HYPERSPACES OF FINITE RAY-GRAPHS

NORAH ESTY

Abstract. In this paper we consider the hyperspace $C_n(X)$ of closed, connected, non-empty subsets of a base space X. The class of base spaces we consider we call finite ray-graphs, and are a noncompact variation on finite graphs. We prove two results about the structure of these hyperspaces.

AMS code 54B20

Keywords: hyperspaces; finite graphs

1. Introduction

The last thirty years has produced a large amount of research in the area of hyperspaces. A hyperspace is a topological space whose points are subsets of a given base space. A general hyperspace is denoted $\mathcal{H}(X)$, where X is the base space. Common hyperspaces include $CL(X)$, the space of non-empty and closed subsets of X, and $C(X)$, the space of non-empty, closed, and connected subsets of X. There are several topologies available for such spaces. When the base space is compact, two of the most popular topologies, the Hausdorff and the Vietoris, agree. However, when the base space is not compact, they differ, and in fact the Vietoris topology is non-metrizable. In contrast, by using a bounded metric on the space, or allowing for infinite-valued metrics, the Hausdorff topology arises from a metric. Most of the study of hypserspaces has been done in the case where the base space X is a continuum.

In 1968, Duda did an examination of the hyperspace of subcontinua of finite connected graphs, and under some minor conditions was able to give a description of $C(X)$ as a polyhedron, decomposable into balls of various dimensions. [3], [4]. A single hyperspace may consist of several sections of different dimension: a two-dimensional disc glued to a three dimensional ball, etc. In particular, for X a finite graph, the hyperspace $C(X)$ is known to be compact and connected.

Uniqueness of hyperspaces is the property that if $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, then X is homeomorphic to Y. This is not true in general, so the question has become for which classes of base spaces it holds. Work by Acosta, Duda, Eberhart, and Nadler has shown that finite graphs (different from an arc and the simple closed curve), hereditarily indecomposable continua, and smooth fans have unique hyperspace $C(X)$. See [3], [4], [1], [2], [5]. In 2002 and 2003, Illanes continued this study, and showed that for finite graphs the hyperspaces $C_n(X)$ of closed, non-empty subsets with at most n connected components are unique. See [6], [7].

In this paper we are mostly interested in the situation where the base space is not compact. We look at a natural generalization of finite graphs which we call finite ray-graphs, which consist of vertices, edges, and rays. Because the graphs are not compact we must choose a particular topology, and in this paper we use the Hausdorff topology, arising from the Hausdorff metric, which we allow to be infinite-valued.

We prove two main results. The first, in section 4, is concerned with the effect to the hyperspace of joining two graphs together at a vertex. We produce a nice algorithm for creating the hyperspace $C(X_1 \vee_p X_2)$, given the hyperspaces $C(X_1)$ and $C(X_2)$ of the two component graphs. This result does not require the graph to be non-compact.

The second theorem, in section 6, is concerned with the number of connected components of the hyperspace of a ray-graph. In particular, we show that in contrast to $C(X)$ for graphs, for a finite, connected ray-graph with N rays, the hyperspace $C(X)$ will have 2^N connected components, and will not be compact.

2. Preliminaries and Notation

2.1. **Notation.** There is not always consistent notation used for the different hyperspaces of a given base space X. We attempt to use those notations from the literature which are least ambiguous. Given a metric space X, we define the following notation for the hyperspaces we will discuss:

- $CL(X) = \{A \subset X : A \text{ closed and } A \neq \emptyset\}$
- $C(X) = \{A \subset X : A \text{ is closed, } A \neq \emptyset, \text{ and connected }\}$

It should be pointed out that much of the literature on hyperspaces assumes that the base space X is compact, in effect making $C(X)$ the hyperspace of subcontinua, but we are not assuming that here. This is also why we write $CL(X)$ rather than 2^X, which is more common, but to many readers may mean *bounded* closed subsets, which we do not mean. When we wish to refer to a general hyperspace, we will write $\mathcal{H}(X)$.

