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ON UNIT ROOT FORMULAS FOR TORIC EXPONENTIAL SUMS

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Abstract. Starting from a classical generating series for Bessel functions due to Schlömilch[4], we use Dwork's relative dual theory to broadly generalize unit-root results of Dwork[3] on Kloosterman sums and Sperber[6] on hyperkloosterman sums. In particular, we express the (unique) p-adic unit root of an arbitrary exponential sum on the torus $\mathbf{T}^n$ in terms of special values of the p-adic analytic continuation of a ratio of A-hypergeometric functions. In contrast with the earlier works, we use noncohomological methods and obtain results that are valid for arbitrary exponential sums without any hypothesis of nondegeneracy.

1. Introduction

The starting point for this work is the classical generating series

$$\exp\frac{1}{2}(\Lambda X - \Lambda/X) = \sum_{i\in\mathbf{Z}} J_i(\Lambda)X^i$$

for the Bessel functions $\{J_i(\Lambda)\}_{i\in\mathbf{Z}}$ due to Schlömilch[4] that was the foundation for his treatment of Bessel functions (see [7, page 14]). Suitably normalized, it also played a fundamental role in Dwork's construction[3] of p-adic cohomology for $J_0(\Lambda)$. Our realization that the series itself (suitably normalized) could be viewed as a distinguished element in Dwork's relative dual complex led us to the present generalization.

Let $A \subseteq \mathbf{Z}^n$ be a finite subset that spans $\mathbf{R}^n$ as real vector space and set

$$f_\Lambda(X) = \sum_{a\in A} \Lambda_a X^a \in \mathbf{Z}[\{\Lambda_a\}_{a\in A}][X_1^{\pm 1}, \dots, X_n^{\pm 1}],$$

where the Λ_a and the X_i are indeterminates and where $X^a = X_1^{a_1}\cdots X_n^{a_n}$ for $a = (a_1, \dots, a_n)$. Let $\mathbf{F}_q$ be the finite field of $q = p^\epsilon$ elements, p a prime, and let $\bar{\mathbf{F}}_q$ be its algebraic closure. For each $\bar\lambda = (\bar\lambda_a)_{a\in A} \in (\bar{\mathbf{F}}_q)^{|A|}$, let

$$f_{\bar\lambda}(X) = \sum_{a\in A} \bar\lambda_a X^a \in \mathbf{F}_q(\bar\lambda)[X_1^{\pm 1}, \dots, X_n^{\pm 1}],$$

a regular function on the n-torus $\mathbf{T}^n$ over $\mathbf{F}_q(\bar\lambda)$. Fix a nontrivial additive character $\Theta : \mathbf{F}_q \to \mathbf{Q}_p(\zeta_p)$ and let $\Theta_{\bar\lambda}$ be the additive character $\Theta_{\bar\lambda} = \Theta \circ \mathrm{Tr}_{\mathbf{F}_q(\bar\lambda)/\mathbf{F}_q}$ of the field $\mathbf{F}_q(\bar\lambda)$. For each positive integer l, let $\mathbf{F}_q(\bar\lambda, l)$ denote the extension of degree l

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of $\mathbf{F}_q(\bar{\lambda})$ and define an exponential sum

$$S_l = S_l(f_{\bar{\lambda}}, \Theta_{\bar{\lambda}}, \mathbf{T}^n) = \sum_{x\in\mathbf{T}^n(\mathbf{F}_q(\bar{\lambda},l))} \Theta_{\bar{\lambda}} \circ \mathrm{Tr}_{\mathbf{F}_q(\bar{\lambda},l)/\mathbf{F}_q(\bar{\lambda})}(f_{\bar{\lambda}}(x)).$$

The associated L-function is

$$L(f_{\bar{\lambda}};T) = L(f_{\bar{\lambda}}, \Theta_{\bar{\lambda}}, \mathbf{T}^n; T) = \exp\biggl(\sum_{l=1}^{\infty} S_l \frac{T^l}{l}\biggr).$$

It is well-known that $L(f_{\bar{\lambda}};T) \in \mathbf{Q}(\zeta_p)(T)$ and that its reciprocal zeros and poles are algebraic integers. We note that among these reciprocal zeros and poles there must be at least one p-adic unit: if $\mathbf{F}_q(\bar{\lambda})$ has cardinality q^κ, then S_l is the sum of $(q^{\kappa l}-1)^n$ p-th roots of unity, so S_l itself is a p-adic unit for every l. On the other hand, a simple consequence of the Dwork trace formula will imply (see Section 3) that there is at most a single unit root, and it must occur amongst the reciprocal zeros (as opposed to the reciprocal poles) of $L(f_{\bar{\lambda}};T)^{(-1)^{n+1}}$. We denote this unit root by $u(\bar{\lambda})$. It is the goal of this work to exhibit an explicit p-adic analytic formula for $u(\bar{\lambda})$ in terms of certain A-hypergeometric functions.

