0 > \tfrac{1}{\nu}\big(\psi^N_\star\|\nabla w\|_{\mathcal{B}^2(L^2)} + c\|F\|_{\mathcal{B}^2(L^2)}\big).$$

Passing to the limit in the Galerkin approximation is standard (the non-linearity contains a finite sum) and follows from uniform bounds (in n) on U^n which are similar to (3.15) and whose proof is formally similar. Hence, we prove (3.15) directly. For each $\xi\in\mu_N(f)$ multiply (3.14) by $\overline{U_\xi}$, integrate by parts on O, sum over $\xi\in\mu_N(f)$, and use Lemma 3.16 to get

$$\begin{aligned} i\sum_{\xi}^{(N)}\xi\|U_\xi\|^2_{L^2} + \nu\sum_{\xi}^{(N)}\|\nabla U_\xi\|^2_{L^2} + \\ + \sum_{\xi}^{(N)}\sum_{\eta+\theta=\xi}^{(N)}\int_O \overline{U_\xi}\cdot(\widetilde{U}_\eta\cdot\nabla)\,w_\theta\,dx = \sum_{\xi}^{(N)}\int_O\overline{U_\xi}\cdot f_\xi. \end{aligned}$$

Hence by Young's inequality for convolutions and taking the real part we get

$$\begin{aligned} \nu\|\nabla U\|^2_{\mathcal{B}^2(L^2)} &\leqslant \sum_{\xi}^{(N)}\sum_{\eta+\theta=\xi}^{(N)}\|\nabla U_\xi\|_{L^2}\|\nabla\widetilde{U}_\eta\|_{L^2}\|\nabla w_\theta\|_{L^2} + c\|F\|_{\mathcal{B}(L^2)}\|\nabla U\|_{\mathcal{B}^2(L^2)} \\ &\leqslant \psi^N_\star\|\nabla w\|_{\mathcal{B}^2(L^2)}\|\nabla U\|_{\mathcal{B}^2(L^2)} + c\|F\|_{\mathcal{B}(L^2)}\|\nabla U\|_{\mathcal{B}^2(L^2)}, \end{aligned}$$

and so inequality (3.15) follows from Lemma 3.15.

To prove (3.16), multiply (3.14) by $\overline{U_\xi}$, integrate by parts on O and *divide* by the non-zero $\|\nabla U_\xi\|_{L^2}$ to get

$$\nu\|\nabla U_\xi\|_{L^2} \leqslant c\|f_\xi\| + \sum_{\eta+\theta=\xi}^{(N)}\big(\|\nabla\widetilde{U}_\eta\|_{L^2}\|\nabla U_\theta\|_{L^2} + \|\nabla\widetilde{U}_\eta\|_{L^2}\|\nabla w_\theta\|_{L^2} + \|\nabla w_\eta\|_{L^2}\|\nabla U_\theta\|_{L^2}\big).$$

Inequality (3.16) follows by summing in ξ, using Young's convolution inequality and Lemma 3.15.

Finally, if $U_1^{(N)}$, $U_2^{(N)}$ are two solutions corresponding to the same data, let $D^{(N)} := U_1^{(N)} - U_2^{(N)}$ and $Q^{(N)} := P_1^{(N)} - P_2^{(N)}$, then

$$i\xi D^{(N)}_\xi - \nu\Delta D^{(N)}_\xi + \sum_{\eta+\theta=\xi}^{(N)}\big((\widetilde{U}_\eta\cdot\nabla)\,D^{(N)}_\theta + (w_\eta\cdot\nabla)\,D^{(N)}_\theta\big) + \nabla Q^{(N)}_\xi = 0,$$

and hence taking the scalar product with $\overline{D^{(N)}_\xi}$

$$\begin{aligned} \nu\|\nabla D^{(N)}\|^2_{\mathcal{B}^2(\mathbf{R};L^2)} &\leqslant \sum_{\xi}^{(N)}\sum_{\eta+\theta=\xi}^{(N)}\|\nabla w_\eta\|_{L^2}\|\nabla D^{(N)}_\xi\|_{L^2}\|\nabla D^{(N)}_\theta\|_{L^2} \\ &\leqslant \Big(\sum_{\xi}^{(N)}\|\nabla w_\eta\|_{L^2}\Big)\|\nabla D^{(N)}\|^2_{\mathcal{B}^2(\mathbf{R};L^2)}, \end{aligned}$$

which, by the assumption on ν, implies that $D\equiv 0$. □

3.4.5. *Second step: existence of a limit as* $N \to \infty$. Let $f \in \mathcal{B}^{1,2}(\mathbf{R}) \cap \mathcal{B}^\star(\mathbf{R})$ and assume additionally that the quantity

$$\phi_\star := \sum_\xi (1 + |\xi|)|f_\xi|$$

is finite. This implies in particular, as in Lemma 3.12, that f and f' have representatives which are Bohr-almost periodic. Let $\widetilde{U} \in \mathcal{B}^2(\mathbf{R}; H^1_0)$, assume that the quantity $\sum_\xi \|\nabla\widetilde{U}_\xi\|_{L^2}$ is also finite and consider the problem

$$i\xi U_\xi - \nu\Delta U_\xi + \sum_{\eta+\theta=\xi} \big[(\widetilde{U}_\eta \cdot \nabla)\, U_\theta + (\widetilde{U}_\eta \cdot \nabla)\, w_\theta + (w_\eta \cdot \nabla)\, U_\theta\big] + \nabla P_\xi = f_\xi, \tag{3.17}$$

for $\xi \in \mu(f)$.

