

Notes on The Feynman Checkerboard Problem

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Abstract

The Feynman checkerboard problem is an interesting path integral approach to the Dirac equation in '1+1' dimensions. I compare two approaches reported in the literature and show how they may be reconciled. Some physical insights may be gleaned from this approach.

1 Introduction

Kauffman and Noyes[4] presented an intriguing derivation of the Feynman checkerboard problem[2]. Of particular value, Kauffman and Noyes provide explicit expressions for the contributions from the various equivalence classes of paths that contribute to the wave function. Explicit evaluation of some simple cases reveals that these expressions are incorrect, unfortunately. Prior to Kauffman and Noyes, Jacobson and Schulman[3] sketched a derivation of the combinatorial factors that occur in the ‘RL’ class of paths discussed in Kauffman and Noyes[4]. By reverse engineering the arguments given in Jacobson and Schulman[3], I was able to write the correct combinatorial factors for the checkerboard problem as developed in Kauffman and Noyes[4]. Section 2 is a précis of Jacobson and Schulman’s constructive procedure[3]. Section 3 shows how to express these results in the notation of Kauffman and Noyes[4]. In Section 4, I show how the propagator for the checkerboard problem may be evaluated for the discrete time step case and the continuum limit. The Appendix contains a graphical representation of all of the possible paths for the specific case of a 3×2 grid so that the combinatorial arguments in the main text may be followed constructively.

2 Jacobson and Schulman’s Argument

For a rectangular grid consisting of r steps of unit length in the ‘R’ direction and l steps of unit length in the ‘L’ direction, a ‘typical’ ‘RL’ path consisting of c corners may be found by noting that for an ‘RL’ path there are exactly $1 + (c - 1)/2$ left turns and exactly $(c - 1)/2$ right turns where the first step must be to the right and the last step must be to the left. Adding the number of left and right turns together, one sees that $1 + (c - 1)/2 + (c - 1)/2 = c$ as it should. Note that, for n steps in a particular direction, there are $n + 1$ grid points, but the first and last grid points are constrained by the equivalence class of the path. Thus, for n steps there are $n - 1$ free grid points at which to place $(c - 1)/2$ turns. As noted above, for the ‘RL’ paths, the last turn must be from right to left so there are $(c - 1)/2$ free turning points among $r - 1$ grid points on the ‘R’ axis and $(c - 1)/2$ free turning points among $l - 1$ grid points on the ‘L’ axis. These considerations allowed Jacobson and Schulman[3] to derive the following count for the number of ‘RL’ paths consisting of c corners

$$N_{RL}(c) = \binom{r - 1}{(c - 1)/2} \binom{l - 1}{(c - 1)/2}. \quad (1)$$

This is related to the expression for N_{RR} or N_{LL} given by Kauffman and Noyes[4] with some important differences. Note the symmetry between l and r in Equation 1. One may verify by explicitly constructing the paths for small dimension grids that the expression for $N_{RL}(c)$ in Equation 1 is in fact the correct one. An example of this procedure for a 3×2 grid is given in the Appendix. Due to the symmetry in the lower argument of the binomial coefficients, one expects that $N_{RL}(c) = N_{LR}(c)$. Explicit construction of the allowed paths for small grids confirms this. In order to understand this result from a geometric perspective, note that each ‘LR’ path may be inferred from a corresponding ‘RL’ path by rotating a given ‘RL’ path about the center of the grid by 180 degrees¹. The reader can construct a few explicit

¹Thanks to Kevin Knuth of the UAlbany Physics Department for pointing this out.

examples of this procedure by considering the diagrams given in the Appendix.

The quantity $(c-1)/2$ is the k index of Kauffman and Noyes[4]. In the notation of Kauffman and Noyes[4]

$$N_{RL}(k) = N_{LR}(k) = C_k^{(r-1)} C_k^{(l-1)}. \quad (2)$$

This expression is seen to be closer to the expressions for $N_{RR}(k)$ and $N_{LL}(k)$ given in Kauffman and Noyes[4]. Equation 2 properly counts the number of allowed paths for both the ‘RL’ and ‘LR’ classes. It is related to the ψ_0 function of Kauffman and Noyes[4]. Similar arguments may be employed to derive the appropriate expressions for $N_{RR}(k) = C_{k+1}^{(r-1)} C_k^{(l-1)}$ and $N_{LL}(k) = C_k^{(r-1)} C_{k+1}^{(l-1)}$. There is an error in the expressions for $N_{RR}(k)$ and $N_{LL}(k)$ in Jacobson and Schulman[3], although the correction is vanishingly small in the large grid, large number of corners limit. The expressions given here may be verified by explicitly counting paths for small grids. An example of this procedure is given in the Appendix for a 3×2 grid. It is seen that $N_{RR}(k)$ is related to the ψ_R wave function and $N_{LL}(k)$ is related to the ψ_L wave function of Kauffman and Noyes[4].