Consider the series

$$\exp f_\Lambda(X) = \prod_{a\in A} \exp(\Lambda_a X^a) \tag{1.1}$$
$$= \sum_{i\in\mathbf{Z}^n} F_i(\Lambda) X^i$$

where the $F_i(\Lambda)$ lie in $\mathbf{Q}[[\Lambda]]$. Explicitly, one has

$$F_i(\Lambda) = \sum_{\substack{u=(u_a)_{a\in A}\\ \sum_{a\in A} u_a a = i}} \frac{\Lambda^u}{\prod_{a\in A}(u_a!)}. \tag{1.2}$$

The A-hypergeometric system with parameter $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{C}^n$ (where $\mathbf{C}$ denotes the complex numbers) is the system of partial differential equations consisting of the operators

$$\Box_\ell = \prod_{\ell_a>0} \biggl(\frac{\partial}{\partial \Lambda_a}\biggr)^{\ell_a} - \prod_{\ell_a<0} \biggl(\frac{\partial}{\partial \Lambda_a}\biggr)^{-\ell_a}$$

for all $\ell = (\ell_a)_{a\in A} \in \mathbf{Z}^{|A|}$ satisfying $\sum_{a\in A} \ell_a a = 0$ and the operators

$$Z_j = \sum_{a\in A} a_j \Lambda_a \frac{\partial}{\partial \Lambda_a} - \alpha_j$$

for $a = (a_1, \dots, a_n) \in A$ and $j = 1, \dots, n$. Using Equations (1.1) and (1.2), it is straightforward to check that for $i \in \mathbf{Z}^n$, $F_i(\Lambda)$ satisfies the A-hypergeometric system with parameter i.

Fix π satisfying $\pi^{p-1} = -p$ and $\Theta(1) \equiv \pi \pmod{\pi^2}$. It follows from Equation (1.2) that the $F_i(\pi\Lambda)$ converge p-adically for all Λ satisfying $|\Lambda_a| < 1$ for all $a \in A$. Let $\mathcal{F}(\Lambda) = F_0(\pi\Lambda)/F_0(\pi\Lambda^p)$. The main result of this paper is the following statement. Note that we make no restriction (such as nondegeneracy) on the choice of $\bar{\lambda} \in (\bar{\mathbf{F}}_q)^{|A|}$.

Theorem 1.3. *The series $\mathcal{F}(\Lambda)$ converges p-adically for $|\Lambda_a| \leq 1$ for all $a \in A$ and the unit root of $L(f_{\bar{\lambda}}; T)$ is given by*

$$u(\bar{\lambda}) = \mathcal{F}(\lambda)\mathcal{F}(\lambda^p)\mathcal{F}(\lambda^{p^2})\cdots\mathcal{F}(\lambda^{p^{\epsilon d(\bar{\lambda})-1}}),$$

where λ denotes the Teichmüller lifting of $\bar{\lambda}$ and $d(\bar{\lambda}) = [\mathbf{F}_q(\bar{\lambda}) : \mathbf{F}_q]$.

2. Analytic continuation

We begin by proving the analytic continuation of the function $\mathcal{F}$ defined in the introduction.

Let $C \subseteq \mathbf{R}^n$ be the real cone generated by the elements of A and let $\Delta \subseteq \mathbf{R}^n$ be the convex hull of the set $A \cup \{(0, \ldots, 0)\}$. Put $M = C \cap \mathbf{Z}^n$. For $\nu \in M$, define the *weight* of ν, $w(\nu)$, to be the least nonnegative real (hence rational) number such that $\nu \in w(\nu)\Delta$. There exists $D \in \mathbf{Z}_{>0}$ such that $w(\nu) \in \mathbf{Q}_{\geq 0} \cap \mathbf{Z}[1/D]$. The weight function w is easily seen to have the following properties:

(i) $w(\nu) \geq 0$ and $w(\nu) = 0$ if and only if $\nu = 0$,

(ii) $w(c\nu) = cw(\nu)$ for $c \in \mathbf{Z}_{\geq 0}$,

(iii) $w(\nu + \mu) \leq w(\nu) + w(\mu)$ with equality holding if and only if ν and μ are cofacial, that is, ν and μ lie in a cone over the same closed face of Δ.

(iv) If $\dim \Delta = n$, let $\{\ell_i\}_{i=1}^N$ be linear forms such that the codimension-one faces of Δ not containing the origin lie in the hyperplanes $\{\ell_i = 1\}_{i=1}^N$. Then $w(\nu) = \max\{\ell_i(\nu)\}_{i=1}^N$.

Let Ω be a finite extension of $\mathbf{Q}_p$ containing π and an element $\tilde{\pi}$ satisfying $\operatorname{ord} \tilde{\pi} = (p-1)/p^2$ (we always normalize the valuation so that $\operatorname{ord} p = 1$). Put

$$R = \left\{ \xi(\Lambda) = \sum_{\nu \in (\mathbf{Z}_{\geq 0})^{|A|}} c_\nu \Lambda^\nu \mid c_\nu \in \Omega \text{ and } \{|c_\nu|\}_\nu \text{ is bounded} \right\},$$

$$R' = \left\{ \xi(\Lambda) = \sum_{\nu \in (\mathbf{Z}_{\geq 0})^{|A|}} c_\nu \Lambda^\nu \mid c_\nu \in \Omega \text{ and } c_\nu \to 0 \text{ as } \nu \to \infty \right\}.$$

Equivalently, R (resp. R') is the ring of formal power series in $\{\Lambda_a\}_{a \in A}$ that converge on the open unit polydisk in $\Omega^{|A|}$ (resp. the closed unit polydisk in $\Omega^{|A|}$). Define a norm on R by setting $|\xi(\Lambda)| = \sup_\nu\{|c_\nu|\}$. Both R and R' are complete in this norm. Note that (1.2) implies that the coefficients $F_i(\pi\Lambda)$ of $\exp \pi f_\Lambda(X)$ belong to R.