Proposition 3.18. *There exists* $\nu_0 > 0$, *with* $\nu_0 = \nu_0(f, O)$, *such that for every* $\nu \geqslant \nu_0$ *there is* $\psi_\star > 0$ *such that*

$$\nu > \psi_\star + c(1 + \tfrac{1}{\nu})\phi_\star,$$

and if

$$\sum_\xi \|\nabla\widetilde{U}_\xi\|_{L^2} \leqslant \psi_\star$$

then there is a unique solution $U \in \mathcal{B}^2(\mathbf{R}; H^1_0(O)) \cap \mathcal{B}^{1/2,2}(\mathbf{R}; L^2(O))$ *to problem* (3.17).

Moreover,

$$\begin{aligned} &\|U\|_{B^{\frac{1}{2},2}(\mathbf{R};L^2)} + \|U\|_{B^2(\mathbf{R};H^1_0)} \leqslant c(\nu, \|f\|_{\mathcal{B}^{1,2}(\mathbf{R})})\psi_\star + c(\nu, \phi_\star, \|f\|_{\mathcal{B}^{1,2}(\mathbf{R})}),\\ &\sum_\xi \|\nabla U_\xi\|_{L^2} \leqslant \psi_\star \end{aligned} \tag{3.18}$$

Proof. Let $(U^{(N)})_{N\geqslant 1}$ be the sequence of solutions to (3.14) provided by Proposition 3.17. By (3.15) it follows that $(U^{(N)})_{N\geqslant 1}$ is bounded in $\mathcal{B}^{1/2,2}(\mathbf{R}; L^2)$ and in $\mathcal{B}^2(\mathbf{R}; H^1_0)$, hence there is a subsequence weakly convergent to a limit point $U \in \mathcal{B}^{1/2,2}(\mathbf{R}; L^2) \cap \mathcal{B}^2(\mathbf{R}; H^1_0)$. Since (3.17), weak convergence is enough to pass to the limit in the equation. Uniqueness follows as in Proposition 3.17, using the bound on the viscosity.

We only have to identify ν_0 and $\psi_\star$. From (3.16) it follows that

$$\big(\nu - \psi_\star - c(1 + \tfrac{1}{\nu})\phi_\star\big)\Big(\sum_\xi \|\nabla U_\xi\|_{L^2}\Big) \leqslant c(1+\nu)\phi_\star + c(1 + \tfrac{1}{\nu})^2(\phi_\star)^2 + c(1 + \tfrac{1}{\nu})\phi_\star\psi_\star,$$

so everything boils down to show that for ν large enough there is $\psi_\star$ such that

$$\frac{c(1+\nu)\phi_\star + c(1 + \frac{1}{\nu})^2(\phi_\star)^2 + c(1 + \frac{1}{\nu})\phi_\star\psi_\star}{\nu - \psi_\star - c(1 + \frac{1}{\nu})\phi_\star} \leqslant \psi_\star,$$

that is

$$\psi_\star^2 - \big(\nu - 2c(1 + \tfrac{1}{\nu})\phi_\star\big)\psi_\star + c(1+\nu)\phi_\star + c^2(1 + \tfrac{1}{\nu})^2\phi_\star^2 \leqslant 0.$$

It is elementary to verify that the above polynomial has two positive solutions for ν large enough. □

Remark 3.19. Clearly, without the assumption on the size of ν in the previous proposition, one can still show existence of at least one solution to (3.17). The size condition on ν is necessary only for proving uniqueness.

3.4.6. *Third step: the fixed point argument.* Under the assumptions of Proposition 3.18 we have a well defined map $\widetilde{U} \mapsto U$, where U is the solution to problem (3.17). Denote the map by $\mathcal{J}$, then it is clear that any fixed point of $\mathcal{J}$ is a solution to (3.13) and hence to (3.11).

Proof of Theorem 3.13. Fix $\nu \geqslant \nu_0$, where ν_0 is given in Proposition 3.18. We prove that the map $\mathcal{J}$ is a contraction on the set $\mathcal{X}$ of all $U \in \mathcal{B}^2(\mathbf{R}; H^1_0(O)) \cap \mathcal{B}^{1/2,2}(\mathbf{R}; L^2(O))$ that verify the bounds (3.18).