3 Kauffman and Noyes’ Wave Functions

In order to write a correct wave function which properly accounts for the number of paths in a given equivalence class for a grid size $r \times l$, it is only necessary to make the following identifications based on the wavefunctions given in Kauffman and Noyes[4]

$$\begin{aligned} \psi_0 &= \sum_{\text{oddc} \geq 1} (-1)^{(c-1)/2} C_{(c-1)/2}^{(r-1)} C_{(c-1)/2}^{(l-1)} \\ \psi_L &= \sum_{\text{evenc} > 0} (-1)^{(c/2)-1} C_{(c/2)-1}^{(r-1)} C_{(c/2)}^{(l-1)} \\ \psi_R &= \sum_{\text{evenc} > 0} (-1)^{(c/2)-1} C_{(c/2)}^{(r-1)} C_{(c/2)-1}^{(l-1)}. \end{aligned}$$

Here $C_{(c-1)/2}^{(r-1)}$ for example is a ‘generalized binomial coefficient’ in the terminology of Kauffman and Noyes[4]. This generalized binomial coefficient may be written as[4] $C_{(c-1)/2}^{(r-1)} \equiv \frac{(r-1-\Delta)!}{((c-1)/2)!(r-1-\Delta-(c-1)/2)!}$.

Note that c ranges over odd values for ψ_0 and even values for ψ_R and ψ_L . For future applications, it is useful to redefine the summation index so that it runs over all non-negative integers regardless of class. For ψ_0 , substitute $(c-1)/2 \rightarrow k$. For ψ_L and ψ_R substitute $(c/2) - 1 \rightarrow k$. With this choice of summation index one finds

$$\psi_0 = \sum_{k \geq 0} (-1)^k \frac{(r-1)!}{k!(r-1-k)!} \frac{(l-1)!}{k!(l-1-k)!} \quad (3)$$

$$\psi_L = \sum_{k \geq 0} (-1)^k \frac{(r-1)!}{k!(r-1-k)!} \frac{(l-1)!}{(k+1)!(l-1-(k+1))!} \quad (4)$$

$$\psi_R = \sum_{k \geq 0} (-1)^k \frac{(r-1)!}{(k+1)!(r-1-(k+1))!} \frac{(l-1)!}{k!(l-1-k)!} \quad (5)$$

As noted above, the summation index k in Equations 3–5 is now over all non-negative integers regardless of path class. With the definitions given in Equations 3–5, the following derivative identities, related to similar expressions in Kauffman and Noyes[4] may be written down. These derivative identities are useful for constructing the discretized version of the ‘1+1’ Dirac equation.

$$\begin{aligned}\frac{\partial\psi_R}{\partial r} &= \psi_0 \\ \frac{\partial\psi_0}{\partial r} &= -\psi_L \\ \frac{\partial\psi_L}{\partial l} &= \psi_0 \\ \frac{\partial\psi_0}{\partial l} &= -\psi_R\end{aligned}$$

In order to compare to the one-dimensional Dirac equation, appropriate linear combinations of Equations 3–5 are needed. Note that the phase of the Dirac equation given in Kauffman and Noyes[4] differs from that used in Jacobson and Schulman[3]. The latter phase convention is more commonly used and will also be used here, as this leads to a propagator of the following form[3]

$$K_{\beta\alpha} = \lim_{n \rightarrow \infty} \sum_{c \geq 0} N_{\beta\alpha}(c)(i\epsilon m_0)^c \quad (6)$$

where $\alpha, \beta \in \{L, R\}$ and c is the number of path changes and n is the number of steps. This is the form of the propagator given by Feynman in his original formulation of the checkerboard problem[2]. This form is chosen for the work presented here in order to facilitate comparison with previously published expressions. As Kauffman and Noyes note[4], other choices are possible for, *e.g.*, real-valued solutions of the Dirac equation.