Let S be the set

$$S = \left\{ \xi(\Lambda, X) = \sum_{\mu \in M} \xi_\mu(\Lambda) \tilde{\pi}^{-w(\mu)} X^{-\mu} \mid \xi_\mu(\Lambda) \in R \text{ and } \{|\xi_\mu(\Lambda)|\}_\mu \text{ is bounded} \right\}.$$

Let S' be defined analogously with the conditions "$\xi_\mu(\Lambda) \in R$" replaced by "$\xi_\mu(\Lambda) \in R'$". Define a norm on S by setting

$$|\xi(\Lambda, X)| = \sup_\mu\{|\xi_\mu(\Lambda)|\}.$$

Both S and S' are complete under this norm.

Define $\theta(t) = \exp(\pi(t - t^p)) = \sum_{i=0}^{\infty} b_i t^i$. One has (Dwork[1, Section 4a)])

$$\operatorname{ord} b_i \geq \frac{i(p-1)}{p^2}. \tag{2.1}$$

Let

$$F(\Lambda, X) = \prod_{a \in A} \theta(\Lambda_a X^a) = \sum_{\mu \in M} B_\mu(\Lambda) X^\mu.$$

Lemma 2.2. *One has $B_\mu(\Lambda) \in R'$ and $|B_\mu(\Lambda)| \leq |\tilde{\pi}|^{w(\mu)}$.*

Proof. From the definition,

$$B_\mu(\Lambda) = \sum_{\nu \in (\mathbf{Z}_{\geq 0})^{|A|}} B_\nu^{(\mu)} \Lambda^\nu,$$

where

$$B_\nu^{(\mu)} = \begin{cases} \prod_{a \in A} b_{\nu_a} & \text{if } \sum_{a \in A} \nu_a a = \mu, \\ 0 & \text{if } \sum_{a \in A} \nu_a a \neq \mu. \end{cases}$$

It follows from (2.1) that $B_\nu^{(\mu)} \to 0$ as $\nu \to \infty$, which shows that $B_\mu(\Lambda) \in R'$. We have

$$\operatorname{ord} B_\nu^{(\mu)} \geq \sum_{a \in A} \operatorname{ord} b_{\nu_a} \geq \sum_{a \in A} \frac{\nu_a(p-1)}{p^2} \geq w(\mu) \frac{p-1}{p^2},$$

which implies $|B_\mu(\Lambda)| \leq |\tilde{\pi}|^{w(\mu)}$. □

By the proof of Lemma 2.2, we may write $B_\nu^{(\mu)} = \tilde{\pi}^{w(\mu)} \tilde{B}_\nu^{(\mu)}$ with $|\tilde{B}_\nu^{(\mu)}| \leq 1$. We may then write $B_\mu(\Lambda) = \tilde{\pi}^{w(\mu)} \tilde{B}_\mu(\Lambda)$ with $\tilde{B}_\mu(\Lambda) = \sum_\nu \tilde{B}_\nu^{(\mu)} \Lambda^\nu$ and $|\tilde{B}_\mu(\Lambda)| \leq 1$. Let

$$\xi(\Lambda, X) = \sum_{\nu \in M} \xi_\nu(\Lambda) \tilde{\pi}^{-w(\nu)} X^{-\nu} \in S.$$

We claim that the product $F(\Lambda, X)\xi(\Lambda^p, X^p)$ is well-defined. Formally we have

$$F(\Lambda, X)\xi(\Lambda^p, X^p) = \sum_{\rho \in \mathbf{Z}^n} \zeta_\rho(\Lambda) X^{-\rho},$$

where

$$\zeta_\rho(\Lambda) = \sum_{\substack{\mu, \nu \in M \\ \mu - p\nu = -\rho}} \tilde{\pi}^{w(\mu) - w(\nu)} \tilde{B}_\mu(\Lambda) \xi_\nu(\Lambda^p). \tag{2.3}$$

To prove convergence of this series, we need to show that $w(\mu) - w(\nu) \to \infty$ as $\nu \to \infty$. By property (iv) of the weight function, for a given $\nu \in M$ we may choose a linear form ℓ (depending on ν) for which $w(\nu) = \ell(\nu)$ while $w(\mu) \geq \ell(\mu)$. Since $\mu = p\nu - \rho$, we get

$$w(\mu) - w(\nu) \geq \ell(\mu - \nu) = \ell((p-1)\nu) - \ell(\rho) = (p-1)w(\nu) - \ell(\rho). \tag{2.4}$$

As $\nu \to \infty$, $(p-1)w(\nu) \to \infty$ while $\ell(\rho)$ takes values in a finite set of rational numbers (there are only finitely many possibilities for ℓ). This gives the desired result.