We will endow our hyperspaces with the Hausdorff topology (τ_H). The Hausdorff topology has the virtue that is arises from a metric, although since we are interested in unbounded base spaces, we allow the metric to be infinite-valued. (The same thing could be done by changing the metric on the base space to be a bounded metric.) Say that $\mathcal{H}(Y)$ is a hyperspace over a base space Y. If $A, B \subset Y$, and $N_Y(A, \epsilon)$ indicates the ϵ-neighborhood in the space Y around the subset A, then the Hausdorff distance in the hyperspace is given by

$$d_{\mathcal{H}(Y)}(A, B) = \inf\{\epsilon : A \subset N_Y(B, \epsilon) \text{ and } B \subset N_Y(A, \epsilon)\}$$

If the elements of the hyperspace are not closed subsets, then it is possible to have the distance between two non-equal sets be zero. However we will deal exclusively with closed sets. One can see from this definition that if A is bounded and B is not, the Hausdorff distance between A and B is infinite.

2.2. **The class of base spaces: finite ray-graphs.** For our base spaces, we will consider a variation on finite graphs, which we will call finite ray-graphs. These graphs will consist of a finite number of vertices (points), edges (homeomorphic to $[0,1]$ and attached at two vertices, or at one vertex twice) and rays (homeomorphic to $[0,\infty)$ and attached at one vertex). We will restrict our attention to finite connected ray-graphs. We give some simple examples in section 3.

Since we are interested in the base spaces only up to homeomorphism, in all cases but two we require that the valency of any given vertex is not equal to two: meaning a vertex with exactly two edges (or rays) attached to it will be removed, and the two edges replaced by one edge instead; see Figure 1. The exception to this will be the spaces $X = S^1$, consisting of one vertex and one edge attached at both ends to that vertex, and $X = \mathbb{R}$, consisting of one vertex and two attached rays.

The metric on these graphs will be that of arc-length, and we will consider all edges as having length one. We shall call the class of all such ray-graphs $\mathcal{X}$ and elements of that class X.

Given such a graph $X \in \mathcal{X}$, we let $\mathcal{V}$ be the set of vertices, $\mathcal{E}$ the set of edges, and $\mathcal{R}$ the set of rays. If $\#\mathcal{R} = n$, we call X an *n-legged* graph. Sometimes we refer to the bounded part of the finite ray-graph, namely $X \backslash \mathcal{R}$, and that we call X_G.

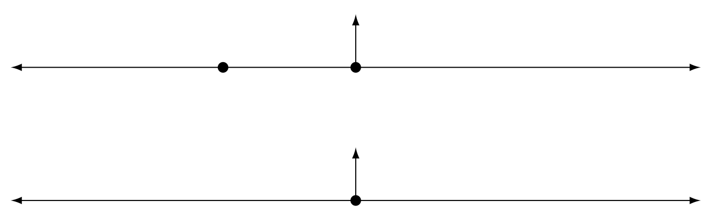

FIGURE 1. Valency 2 vertices are omitted

We will often be interested in the *containment hyperspace* $C_A(X) = \{B \in C(X) : A \subset B\}$. This concept is especially useful to us when A is a vertex of the graph. If $A = \{p\}$ we may write $C_p(X)$ rather than $C_{\{p\}}(X)$.

3. Some Basic Examples of $(C(X), \tau_H)$

In this section we will discuss a few known examples of the hyperspace $(C(X), \tau_H)$ for specific $X \in \mathcal{X}$. For more detail on the hyperspaces in the first two subsections, see [8].

3.1. $X \approx [0,1]$**.** If X is a segment homeomorphic to a closed interval then any element $A \in C(X)$ is of the form $[a,b]$. Let us assume $X = [0,1]$, and then we have $0 \leq a \leq b \leq 1$. There is a homeomorphism from the hyperspace $C(X)$ to the solid triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(0,1)$ and $(1,1)$ which takes an interval $[a,b]$ to the point (a,b). (Here we are abusing the notation to say that $[a,a] = \{a\}$.) See Figure 2. Notice that the left edge of the triangle corresponds to subsets of X which contain 0, i.e. the containment hyperspace $C_{\{0\}}(X)$. The top edge corresponds to subsets which contain 1, $C_{\{1\}}(X)$, and the hypotenuse corresponds to single-element sets. We will refer to this triangle as T.

