NEIGHBOURHOODS AND ISOTOPIES OF KNOTS IN CONTACT 3-MANIFOLDS

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ABSTRACT. We prove a neighbourhood theorem for arbitrary knots in contact 3-manifolds. As an application we show that two topologically isotopic Legendrian knots in a contact 3-manifold become Legendrian isotopic after suitable stabilisations.

1. INTRODUCTION

For an oriented Legendrian knot K in a 3-dimensional contact manifold (M, ξ), i.e. a knot everywhere tangent to the contact structure ξ, there is a well-defined notion of positive or negative stabilisation. By the Darboux theorem, (M, ξ) is locally diffeomorphic to $\mathbb{R}^3$ with its standard contact structure $\xi_{\mathrm{st}} = \ker(dz + x\, dy)$. In such a neighbourhood, the Legendrian knot K can be represented by its front projection to the yz-plane; the x-coordinate can be recovered from the front as $x = -dz/dy$. In this local picture, the stabilisation $S_{\pm}K$ of K is obtained by adding a zigzag to the front, oriented downwards (resp. upwards) for the positive (resp. negative) stabilisation. Positive and negative stabilisations commute with each other, and we write $S_{+}^{m} S_{-}^{n} K$ for an m-fold positive and n-fold negative stabilisation of K. For more background information see [2] and [4].

The following theorem says that, up to stabilisation, the classification of Legendrian knots is purely topological.

Theorem 1. *If two oriented Legendrian knots K_0 and K_1 in a 3-dimensional contact manifold (M, ξ) are topologically isotopic, one can find Legendrian isotopic stabilisations $S_{+}^{m_0} S_{-}^{n_0} K_0$ and $S_{+}^{m_1} S_{-}^{n_1} K_1$.*

For $(M, \xi) = (\mathbb{R}^3, \xi_{\mathrm{st}})$ this theorem was proved by Fuchs and Tabachnikov [3, Theorem 4.4]. Dymara [1] has suggested to prove the theorem for general (M, ξ) by reducing it to that special case using the Darboux theorem, without providing details. In order to make such an argument precise, one needs considerations very much like those in the proofs of Lemma 6 and Theorem 1 below.

In the present note we give a proof based on convex surface theory and a neighbourhood theorem for arbitrary knots in contact 3-manifolds; this argument does not depend on the result of Fuchs and Tabachnikov.

Remark. For homologically trivial Legendrian knots one can define the Thurston–Bennequin invariant $\mathtt{tb}$ and the rotation number $\mathtt{rot}$ (relative to a choice of Seifert surface). The parity of the sum $\mathtt{tb} + \mathtt{rot}$ is invariant under stabilisation, so the

theorem implies that this parity is constant within any (homologically trivial) knot type. See [4, Remark 4.6.35] for a more general statement of this parity condition.

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2. A NEIGHBOURHOOD THEOREM

Denote the obvious coordinates on the manifold $S^1 \times \mathbb{R}^2$ by θ, x, y. Throughout this note we write S^1 as shorthand for $S^1 \times \{0\} \subset S^1 \times \mathbb{R}^2$.

Lemma 2. *Let α be a contact form defined near $S^1 \subset S^1 \times \mathbb{R}^2$. Then there is a neighbourhood U of S^1 and a smooth function $\lambda \colon U \to \mathbb{R}^+$ with $\lambda|_{S^1} \equiv 1$ and such that along S^1 the Reeb vector field of the contact form $\lambda\alpha|_U$ is transverse to S^1.*

Proof. We make the ansatz $\lambda(\theta, x, y) = 1 + ax + by$, where a, b are real constants that we shall have to choose judiciously. Write $\beta := \lambda\alpha$. Then

$$d\beta = (1 + ax + by)\, d\alpha + a\, dx \wedge \alpha + b\, dy \wedge \alpha.$$

We want to choose $a, b \in \mathbb{R}$ such that $i_{\partial_\theta} d\beta$ does not vanish along S^1. We define three smooth functions on S^1 by

$$\lambda_1 := d\alpha(\partial_\theta, \partial_x)|_{S^1}, \quad \lambda_2 := d\alpha(\partial_\theta, \partial_y)|_{S^1}, \quad \mu := \alpha(\partial_\theta)|_{S^1}.$$

Then

$$i_{\partial_\theta} d\beta|_{S^1} = (\lambda_1(\theta) - a\mu(\theta))\, dx + (\lambda_2(\theta) - b\mu(\theta))\, dy.$$