For a formal series $\sum_{\rho \in \mathbf{Z}^n} \zeta_\rho(\Lambda) X^{-\rho}$ with $\zeta_\rho(\Lambda) \in \Omega[[\Lambda]]$, define

$$\gamma'\biggl(\sum_{\rho \in \mathbf{Z}^n} \zeta_\rho(\Lambda) X^{-\rho} \biggr) = \sum_{\rho \in M} \zeta_\rho(\Lambda) X^{-\rho}$$

and define for $\xi(\Lambda, X) \in S$

$$\begin{aligned}\alpha^*(\xi(\Lambda, X)) &= \gamma'(F(\Lambda, X)\xi(\Lambda^p, X^p)) \\ &= \sum_{\rho \in M} \zeta_\rho(\Lambda) X^{-\rho}.\end{aligned}$$

For $\rho \in M$ put $\eta_\rho(\Lambda) = \tilde{\pi}^{w(\rho)}\zeta_\rho(\Lambda)$, so that

$$\alpha^*(\xi(\Lambda, X)) = \sum_{\rho \in M} \eta_\rho(\Lambda)\tilde{\pi}^{-w(\rho)} X^{-\rho} \tag{2.5}$$

with

$$\eta_\rho(\Lambda) = \sum_{\substack{\mu,\nu \in M \\ \mu - p\nu = \rho}} \tilde{\pi}^{w(\rho)+w(\mu)-w(\nu)} \tilde{B}_\mu(\Lambda)\xi_\nu(\Lambda^p). \tag{2.6}$$

Since $w(\rho) \geq \ell(\rho)$ for $\rho \in M$, Equation (2.4) implies that

$$w(\rho) + w(\mu) - w(\nu) \geq (p-1)w(\nu), \tag{2.7}$$

so by Equation (2.6), $|\eta_\rho(\Lambda)| \leq |\xi(\Lambda, X)|$ for all $\rho \in M$. This shows $\alpha^*(\xi(\Lambda, X)) \in S$ and

$$|\alpha^*(\xi(\Lambda, X))| \leq |\xi(\Lambda, X)|.$$

Furthermore, this argument also shows that $\alpha^*(S') \subseteq S'$.

Lemma 2.8. *If $|\xi_0(\Lambda)| \leq |\tilde{\pi}|^{(p-1)/D}$, then $|\alpha^*(\xi(\Lambda, X))| \leq |\tilde{\pi}|^{(p-1)/D}|\xi(\Lambda, X)|$.*

Proof. This follows immediately from Equations (2.6) and (2.7) since $w(\nu) \geq 1/D$ for $\nu \neq 0$. □

From Equation (2.6), we have

$$\eta_0(\Lambda) = \sum_{\nu \in M} \tilde{B}_{p\nu}(\Lambda)\xi_\nu(\Lambda^p)\tilde{\pi}^{(p-1)w(\nu)}. \tag{2.9}$$

Note that $\tilde{B}_0(\Lambda) = B_0(\Lambda) \equiv 1 \pmod{\tilde{\pi}}$ since $\operatorname{ord} b_i > 0$ for all $i > 0$ implies $\operatorname{ord} B_\nu^{(0)} > 0$ for all $\nu \neq 0$. Thus $B_0(\Lambda)$ is an invertible element of R'. The following lemma is then immediate from Equation (2.9).

Lemma 2.10. *If $\xi_0(\Lambda)$ is an invertible element of R (resp. R'), then so is $\eta_0(\Lambda)$.*

Put

$$T = \{\xi(\Lambda, X) \in S \mid |\xi(\Lambda, X)| \leq 1 \text{ and } \xi_0(\Lambda) = 1\}$$

and put $T' = T \cap S'$. Using the notation of Equation (2.5), define $\beta : T \to T$ by

$$\beta(\xi(\Lambda, X)) = \frac{\alpha^*(\xi(\Lambda, X))}{\eta_0(\Lambda)}.$$

Note that $\beta(T') \subseteq T'$.

Proposition 2.11. *The operator β is a contraction mapping on the complete metric space T. More precisely, if $\xi^{(1)}(\Lambda, X), \xi^{(2)}(\Lambda, X) \in T$, then*

$$|\beta(\xi^{(1)}(\Lambda, X)) - \beta(\xi^{(2)}(\Lambda, X))| \leq |\tilde{\pi}|^{(p-1)/D}|\xi^{(1)}(\Lambda, X) - \xi^{(2)}(\Lambda, X)|.$$

Proof. We have (in the obvious notation)

$$\begin{aligned}\beta(\xi^{(1)}(\Lambda,X))-\beta(\xi^{(2)}(\Lambda,X)) &= \frac{\alpha^*(\xi^{(1)}(\Lambda,X))}{\eta_0^{(1)}(\Lambda)}-\frac{\alpha^*(\xi^{(2)}(\Lambda,X))}{\eta_0^{(2)}(\Lambda)}\\ &= \frac{\alpha^*(\xi^{(1)}(\Lambda,X)-\xi^{(2)}(\Lambda,X))}{\eta_0^{(1)}(\Lambda)}\\ &\quad -\alpha^*(\xi^{(2)}(\Lambda,X))\frac{\eta_0^{(1)}(\Lambda)-\eta_0^{(2)}(\Lambda)}{\eta_0^{(1)}(\Lambda)\eta_0^{(2)}(\Lambda)}.\end{aligned}$$