Using the phase convention implied by Equation 6, one finds

$$\begin{pmatrix} i\psi_2 \\ i\psi_1 \end{pmatrix} = \begin{pmatrix} \partial\psi_1/\partial r \\ \partial\psi_2/\partial l \end{pmatrix}. \quad (7)$$

The phase convention implied by Equation 7 leads to the following choices for ψ_1 and ψ_2 :

$$\psi_1 = i\psi_0 - \psi_R \quad (8)$$

$$\psi_2 = i\psi_0 - \psi_L. \quad (9)$$

Equations 8 and 9 are correctly phased with respect to the propagator defined by Equation 6.

4 Constructing the Propagator

In order to test whether the phases and wavefunctions chosen here are consistent with previously published results, it is useful to perform a consistency check. For this reason, the propagator for the checkerboard problem will be derived with the wavefunctions defined

by Equations 3–5. In the limit as the number of corners in the path goes to infinity the continuum limit for the propagator may be derived. Observe (cf. Equation 6)

$$K_{RL} = K_{LR} = \lim_{n \rightarrow \infty} \sum_{c \geq 0} N_{RL}(c) (i\epsilon m_0)^c \quad (10)$$

where c is the number of corners and m_0 is the particle mass. Note that the path specific summation index c is used here instead of the generalized k index. This choice facilitates comparison with Jacobson and Schulman’s derivation[3]. In a system of units where \hbar and the speed of light are unity, m_0 has units of 1/length. Recall that for ‘RL’ or ‘LR’ paths c is odd, N_{RL} gives the number of paths connecting the endpoints of the path with c corners and n is the total number of steps in the path. It is useful to define the subsidiary quantity m such that $r = (n + m)/2$ and $l = (n - m)/2$. Following Jacobson and Schulman’s exposition[3], note that

$$rl = \frac{n^2}{4} \left(1 - \left(\frac{m}{n} \right)^2 \right). \quad (11)$$

Defining the quantity $\gamma = \sqrt{1 - (m/n)^2}$, the product rl in Equation 11 may be written as $rl = (n/(2\gamma))^2$. The quantity m/n may be interpreted as a dimensionless velocity, as $n > 0$ and m can have either sign. Note that $m = 0$ corresponds to a particle at rest. If the path difference $b - a$ is traversed in a time $t_b - t_a$ then $(m/n)^2 = (b - a)^2 / (t_b - t_a)^2$ where time is measured in light meters, say. Performing time slicing in the usual way, note that $\epsilon = (t_b - t_a)/n$. For a large number of time steps such that $c/n \xrightarrow{n \rightarrow \infty} 0$ the combinatorial factors in $N_{RL}(c)$ may be evaluated asymptotically. The steps involved are similar to the ones used to obtain a Poisson distribution from a binomial distribution[5]. Note that

$$N_{RL}(c) = \binom{r-1}{(c-1)/2} \binom{l-1}{(c-1)/2}. \quad (12)$$

When $r, l \gg c$ in Equation 12 the ratios of factorials appearing in the binomial coefficients may be approximated as

$$\begin{aligned} N_{RL}(c) &= \frac{(r-1)!}{[(c-1)/2]! [r-1-(c-1)/2]!} \frac{(l-1)!}{[(c-1)/2]! [l-1-(c-1)/2]!} \\ &\approx \frac{r^{(c-1)/2} l^{(c-1)/2}}{[(c-1)/2]! [(c-1)/2]!} \\ &= \frac{(lr)^{(c-1)/2}}{[(c-1)/2]!^2}. \end{aligned} \quad (13)$$

The asymptotic propagator using Equation 13 may thus be written

$$K_{RL} = (i\epsilon m_0) \sum_c (i\epsilon m_0)^{c-1} \left(\frac{n}{2\gamma} \right)^{(c-1)}.$$

Using the definition of ϵ and introducing the abbreviation $z = m_0(t_b - t_a)/\gamma$, the expression for the propagator may be put in to the following form

$$K_{RL} = (i\epsilon m_0) \sum_{k=0}^{\infty} (-1)^k (z/2)^{2k} / (k!)^2 \quad (14)$$

where $k = (c - 1)/2$. Note that[1]

$$J_\alpha(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(s + \alpha + 1)} \left(\frac{x}{2}\right)^{2s+\alpha} \quad (15)$$

where $\Gamma(s + \alpha + 1) = (s + \alpha)!$ for integer α such that $s + \alpha \geq 0$. Using Equation 15, the asymptotic form of the K_{RL} propagator may be written.