Since α is a contact form, we have

$$(\lambda_1(\theta), \lambda_2(\theta)) \neq (0, 0) \quad \text{on} \quad \{\theta \in S^1 \colon \mu(\theta) = 0\}.$$

Hence, any point $(a, b) \in \mathbb{R}^2$ not in the image of the map

$$\begin{array}{ccc} \{\theta \in S^1 \colon \mu(\theta) \neq 0\} & \longrightarrow & \mathbb{R}^2 \\ \theta & \longmapsto & (\lambda_1(\theta), \lambda_2(\theta))/\mu(\theta) \end{array}$$

will satisfy our requirements. By Sard's theorem such points exist in abundance. $\square$

The following proposition includes as special cases the neighbourhood theorems for, respectively, Legendrian and transverse knots, cf. [4, Section 2.5].

Proposition 3. *Suppose $\xi_i = \ker \alpha_i$, $i = 1, 2$, are two positive contact structures defined near $S^1 \subset S^1 \times \mathbb{R}^2$ with the property that there is a smooth function $\mu \colon S^1 \to \mathbb{R}^+$ such that $\alpha_1(\partial_\theta)|_{S^1} = \mu\alpha_2(\partial_\theta)|_{S^1}$. Then there is a neighbourhood U of S^1 and a contactomorphism $f \colon (U, \xi_1) \to (f(U), \xi_2)$ equal to the identity on S^1.*

Proof. By extending μ to a smooth positive function on $S^1 \times \mathbb{R}^2$ and replacing α_2 by $\mu\alpha_2$ we may assume that $\alpha_1(\partial_\theta)|_{S^1} = \alpha_2(\partial_\theta)|_{S^1}$. Moreover, the lemma allows us to assume that the Reeb vector field R_i of α_i is transverse to S^1 for $i = 1, 2$.

Then the vector field

$$X_i := \big(\partial_\theta - \alpha_i(\partial_\theta)R_i\big)|_{S^1}$$

is a non-zero section of $\xi_i|_{S^1}$. Choose a section Y_i of $\xi_i|_{S^1}$ linearly independent of X_i and such that R_i, X_i, Y_i constitutes a positive frame of $T(S^1 \times \mathbb{R}^2)|_{S^1}$. Then $R_i, \partial_\theta, Y_i$ is likewise a positive frame of $T(S^1 \times \mathbb{R}^2)|_{S^1}$.

We now find a germ of an orientation-preserving diffeomorphism g near S^1 with the properties

(i) $g|_{S^1} = \mathrm{id}$,
(ii) $Tg(R_1) = R_2$ and $Tg(Y_1) = Y_2$ along S^1.

Then also $Tg(X_1) = X_2$, so α_1 and $\beta_1 := g^*\alpha_2$ are contact forms near S^1 that coincide along S^1. Hence, in a sufficiently small neighbourhood of S^1, we have a 1-parameter family $(1-t)\alpha_1 + t\beta_1$ of contact forms; this homotopy of contact forms is stationary along S^1. Gray stability [4, Theorem 2.2.2] gives us a germ of a diffeomorphism h near S^1 sending $\ker\alpha_1$ to $\ker\beta_1$ and equal to the identity along S^1. The composition $f := g \circ h$ is the desired germ of a diffeomorphism near S^1. $\qquad\square$

Corollary 4. *Any knot K in a 3-dimensional contact manifold (M, ξ) has a neighbourhood U such that $\xi|_U$ is tight.*

Proof. Identify a neighbourhood of $K \subset M$ with a neighbourhood of $S^1 \subset S^1 \times \mathbb{R}^2$ such that K becomes identified with S^1. We continue to write $\xi = \ker\alpha$ for the contact structure in this neighbourhood; the identification of neighbourhoods may be done in such a way that ξ is a positive contact structure near $S^1 \subset S^1 \times \mathbb{R}^2$. Define a smooth function $\mu\colon S^1 \to \mathbb{R}$ by $\mu = \alpha(\partial_\theta)|_{S^1}$.