Since $\eta_0^{(1)}(\Lambda)-\eta_0^{(2)}(\Lambda)$ is the coefficient of X^0 in $\alpha^*(\xi^{(1)}(\Lambda,X)-\xi^{(2)}(\Lambda,X))$, we have

$$|\eta_0^{(1)}(\Lambda)-\eta_0^{(2)}(\Lambda)|\leq|\alpha^*(\xi^{(1)}(\Lambda,X)-\xi^{(2)}(\Lambda,X))|.$$

And since the coefficient of X^0 in $\xi^{(1)}(\Lambda,X)-\xi^{(2)}(\Lambda,X)$ equals 0, the proposition follows from Lemma 2.8. □

Remark: Proposition 2.11 implies that β has a unique fixed point in T. And since β is stable on T', that fixed point must lie in T'. Let $\xi(\Lambda,X)\in T'$ be the unique fixed point of β. The equation $\beta(\xi(\Lambda,X))=\xi(\Lambda,X)$ is equivalent to the equation

$$\alpha^*(\xi(\Lambda,X))=\eta_0(\Lambda)\xi(\Lambda,X).$$

Since α^* is stable on S', it follows that

$$\eta_0(\Lambda)\xi_\mu(\Lambda)\in R'\quad\text{for all }\mu\in M. \tag{2.12}$$

In particular, since $\xi_0(\Lambda)=1$, we have $\eta_0(\Lambda)\in R'$.

Put $C_0=C\cap(-C)$, the largest subspace of $\mathbf{R}^n$ contained in C, and put $M_0=\mathbf{Z}^n\cap C_0$, a subgroup of M. For a formal series $\sum_{\mu\in\mathbf{Z}^n}c_\mu(\Lambda)X^\mu$ with $c_\mu(\Lambda)\in\Omega[[\Lambda]]$ we define

$$\gamma\biggl(\sum_{\mu\in\mathbf{Z}^n}c_\mu(\Lambda)X^\mu\biggr)=\sum_{\mu\in M_0}c_\mu(\Lambda)X^\mu$$

and set

$$\zeta(\Lambda,X)=\gamma(\exp(\pi f_\Lambda(X))).$$

Of course, when the origin is an interior point of Δ, then $M_0=\mathbf{Z}^n$ and $\zeta(\Lambda,X)=\exp(\pi f_\Lambda(X))$. In any case, the coefficients of $\zeta(\Lambda,X)$ belong to R.

Since $\exp(\pi f_\Lambda(X))=\prod_{a\in A}\exp(\pi\Lambda_aX^a)$, we can expand this product to get

$$\begin{aligned}\zeta(\Lambda,X)&=\gamma\biggl(\prod_{a\in A}\sum_{\nu_a=0}^\infty\frac{(\pi\Lambda_aX^a)^{\nu_a}}{\nu_a!}\biggr)\\ &=\sum_{\mu\in M_0}G_\mu(\Lambda)\tilde{\pi}^{-w(\mu)}X^{-\mu},\end{aligned}$$

where $G_\mu(\Lambda)=\sum_{\nu\in(\mathbf{Z}_{\geq0})^{|A|}}G_\nu^{(\mu)}\Lambda^\nu$ with

$$G_\nu^{(\mu)}=\begin{cases}\tilde{\pi}^{w(\mu)}\prod_{a\in A}\frac{\pi^{\nu_a}}{\nu_a!} & \text{if }\sum_{a\in A}\nu_aa=-\mu,\\ 0 & \text{if }\sum_{a\in A}\nu_aa\neq-\mu.\end{cases}$$

Since $\operatorname{ord}\pi^i/i! > 0$ for all $i > 0$, it follows that $G_\mu(\Lambda) \in R$, $|G_\mu(\Lambda)| \leq |\tilde{\pi}|^{w(\mu)}$, and $G_0(\Lambda)$ is invertible in R. This implies that $\zeta(\Lambda, X)/G_0(\Lambda) \in T$. Note also that since $F(\Lambda, X) = \exp(\pi f_\Lambda(X))/\exp(\pi f_{\Lambda^p}(X^p))$, it is straightforward to check that

$$\gamma'(F(\Lambda, X)) = \gamma(F(\Lambda, X)) = \gamma\left(\frac{\exp \pi f_\Lambda(X)}{\exp \pi f_{\Lambda^p}(X^p)}\right) = \frac{\zeta(\Lambda, X)}{\zeta(\Lambda^p, X^p)}.$$

It follows that if $\xi(\Lambda, X)$ is a series satisfying $\gamma(\xi(\Lambda, X)) \in S$, then

$$\begin{aligned}\alpha^*(\gamma(\xi(\Lambda, X))) &= \gamma'(F(\Lambda, X)\gamma(\xi(\Lambda^p, X^p))) = \gamma(F(\Lambda, X))\gamma(\xi(\Lambda^p, X^p)) \\ &= \frac{\zeta(\Lambda, X)\gamma(\xi(\Lambda^p, X^p))}{\zeta(\Lambda^p, X^p)}.\end{aligned} \tag{2.13}$$