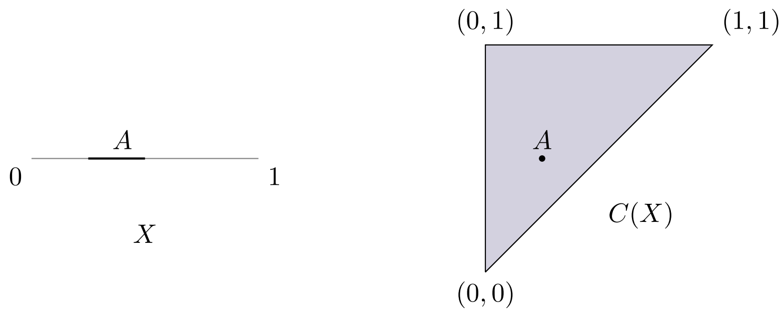

FIGURE 2. $X = [0, 1]$ and $C(X)$, as well as an element $A \in C(X)$

3.2. $X \approx S^1$. If X is a simple closed curve, than elements of $C(X)$ can be categorized by their midpoint and their length. Let $X = S^1$. We can make a homeomorphism from $C(X)$ to the solid unit disc by mapping an arc with length l and midpoint p to the point which sits on the radial line through p, and whose distance from the origin is $1 - \frac{l}{2\pi}$. See Figure 3. Notice that the boundary of the disc corresponds to single-element sets, and the center point of the disc corresponds to the full circle. We will refer to this disc as D.

Although $[0, 1] \not\approx S^1$, their two hyperspaces D and T are homeomorphic. It is known that for finite graphs this is the only such example [3].

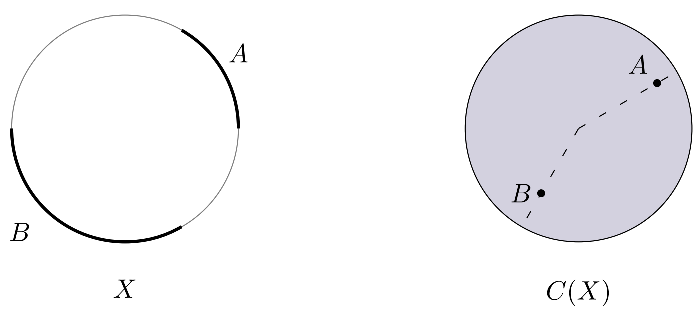

FIGURE 3. $X = S^1$ and $C(X)$, as well as two elements $A, B \in C(X)$

3.3. $X \approx [0, \infty)$. If X is a ray homeomorphic to $[0, \infty)$, then elements of $C(X)$ are either bounded intervals of the form $[a, b]$ or unbounded intervals of the form $[a, \infty)$. We can make a homeomorphism from $C(X)$ to the space $T^\infty \sqcup [0, \infty)$, where $T^\infty = \{(a, b) \in \mathbb{R}^2 : 0 \leq a \leq b\}$ is an "infinite triangle."1 This is done by mapping $[a, b]$ to $(a, b) \in T^\infty$ and $[a, \infty)$ to $a \in [0, \infty)$. See Figure 4.

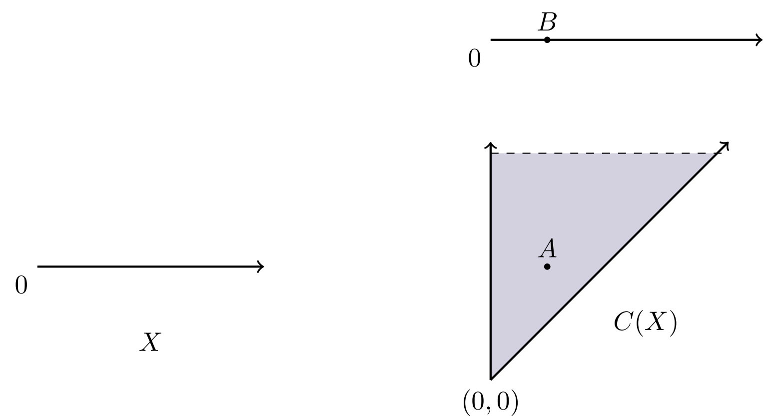

FIGURE 4. $X = [0, \infty)$ and $C(X)$, as well as the elements $A = [.25, .5]$ and $B = [.25, \infty)$, both in $C(X)$

Notice that elements of T^∞ correspond to bounded subsets of X, and in particular, the left edge of T^∞ corresponds to bounded elements which contain 0, and the hypotenuse corresponds to single-element sets. For a fixed horizontal value a, increasing the vertical value b corresponds to longer bounded intervals. Since the second component, $[0, \infty)$, corresponds to unbounded intervals, it can be loosely thought of as the "top" of the infinite triangle. In this example, unlike before, the containment hyperspace $C_{\{0\}}(X)$ has two components: the left edge of the triangle *and* the leftmost point $0 \in [0, \infty)$. Clearly this $C(X)$ is not compact.