The 1-form $\alpha_0 = dy - x\,d\theta$ defines the standard positive tight contact structure $\xi_0 = \ker\alpha_0$ on $S^1 \times \mathbb{R}^2$. Now consider the embedding $i\colon S^1 \to S^1 \times \mathbb{R}^2$ given by $\theta \mapsto (\theta, -\mu(\theta), 0)$. Then

$$i^*\alpha_0(\partial_\theta) = \mu = \alpha(\partial_\theta)|_{S^1}.$$

By the preceding proposition there is a neighbourhood of K contactomorphic to a neighbourhood of $i(S^1)$ in the tight contact manifold $(S^1 \times \mathbb{R}^2, \xi_0)$. $\qquad\square$

3. Proof of the isotopy theorem

We first want to prove a local version of Theorem 1 (see Lemma 6 below). We begin with one of the two model situations of such a local isotopy. In $S^1 \times \mathbb{R}^2$ with the standard contact structure $\xi_0 = \ker(dy - x\,d\theta)$ we have for each $s \in \mathbb{R}$ a Legendrian knot $\Lambda_s := S^1 \times \{(0, s)\}$. In the front projection to the (θ, y)-plane, where we think of S^1 as $\mathbb{R}/2\pi\mathbb{Z}$, the knot Λ_s is represented by a horizontal line at level $y = s$ (see Figure 1). We give Λ_s the orientation corresponding to the positive θ-direction.

The annulus

$$A_0 := \{(\theta, x, y) \in S^1 \times \mathbb{R}^2 \colon x^2 + y^2 = 1,\ x \geq 0\}$$

with boundary $\Lambda_1 \sqcup \Lambda_{-1}$ (one of them with reversed orientation) is transverse to the contact vector field $x\partial_x + y\partial_y$ and hence a convex surface in the sense of Giroux [5]. The dividing set of A_0, i.e. the set of points where the contact vector field is tangential to the contact structure, consists of a single circle $A_0 \cap \{y = 0\}$.

Lemma 5. *The Legendrian knots $S_+\Lambda_1$ and $S_+\Lambda_{-1}$ are Legendrian isotopic inside any given neighbourhood of the annulus A_0.*

Remark. It follows from a result of Traynor [7] that no such isotopy exists between the unstabilised knots Λ_1 and Λ_{-1}.

Proof of Lemma 5. The x-coordinate of a point on a Legendrian knot is given as the slope $dy/d\theta$ of the front projection at the corresponding point in the (θ, y)-plane. Hence, the condition that a Legendrian knot be close to the annulus A_0 translates into $y^2 + (dy/d\theta)^2$ being close to 1 for all points on the front projection of the knot. An isotopy of the front of $S_+\Lambda_1$ to that of $S_+\Lambda_{-1}$ via fronts that satisfy this condition is shown in Figure 1. $\qquad\square$
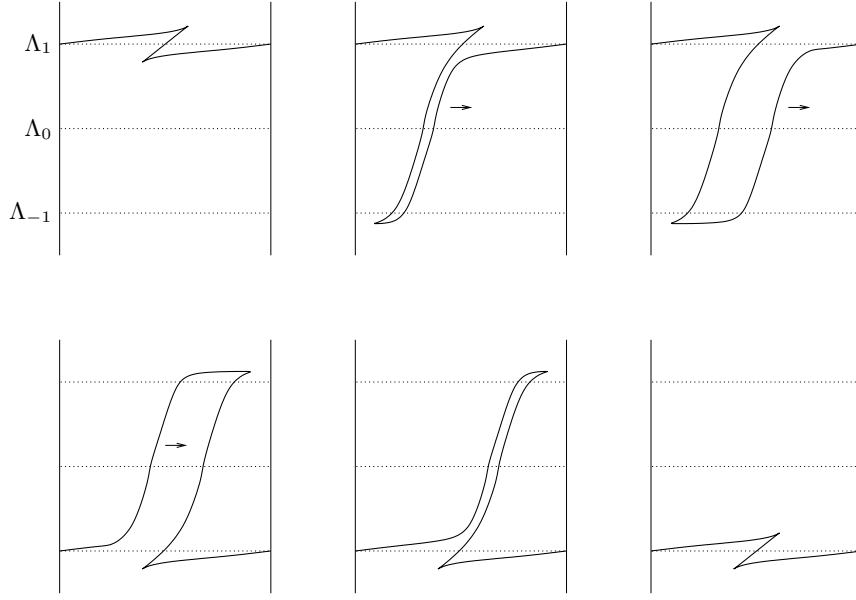
FIGURE 1. Isotoping $S_+(\Lambda_1)$ to $S_+(\Lambda_{-1})$ near A_0.

Lemma 6. *Let ξ be a tight contact structure on $S^1 \times \mathbb{R}^2$, and K_0, K_1 oriented Legendrian knots in $(S^1 \times \mathbb{R}^2, \xi)$ topologically isotopic to S^1. Then one can find Legendrian isotopic stabilisations of K_0 and K_1.*

Proof. Write D_r for the open 2-disc of radius r in $\mathbb{R}^2$, and $\overline{D}_r$ for its closure. Choose $R > 0$ sufficiently large such that K_0 and K_1 are topologically isotopic inside $S^1 \times D_R$. Let K be an oriented Legendrian knot topologically isotopic to $S^1 \times \{(0, 3R)\}$ inside $S^1 \times (\mathbb{R}^2 \setminus \overline{D}_{2R})$. We claim that suitable stabilisations of K_0 and K_1 are Legendrian isotopic to a stabilisation of K, and hence Legendrian isotopic to each other.