Remark: In terms of the A-hypergeometric functions $\{F_i(\Lambda)\}_{i\in M}$ defined in Equation (1.1), we have $\exp(\pi f_\Lambda(X)) = \sum_{i\in M} F_i(\pi\Lambda)X^i$, so for $i \in M_0$ we have the relation

$$F_i(\pi\Lambda) = \tilde{\pi}^{-w(-i)}G_{-i}(\Lambda). \tag{2.14}$$

Proposition 2.15. *The unique fixed point of β is $\zeta(\Lambda, X)/G_0(\Lambda)$.*

Proof. By Equation (2.13), we have

$$\alpha^*\left(\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\right) = \frac{G_0(\Lambda)}{G_0(\Lambda^p)}\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}, \tag{2.16}$$

which is equivalent to the assertion of the proposition. □

By the Remark following Proposition 2.11, $\zeta(\Lambda, X)/G_0(\Lambda) \in T'$. This gives the following result.

Corollary 2.17. *For all $\mu \in M_0$, $G_\mu(\Lambda)/G_0(\Lambda) \in R'$.*

In the notation of the Remark following Proposition 2.11, one has $\xi(\Lambda, X) = \zeta(\Lambda, X)/G_0(\Lambda)$ and $\eta_0(\Lambda) = G_0(\Lambda)/G_0(\Lambda^p)$, so Equation (2.12) implies the following result.

Corollary 2.18. *For all $\mu \in M_0$, $G_\mu(\Lambda)/G_0(\Lambda^p) \in R'$.*

In view of Equation (2.14), Corollary 2.18 implies that the function $\mathcal{F}(\Lambda) = F_0(\pi\Lambda)/F_0(\pi\Lambda^p)$ converges on the closed unit polydisk, which was the first assertion of Theorem 1.3.

3. p-adic Theory

Fix $\bar{\lambda} = (\bar{\lambda}_a)_{a\in A} \in (\bar{\mathbf{F}}_q)^{|A|}$ and let $\lambda = (\lambda_a)_{a\in A} \in (\bar{\mathbf{Q}}_p)^{|A|}$, where λ_a is the Teichmüller lifting of $\bar{\lambda}_a$. We recall Dwork's description of $L(f_{\bar{\lambda}}; T)$. Let $\Omega_0 = \mathbf{Q}_p(\lambda, \zeta_p, \tilde{\pi})$ $(= \mathbf{Q}_p(\lambda, \pi, \tilde{\pi}))$ and let $\mathcal{O}_0$ be the ring of integers of Ω_0.

We consider certain spaces of functions with support in M. We will assume that Ω_0 has been extended by a finite totally ramified extension so that there is an element $\tilde{\pi}_0$ in Ω_0 satisfying $\tilde{\pi}_0^D = \tilde{\pi}$. We shall write $\tilde{\pi}^{w(\nu)}$ and mean by it $\tilde{\pi}_0^{Dw(\nu)}$ for $\nu \in M$. Using this convention to simplify notation, we define

$$B = \left\{\sum_{\nu\in M} A_\nu\tilde{\pi}^{w(\nu)}X^\nu \mid A_\nu \in \Omega_0,\ A_\nu \to 0 \text{ as } \nu \to \infty\right\}. \tag{3.1}$$

Then B is an Ω_0-algebra which is complete under the norm

$$\Big|\sum_{\nu\in M} A_\nu \tilde{\pi}^{w(\nu)} X^\nu\Big| = \sup_{\nu\in M} |A_\nu|.$$

We construct a Frobenius map with arithmetic import in the usual way. Let

$$F(\lambda, X) = \prod_{a\in A} \theta(\lambda_a X^a) = \sum_{\mu\in M} B_\mu(\lambda) X^\mu,$$

i.e., $F(\lambda, X)$ is the specialization of $F(\Lambda, X)$ at $\Lambda = \lambda$, which is permissible by Lemma 2.2. Note also that Lemma 2.2 implies

$$\operatorname{ord} B_\mu(\lambda) \geq \frac{w(\mu)(p-1)}{p^2},$$

so we may write $B_\mu(\lambda) = \tilde{\pi}^{w(\mu)} \tilde{B}_\mu(\lambda)$ with $\tilde{B}_\mu(\lambda)$ p-integral.

Let

$$\Psi(X^\mu) = \begin{cases} X^{\mu/p} & \text{if } p|\mu_i \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

We show that $\Psi \circ F(\lambda, X)$ acts on B. If $\xi = \sum_{\nu\in M} A_\nu \tilde{\pi}^{w(\nu)} X^\nu \in B$, then

$$\Psi\bigg(\Big(\sum_{\nu\in M} \tilde{\pi}^{w(\mu)} \tilde{B}_\mu(\lambda) X^\mu\Big)\Big(\sum_{\nu\in M} A_\nu \tilde{\pi}^{w(\nu)} X^\nu\Big)\bigg) = \sum_{\omega\in M} C_\omega(\lambda) \tilde{\pi}^{w(\omega)} X^\omega$$

where

$$C_\omega(\lambda) = \sum_{\nu} \tilde{\pi}^{w(p\omega-\nu)+w(\nu)-w(\omega)} \tilde{B}_{p\omega-\nu}(\lambda) A_\nu$$

(a finite sum). We have

$$pw(\omega) = w(p\omega) \leq w(p\omega - \nu) + w(\nu)$$

so that

$$\operatorname{ord} C_\omega(\lambda) \geq \inf_\nu\{\operatorname{ord} \tilde{\pi}^{(p-1)w(\omega)} A_\nu\} = \frac{(p-1)^2 w(\omega)}{p^2} + \inf_\nu\{\operatorname{ord} A_\nu\}. \tag{3.2}$$

This implies that $\Psi(F(\lambda, X)\xi) \in B$.