With these three examples we can form several more examples by understanding what happens to the hyperspace when you attach two graphs together in a specific way.

4. Joining graphs together

We begin by describing the hyperspace of the compound ray-graph $X = X_1 \vee_p X_2$, made up of two subgraphs attached at a vertex, $X_1 \cap X_2 = \{p\}$, when the two hyperspaces $C(X_1)$ and $C(X_2)$ are already understood.

It is clear that the hyperspace $C(X)$ will contain all the elements which are in $C(X_1)$ and $C(X_2)$. It will also contain elements which correspond to subsets of X that contain the joining point p and part of X_1 and X_2. In fact, to any subset $A \subset X_1$ which contains p, we can union a subset of X_2 which contains p, and arrive at an element of $C(X)$. This shows that $C(X)$ will contain a cross product of $C_p(X_1)$ and $C_p(X_2)$.

Theorem 1. *For $X_1, X_2 \in \mathcal{X}$ and the compound graph $X = X_1 \vee_p X_2$ (where p is a vertex of X_1 and X_2), then*

$$C(X) \approx \frac{C(X_1) \sqcup C_p(X_1) \times C_p(X_2) \sqcup C(X_2)}{(C_p(X_1) \sim C_p(X_1) \times \{p\} \text{ and } \{p\} \times C_p(X_2) \sim C_p(X_2))}$$

We delay the proof of the theorem until section 5 in order to first give several examples, which may help the reader to follow the discussion. The theorem gives us the following nice algorithm for drawing $C(X)$:

(1) Draw $C_p(X_1) \times C_p(X_2)$.
(2) Attach the rest of $C(X_1)$ to the figure by identifying its subset $C_p(X_1)$ with the slice $C_p(X_1) \times \{p\}$ in the cross product.
(3) Attach the rest of $C(X_2)$ to the figure by identifying its subset $C_p(X_2)$ with the slice $\{p\} \times C_p(X_2)$ in the cross product.

4.1. **The noose.** We begin with a simple known example, the noose. The space is a circle joined to an interval. To follow the steps outlined in section 4 for drawing the hyperspace, let $X_1 \approx S^1$ and $X_2 \approx [0,1]$, and let p be the point where they intersect. Begin by noting that $C_p(X_1)$ is homeomorphic to a subdisc inside the disc D, which includes the point p. (In fact, the shape is a cardiod, but for ease of representation we will draw it as a disc.) $C_p(X_2)$ is the left edge of the triangle T.

When we cross $C_p(X_1)$ with $C_p(X_2)$ we get a solid cylinder. Then we attach the rest of the disc $D = C(X_1)$ along the slice $C_p(X_1) \times \{p\}$, which is the bottom of the cylinder. Finally, we attach the rest of the triangle $T = C(X_2)$ along the slice $\{p\} \times C_p(X_2)$, which is a vertical line on the boundary of the cylinder. Note that the resulting space has both two- and three-dimensional sections. See Figure 5.

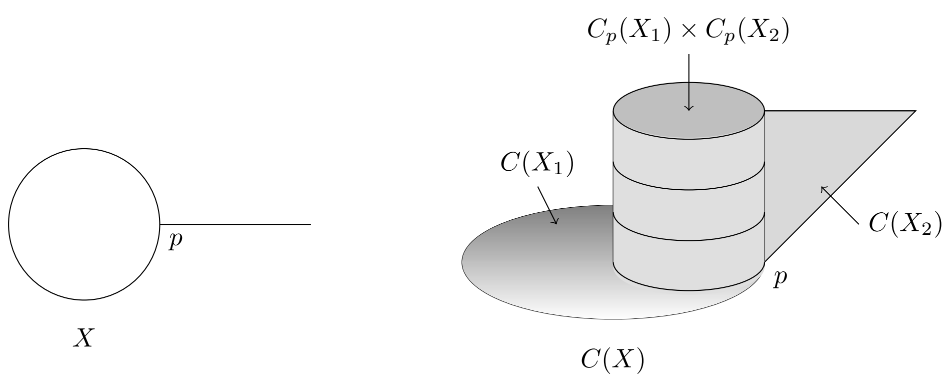

FIGURE 5. X a noose, and $C(X)$

4.2. n**-od.** An n-od is a point p with n intervals attached such that they intersect only at p. We begin with the 2-od. Of course the 2-od is homeomorphically the same as an interval, and so we should arrive at $C(X) \approx T$. But for the purposes of the construction, we will go through it nonetheless. In this case $X_1 = X_2 \approx [0,1]$ and $C_p(X_1) = C_p(X_2)$ is the left edge of the triangle T.