$$K_{RL} = (i\epsilon m_0) J_0(z) \quad (16)$$

A similar procedure can be used to evaluate the K_{RR} and K_{LL} contributions to the propagator. For K_{RR} paths, one has

$$K_{RR} = \lim_{n \rightarrow \infty} \sum_c \binom{r-1}{c/2} \binom{l-1}{c/2-1} (i\epsilon m_0)^c$$

where the sum is over all even integers greater than zero. The expression for K_{LL} may be found from the substitution $r \rightarrow l \rightarrow r$. Using the same asymptotic analysis as before, one finds

$$K_{RR} \approx -2r\epsilon m_0 \frac{\gamma}{n} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+1}}{k!(k+1)!},$$

where the summation index $k = (c/2) - 1$. Using the definition of J_α in Equation 15, this may be rewritten

$$K_{RR} \approx -2r\epsilon m_0 \frac{\gamma}{n} J_1(z).$$

Recalling that $r = (n + m)/2$ and $\gamma = 1/\sqrt{1 - (m/n)^2}$, and defining $t = n\epsilon$ and $x = m\epsilon$, where $|x| < t$, one may show

$$K_{RR} \rightarrow -\epsilon m_0 \frac{t+x}{\tau} J_1(z), \quad (17)$$

where $\tau^2 = t^2 - x^2$. Making the substitution $r \rightarrow l \rightarrow r$ one may also write down

$$K_{LL} \rightarrow -\epsilon m_0 \frac{t-x}{\tau} J_1(z), \quad (18)$$

Noting that the number of corners is even for ‘RR’ or ‘LL’ paths and odd for ‘RL’ or ‘LR’ paths, one may write down the continuum limit for the propagator[3] by dividing through by 2ϵ . Combining Equations 16, 17 and 18 one may write down the continuum propagator in a concise matrix form as follows

$$K = \begin{pmatrix} K_{RR} & K_{RL} \\ K_{LR} & K_{LL} \end{pmatrix} = \frac{m_0}{2} \begin{pmatrix} -\frac{t+x}{\tau} J_1(z) & iJ_0(z) \\ iJ_0(z) & -\frac{t-x}{\tau} J_1(z) \end{pmatrix} \quad (19)$$

Equation 19 reproduces Jacobson and Schulman’s result[3]. This form is not strictly necessary, but it does permit an efficient summary of the results.

Acknowledgements

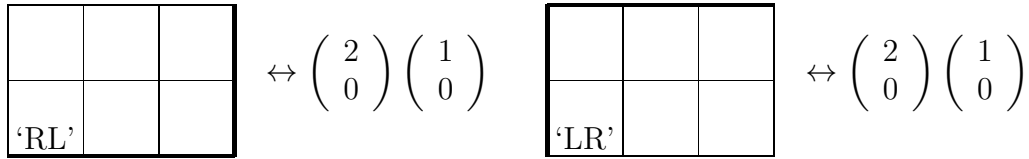
KAE thanks the University at Albany for partial support of this work through its Faculty Research Award Program. KAE also thanks ACERT (NIH NCRR P41 RR016292) for the use of its computational resources.

References

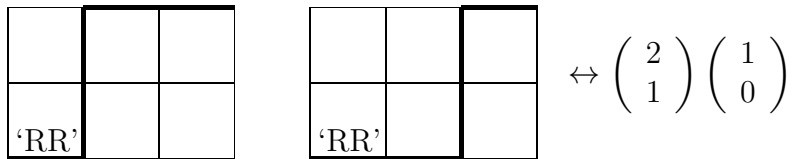
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A Explicit Construction on a 3×2 Grid

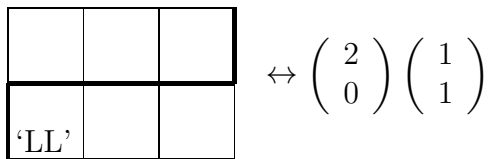
In order to get a feel for the practical aspects of the combinatorial factors used here, consider the problem of enumerating all paths with one, two and three corners on a 3×2 grid. Here are, respectively, the 'RL' and 'LR' paths with one corner



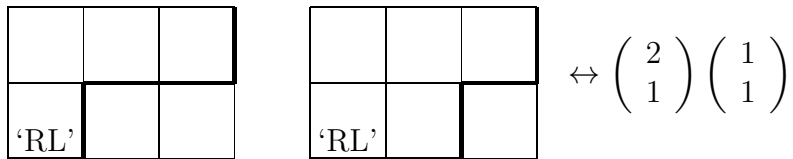
Here are the 'RR' paths with two corners



Here is the unique 'LL' path with 2 corners



Here are the 'RL' paths with 3 corners



Here are 'LR' paths with three corners