Let $d(\bar{\lambda}) = [\mathbf{F}_q(\bar{\lambda}) : \mathbf{F}_q]$, so that $\lambda^{p^{\epsilon d(\bar{\lambda})}} = \lambda$. Put

$$\alpha_\lambda = \Psi^{\epsilon d(\bar{\lambda})} \circ \bigg(\prod_{i=0}^{\epsilon d(\bar{\lambda})-1} F(\lambda^{p^i}, X^{p^i})\bigg).$$

For any power series $P(T)$ in the variable T with constant term 1, define $P(T)^{\delta_{\bar{\lambda}}} = P(T)/P(p^{\epsilon d(\bar{\lambda})} T)$. Then α_λ is a completely continuous operator on B and the Dwork Trace Formula (see Dwork[1], Serre[5]) gives

$$L(f_{\bar{\lambda}}, \Theta_{\bar{\lambda}}, \mathbf{T}^n; T)^{(-1)^{n+1}} = \det(I - T\alpha_\lambda | B)^{\delta_{\bar{\lambda}}^n}. \tag{3.3}$$

By Equation (3.2), the (ω, ν)-entry of the matrix of α_λ ([5, Section 2]) has ord > 0 unless $\omega = \nu = 0$. The formula for $\det(I - T\alpha_\lambda)$ ([5, Proposition 7a)]) then shows that this Fredholm determinant can have at most a single unit root. Since $L(f_{\bar{\lambda}}; T)$ has at least one unit root (Section 1), Equation (3.3) proves that $L(f_{\bar{\lambda}}; T)$ has exactly one unit root.

4. Dual theory

It will be important to consider the trace formula in the dual theory as well. The basis for this construction goes back to [2] and [5]. We define

$$B^* = \left\{ \xi^* = \sum_{\mu\in M} A^*_\mu \tilde{\pi}^{-w(\mu)} X^{-\mu} \mid \{A^*_\mu\}_{\mu\in M} \text{ is a bounded subset of } \Omega_0 \right\},$$

a p-adic Banach space with the norm $|\xi^*| = \sup_{\mu\in M}\{|A^*_\mu|\}$. We define a pairing $\langle\ ,\ \rangle : B^*\times B \to \Omega_0$: if $\xi = \sum_{\mu\in M} A_\mu \tilde{\pi}^{w(\mu)} X^\mu$, $\xi^* = \sum_{\mu\in M} A^*_\mu \tilde{\pi}^{-w(\mu)} X^{-\mu}$, set

$$\langle \xi^*, \xi\rangle = \sum_{\mu\in M} A_\mu A^*_\mu \in \Omega_0.$$

The series on the right converges since $A_\mu \to 0$ as $\mu\to\infty$ and $\{A^*_\mu\}_{\mu\in M}$ is bounded. This pairing identifies B^* with the dual space of B, i.e., the space of continuous linear mappings from B to Ω_0 (see [5, Proposition 3]).

Let Φ be the endomorphism of the space of formal series defined by

$$\Phi\left(\sum_{\mu\in\mathbf{Z}^n} c_\mu X^{-\mu}\right) = \sum_{\mu\in\mathbf{Z}^n} c_\mu X^{-p\mu},$$

and let γ' be the endomorphism

$$\gamma'\left(\sum_{\mu\in\mathbf{Z}^n} c_\mu X^{-\mu}\right) = \sum_{\mu\in M} c_\mu X^{-\mu}.$$

Consider the formal composition $\alpha^*_\lambda = \gamma'\circ\left(\prod_{i=0}^{\epsilon d(\bar{\lambda})-1} F(\lambda^{p^i}, X^{p^i})\right)\circ\Phi^{\epsilon d(\bar{\lambda})}$.