When we cross $C_p(X_1)$ with $C_p(X_2)$ we get a square; we then attach the triangles $C(X_i)$ along the appropriate edges. The result is, of course, another triangle. See Figure 6.

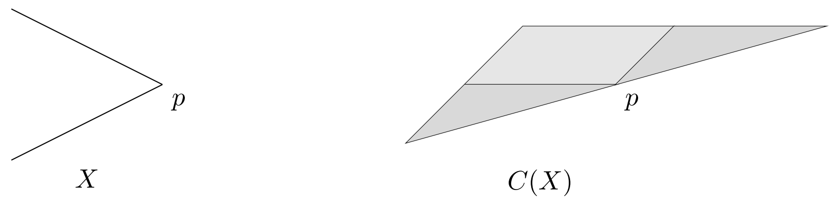

FIGURE 6. X a 2-od and $C(X)$

Now we will use the 2-od to construct the hyperspace of the triod, by letting X_1 be the 2-od and X_2 be the interval. $C_p(X_1)$ is now the square from before, and $C_p(X_2)$ is again the left edge of T. The cross product therefore results in a cube. The two fins from the hyperspace of the 2-od are attached along the bottom of the cube, and another fin (the rest of $C(X_2)$) is attached along the front right edge. See Figure 7.

It is easy to see that continuing in this manner will result in the hyperspace of the n-od consisting of the n-cube I_n, with n fins attached along its edges, each with a corner at p.

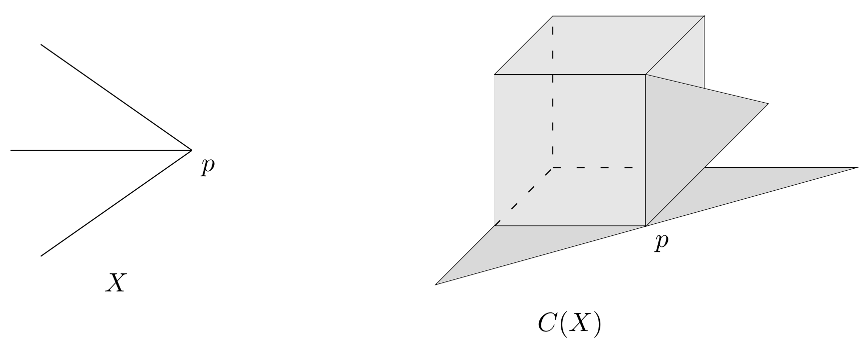

FIGURE 7. X a triod and $C(X)$, a cube with three fins.

It is interesting to point out that although the hyperspaces of the circle and the interval are homeomorphic, the hyperspaces you get by attaching an interval to them at a point (creating a noose or a triod, respectively) are not. They are similar, but the hyperspace of the noose is structured so that removing the cross product and $C(X_2)$ leaves one component, but doing the same to they hyperspace of the triod leaves two components.

4.3. **The figure eight.** This is an example where the hyperspace will be four-dimensional. If $X_1 = X_2 \approx S^1$, attached at the point p, at angle 0 for X_1 and at angle π for X_2, then $C_p(X_1)$ and $C_p(X_2)$ are both cardiods inside discs, one intersecting the boundary of its disc at 0, and the other at π. The cross product is four-dimensional, and then the two discs are attached along the "bottom" and "side" of that object.

4.4. **The infinite noose.** Let X be the infinite noose, made up of $X_1 \approx S^1$ and $X_2 \approx [0, \infty)$, joined at the point p. $C_p(X_1)$, as we have already noted in section 4.1, is a cardiod inside the unit disc, which we will draw as a subdisc. Recall from section 3.3 that $C_p(X_2)$ has two components: the left edge of the infinite triangle T^∞, and the left-most point of the ray. The point of $C(X_2)$ which corresponds to the single-point set $\{p\} \subset X_2$ is at the bottom of the triangle.