The fact that $\mathcal{J}$ maps $\mathcal{X}$ into $\mathcal{X}$ clearly follows from Proposition 3.18, so we only need to prove that $\mathcal{J}$ is a contraction. This is obtained as in the proof of uniqueness of Proposition (3.17). Indeed, if $E := \widetilde{U}_1 - \widetilde{U}_2$ and $D := \mathcal{J}(\widetilde{U}_1) - \mathcal{J}(\widetilde{U}_2)$, then

$$i\xi E_\xi - \nu\Delta E_\xi + \sum_{\eta+\theta=\xi} (E_\eta \cdot \nabla)\, U^1_\theta + (\widetilde{U}^2_\eta \cdot \nabla) D_\theta + (E_\eta \cdot \nabla)\, w_\theta + \nabla Q_\xi = 0,$$

with a suitable Q. By multiplying by $\overline{D_\xi}$, integrating by parts, and summing over ξ we get

$$(\nu - \psi_\star)\|\nabla E\|_{\mathcal{B}^2(L^2)} \leqslant \big(\psi_\star + c(1 + \tfrac{1}{\nu})\phi_\star\big)\|\nabla E\|_{\mathcal{B}^2(L^2)}$$

that is $\|\nabla E\|_{\mathcal{B}^2(L^2)} \leqslant K_0 \|\nabla E\|_{\mathcal{B}^2(L^2)}$, with

$$K_0 := \frac{\psi_\star + c(1 + \frac{1}{\nu})\phi_\star}{\nu - \psi_\star}.$$

Likewise we also have $\|D\|_{\mathcal{B}^{\frac{1}{2},2}(L^2)} \leqslant \nu K_0^2 \|\nabla E\|_{\mathcal{B}^2(L^2)}$. Finally, by multiplying by $\overline{D_\xi}$, dividing by $\|\nabla D_\xi\|_{L^2}$ and summing over ξ we get

$$\sum_\xi \|\nabla E_\xi\|_{L^2} \leqslant K_0 \sum_\xi \|\nabla E_\xi\|_{L^2}.$$

In conclusion, if ν is large enough it follows that $K_0 < 1$ and $\nu K_0^2 < 1$ and the map $\mathcal{J}$ is a contraction.

In order to conclude the proof, we need to show that if $(U_\xi)_{\xi\in\mu(f)}$ is solution to (3.13), then the Besicovitch almost periodic vector field U having Fourier coefficients $(U_\xi)_{\xi\in\mu(f)}$ is a weak solution to (3.11), namely that for every divergence-free $\varphi \in C^\infty_c(O; \mathbf{R}^3)$ and every $\xi \in \mathbf{R}$,

$$\mathcal{M}\Big[\Big(\langle \partial_t U, \varphi\rangle - \nu\langle U, \Delta\varphi\rangle - \langle U, \big((U + w)\cdot\nabla\big)\, \varphi\rangle - \langle w, (U\cdot\nabla)\, \varphi\rangle\Big)\, e^{i\xi t}\Big] = 0,$$

which is an easy consequence due to the bounds (3.18). □

3.4.7. *Final considerations.* Apparently the assumption (3.12) seems to be essential for the proof in the Besicovitch setting to work. The technical problem is essentially related to the term

$$\sum_\xi \sum_{\eta+\theta=\xi} U_\xi \cdot (V_\eta \cdot \nabla) W_\theta$$

which is of order three, although all bounds on U, V, W are of order two, if one works in the framework of $\mathcal{B}^2$ spaces. Young's convolution inequalities tell us that in general there is no possibility to bound the above term under these assumptions. In terms of the time variable, we are trying to bound the Navier-Stokes nonlinearity *over the whole* $\mathbf{R}$.

Another possibility would be to use the other a-priori estimate, namely the bound in $\mathcal{B}^{1/2,2}(\mathbf{R}; L^2)$, which plays no role in the proof of Theorem 3.13, using for instance the results in Section 3.3. *This possibility is ruled out by the non-linear term.* In fact, in the standard case of Leray-Hopf weak solutions one has a better knowledge of the time derivative and this can be used for instance with the Aubin-Lions compactness lemma to handle the non linear term.

Indeed the non-linearity reads in Fourier variables (in time) as a convolution and, whatever is the spectrum $\sigma(f)$ of the flux, the spectrum of the solution to the non-linear problem will have the **Z**-module $\mu(f)$ as its spectrum. In different words, the non-linearity creates a full set of harmonic resonances in the time frequency. The structure of **Z**-modules in **R** shows that the only possibility to use a bound on the derivatives (while obtaining a useful information for all times $t \in \mathbf{R}$) is the periodic case, previously studied in [3]. Indeed it is easy to verify the following result.

Proposition 3.20. *Let* $G \subset \mathbf{R}$ *be a* **Z**-*module. Then, either* $G = \kappa\mathbf{Z}$ *for some* $\kappa \in \mathbf{R}$ *or* G *is dense in* **R**.

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