Proposition 4.1. *The operator α^*_λ is an endomorphism of B^* which is adjoint to $\alpha_\lambda : B\to B$.*

Proof. As α^*_λ is the composition of the operators $\gamma'\circ F(\lambda^{p^i}, X)\circ\Phi$ and α_λ is the composition of the operators $\Psi\circ F(\lambda^{p^i}, X)$, $i = 0,\dots,\epsilon d(\bar{\lambda})-1$, it suffices to check that $\gamma'\circ F(\lambda, X)\circ\Phi$ is an endomorphism of B^* adjoint to $\Psi\circ F(\lambda, X) : B\to B$. Let $\xi^*(X) = \sum_{\mu\in M} A^*_\mu\tilde{\pi}^{-w(\mu)}X^{-\mu}\in B^*$. The proof that the product $F(\lambda, X)\xi^*(X^p)$ is well-defined is analogous to the proof of convergence of the series (2.3). We have

$$\gamma'(F(\lambda, X)\xi^*(X^p)) = \sum_{\omega\in M} C_\omega(\lambda)\tilde{\pi}^{-w(\omega)}X^{-\omega},$$

where

$$C_\omega(\lambda) = \sum_{\mu - p\nu = -\omega} \tilde{B}_\mu(\lambda) A^*_\nu \tilde{\pi}^{w(\omega)+w(\mu)-w(\nu)}. \tag{4.2}$$

Note that

$$pw(\nu) = w(p\nu) \le w(\omega) + w(\mu)$$

since $p\nu = \omega+\mu$. Thus

$$(p-1)w(\nu) \le w(\omega) + w(\mu) - w(\nu),$$

which implies that the series on the right-hand side of (4.2) converges and that $|C_\omega(\lambda)| \le |\xi^*|$ for all $\omega\in M$. It follows that $\gamma'(F(\lambda, X)\xi^*(X^p))\in B^*$. It is straightforward to check that $\langle\Phi(X^{-\mu}), X^\nu\rangle = \langle X^{-\mu}, \Psi(X^\nu)\rangle$ and that

$$\langle\gamma'(F(\lambda, X)X^{-\mu}), X^\nu\rangle = \langle X^{-\mu}, F(\lambda, X)X^\nu\rangle$$

for all $\mu, \nu \in M$, which implies the maps are adjoint. □

By [5, Proposition 15] we have $\det(I - T\alpha_{\bar\lambda}^* \mid B^*) = \det(I - T\alpha_{\bar\lambda} \mid B)$, so Equation (3.3) implies

$$L(f_{\bar\lambda}, \Theta_{\bar\lambda}, \mathbf{T}^n; T)^{(-1)^{n+1}} = \det(I - T\alpha_{\bar\lambda}^* \mid B^*)^{\delta_{\bar\lambda}^n}. \tag{4.3}$$

From Equations (2.14) and (2.16), we have

$$\alpha^*\left(\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\right) = \mathcal{F}(\Lambda)\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}.$$

It follows by iteration that for $m \geq 0$,

$$(\alpha^*)^m\left(\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\right) = \left(\prod_{i=0}^{m-1} \mathcal{F}(\Lambda^{p^i})\right)\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}. \tag{4.4}$$

We have

$$\frac{\zeta(\Lambda, X)}{G_0(\Lambda)} = \sum_{\mu \in M_0} \frac{G_\mu(\Lambda)}{G_0(\Lambda)}\tilde\pi^{-w(\mu)}X^{-\mu},$$

so by Corollary 2.17 we may evaluate at $\Lambda = \lambda$ to get an element of B^*:

$$\left.\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\right|_{\Lambda=\lambda} = \sum_{\mu \in M_0} \left.\frac{G_\mu(\Lambda)}{G_0(\Lambda)}\right|_{\Lambda=\lambda}\tilde\pi^{-w(\mu)}X^{-\mu} \in B^*.$$

It is straightforward to check that the specialization of the left-hand side of Equation (4.4) with $m = \epsilon d(\bar\lambda)$ at $\Lambda = \lambda$ is exactly $\alpha_{\bar\lambda}^*((\zeta(\Lambda, X)/G_0(\Lambda))|_{\Lambda=\lambda})$, so specializing Equation (4.4) with $m = \epsilon d(\bar\lambda)$ at $\Lambda = \lambda$ gives

$$\alpha_{\bar\lambda}^*\left(\left.\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\right|_{\Lambda=\lambda}\right) = \left(\prod_{i=0}^{\epsilon d(\bar\lambda)-1} \mathcal{F}(\lambda^{p^i})\right)\left.\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\right|_{\Lambda=\lambda}. \tag{4.5}$$

Equation (4.5) shows that $\prod_{i=0}^{\epsilon d(\bar\lambda)-1} \mathcal{F}(\lambda^{p^i})$ is a (unit) eigenvalue of $\alpha_{\bar\lambda}^*$, hence by Equation (4.3) it is the unique unit eigenvalue of $L(f_{\bar\lambda}; T)$.

References

[1] B. Dwork, On the zeta function of a hypersurface, Inst. Hautes Études Sci. Publ. Math. **12** (1962), 5–68.

[2] B. Dwork, On the zeta function of a hypersurface: II. Ann. of Math. **80** (1964), 227–299.

[3] B. Dwork, Bessel functions as p-adic functions of the argument. Duke Math. J. **41** (1974), 711–738.

[4] O. Schlömilch, Ueber die Bessel'sche Funktion. Zeitschrift für Math. und Phys. **2** (1857), 137–165.

[5] J.-P. Serre, Endomorphismes complètement continus des espaces de Banach p-adiques, Inst. Hautes Études Sci. Publ. Math. **12** (1962), 69–85.

[6] S. Sperber, p-adic hypergeometric functions and their cohomology. Thesis, University of Pennsylvania, 1975.

[7] G. Watson, A treatise on the theory of Bessel functions. Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1944.

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