Crossing $C_p(X_1)$ with $C_p(X_2)$, we get an infinite cylinder and a disc. We attach $C(X_1)$ to the slice $C_p(X_1) \times \{p\}$, along the bottom of the cylinder. We attach $C(X_2)$ along $\{p\} \times C_p(X_2)$, producing an infinite fin off the side of the cylinder and a ray off the side of the disc. See Figure 8.

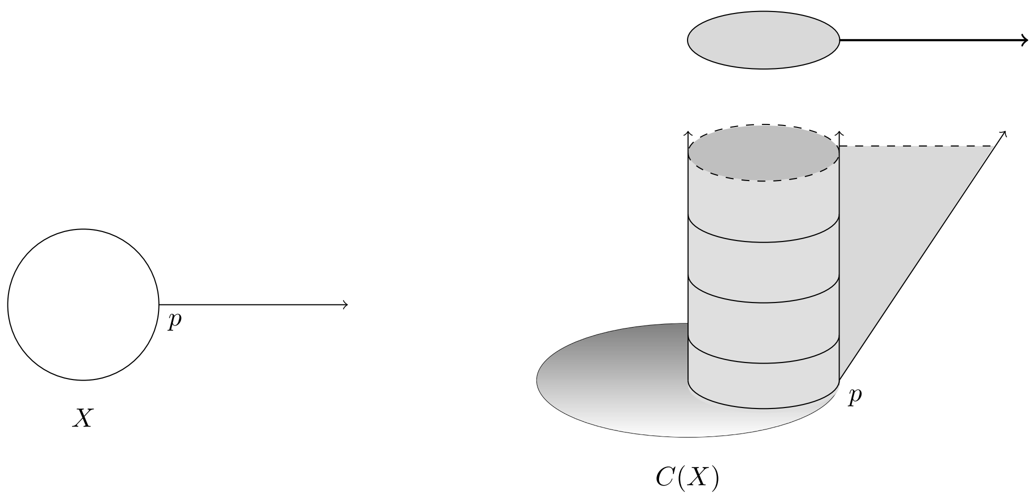

FIGURE 8. X is the infinite noose. $C(X)$ has two components.

4.5. **The real line.** Let $X_1 = X_2 \approx [0, \infty)$, both just a single vertex with a ray attached. Then we can think of $X \approx \mathbb{R}$ as the result of attaching these two subgraphs along their vertex. Both subgraphs have hyperspaces which consist of $T^\infty \sqcup [0, \infty)$, and the containment hyperspace for the vertex is the union of the left edge of the triangle and the leftmost point of the ray.

Following the algorithm, $C_p(X_1) \times C_p(X_2)$ gives us four components: an infinite square, two rays, and a point. When we attach the rest of $C(X_1)$ along the correct slice, it attaches the rest of the triangle along one side of the infinite square, and the rest of a ray along one of the rays. Similarly when we attach the rest of $C(X_2)$ it attaches the rest of an infinite triangle along the other side of the infinite square, and another ray along the second ray. The end result is four components: a half-plane, two real lines, and a point.

To see this more algebraically, we will briefly construct a homeomorphism to show:

$$(C(X), \tau_H) \approx \{0\} \sqcup \mathbb{R} \sqcup \mathbb{R} \sqcup (\mathbb{R} \times [0, \infty))$$

We will write different sections of the disjoint union with a subscript to distinguish them, e.g. $\mathbb{R}_1$ and $\mathbb{R}_2$. Because elements of $C(X)$ are connected closed subsets of X, they are closed intervals, and are therefore one of four types: $A = X = \mathbb{R}$, $A = [a, \infty)$ ("unbounded to the right"), $A = (-\infty, a]$ ("unbounded to the left") or $A = [a, b]$ ("bounded").

Define a map ϕ from $C(X) \to \{0\} \sqcup \mathbb{R}_1 \sqcup \mathbb{R}_2 \sqcup (\mathbb{R} \times [0, \infty))$ in the following way:

$$\phi(A) = \begin{cases} \{0\} & \text{if } A = X \\ a \in \mathbb{R}_1 & \text{if } A = [a, \infty) \\ a \in \mathbb{R}_2 & \text{if } A = (-\infty, a] \\ (a, a' - a) \in \mathbb{R} \times [0, \infty) & \text{if } A = [a, a'] \end{cases}$$

This is clearly a homeomorphism. Notice that the four components of the hyperspace correspond to the four different ways in which it is possible for a subset to be unbounded. In section 6 we will show this is not a coincidence.