The key to proving this claim (for K_0, say) is that, by construction, K and K_0 (one of them with reversed orientation) bound an embedded annulus A in $S^1 \times \mathbb{R}^2$. Beware that K_0 and K_1 do not, in general, bound an annulus; an example is given by the Whitehead link.

Write $t_A(K), t_A(K_0)$ for the twisting of the contact planes along K, K_0, respectively, relative to the framing induced by A. By stabilising K and K_0, if necessary, we may assume that $t_A(K), t_A(K_0) \leq 0$. Then A can be perturbed (relative to its boundary $\partial A = K \sqcup K_0$) into a convex surface, see [6, Proposition 3.1]. We continue to write A for the annulus after this and the following perturbations. If there is a boundary parallel dividing curve on A, then the corresponding boundary component can be destabilised without affecting the convexity of A, see [6, Proposition 3.18].

So we may assume that K and K_0 are connected by a convex annulus A without boundary parallel dividing curves. The Giroux criterion [4, Proposition 4.8.13] tells us that, since ξ is tight, there are no homotopically trivial closed curves in the dividing set of A. Thus, the dividing set consists either of an even number of curves connecting K with K_0, or a collection of simple closed curves parallel to K

and K_0. We now use the Giroux flexibility theorem [5, Proposition II.3.6], cf. [6, Theorem 3.4] and [4, Theorem 4.8.11], to bring the annulus A into standard form.

In the first case we can perturb A such that its characteristic foliation is given by curves parallel to K and K_0; this Legendrian ruling of A defines a Legendrian isotopy between K and K_0.

In the second case, which occurs if $t_A(K) = t_A(K_0) = 0$, we can assume that the characteristic foliation consists of curves going from K to K_0, with K and K_0 Legendrian divides (i.e. curves in the characteristic foliation consisting entirely of singular points, where the contact planes coincide with the tangent planes to A), and one further Legendrian divide between each pair of dividing curves. Then each of the annuli between two adjacent Legendrian divides has a characteristic foliation like our model annulus A_0. Since the characteristic foliation determines the germ of the contact structure near the surface, cf. [4, Theorem 2.5.22], Lemma 5 tells us that the stabilised knots S_+K and S_+K_0 are Legendrian isotopic. $\qquad\square$

Proof of Theorem 1. Let $\phi_t\colon S^1 \to M$, $t \in [0,1]$, be an isotopy of topological embeddings with $\phi_i(S^1) = K_i$ for $i = 0, 1$. By Corollary 4, for each $t \in [0,1]$ there is a neighbourhood U_t of $\phi_t(S^1)$, diffeomorphic to $S^1 \times \mathbb{R}^2$ under a diffeomorphism sending $\phi_t(S^1)$ to S^1, with $\xi|_{U_t}$ tight, and a real number $\varepsilon_t > 0$ such that $\phi_s(S^1) \subset U_t$ for all $s \in (t - \varepsilon_t, t + \varepsilon_t) \cap [0,1]$.

By the Lebesgue lemma on open coverings of compact metric spaces, there is a positive integer N such that for each $j \in \{1, \ldots, N\}$ the interval $[(j-1)/N, j/N]$ is contained in $(t_j - \varepsilon_{t_j}, t_j + \varepsilon_{t_j})$ for some $t_j \in [0,1]$. We abbreviate U_{t_j} to U_j. Notice that $\phi_{j/N}(S^1) \subset U_j \cap U_{j+1}$.

Relabel K_1 as K_N. For $j \in \{1, \ldots, N-1\}$, let K_j be a Legendrian approximation of $\phi_{j/N}(S^1)$ contained in the neighbourhood $U_j \cap U_{j+1}$; such a C^0-close Legendrian approximation exists by [4, Theorem 3.3.1].

By the preceding lemma, applied to the Legendrian knots K_{j-1} and K_j in U_j, suitable stabilisations of K_{j-1} and K_j are Legendrian isotopic, $j = 1, \ldots, N$. It follows that some stabilisation of K_0 is Legendrian isotopic to some stabilisation of K_N (which was the K_1 in the statement of the theorem). $\qquad\square$

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