5. Proof of theorem 1

In order to prove Theorem 1, we will need some understanding of how the metric on X_1 and X_2 (arc-length) relates to the metric on the space X. Because the same set may sit in both X_1 and X, for instance, we will indicate the ambient space of the metric by a subscript, e.g. $d_{C(X_1)}(A, B)$. A subscript of a hyperspace indicates the metric is the Hausdorff metric; a subscript of a graph indicates the metric is arc length. Note that $d_X(A, B) \neq d_{C(X)}(A, B)$ unless both sets are singleton sets.

Lemma 2. *Let X_1, X_2, and $X \in \mathcal{X}$ be three graphs all using the metric of arc length, with $X = X_1 \vee_p X_2$, where p is a vertex. Denote by A_i the intersection $A \cap X_i$. We have:*

1. *If $A, B \in C(X_1)$, then $d_{C(X)}(A, B) = d_{C(X_1)}(A, B)$.*
2. *If $A, B \in C(X)$, and*
   - (a) *if $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$, then $d_{C(X)}(A, B) \geq d_{C(X_1)}(A_1, B_1)$.*
   - (b) *if for each $i = 1, 2$, each of A_i and B_i is nonempty, then $d_{C(X)}(A, B) = \max\{d_{C(X_1)}(A_1, B_1), d_{C(X_2)}(A_2, B_2)\}$.*
3. *If $A \in C(X_1)$, then $d_{C(X_1)}(A, C_p(X_1)) = d_{X_1}(A, p)$.*
4. *If $A, B \in C(X)$ and if $A_1 \neq \emptyset$ and $B_2 \neq \emptyset$, then $d_{C(X)}(A, B) \geq d_{C(X_1)}(A_1, C_p(X_1)) + d_{C(X_2)}(B_2, C_p(X_2))$.*
5. *If $A_1 \neq \emptyset$ and $B \subset X_2$, then $d_{C(X)}(A, B) \geq d_{C(X_1)}(A_1, \{p\})$.*

For the proof of the lemma, it is helpful to recall the definition of the Hausdorff metric. If $A, B \subset Y$, and $N_Y(A, \epsilon)$ indicates the ϵ-neighborhood in the space Y around the subset A, then

$$d_{\mathcal{H}(Y)}(A, B) = \inf\{\epsilon : A \subset N_Y(B, \epsilon) \text{ and } B \subset N_Y(A, \epsilon)\}$$

Proof: For part 1, assume that $A, B \in C(X_1)$. Any neighborhood of size ϵ in the space X_1 is automatically contained inside a neighborhood of size ϵ in X. Therefore if $A \subset N_{X_1}(B, \epsilon)$ then $A \subset N_X(B, \epsilon)$. This means

$$\{\epsilon : A \subset N_{X_1}(B, \epsilon) \text{ and } B \subset N_{X_1}(A, \epsilon)\} \subset \{\epsilon : A \subset N_X(B, \epsilon) \text{ and } B \subset N_X(A, \epsilon)\}$$

which implies that the infimum of the set on the left is greater than or equal to the infimum of the set on the right. So we have $d_{C(X_1)}(A, B) \geq d_{C(X)}(A, B)$.

On the other hand, if $A \subset N_X(B, \epsilon)$, since A and B are both contained entirely in X_1, it must be that $A \subset N_X(B, \epsilon) \cap X_1 = N_{X_1}(B, \epsilon)$. With respect to the two sets of epsilons above, this gives us containment the other direction, and we conclude $d_{C(X_1)}(A, B) = d_{C(X)}(A, B)$.

For part 2a, suppose $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. If $A \subset N_X(B, \epsilon)$, then $A_1 \subset N_{X_1}(B_1, \epsilon)$ by intersecting with X_1. Similarly, $B \subset N_X(A, \epsilon)$ implies $B_1 \subset N_{X_1}(A_1, \epsilon)$. Therefore

$$\{\epsilon : A \subset N_X(B, \epsilon) \text{ and } B \subset N_X(A, \epsilon)\} \subset \{\epsilon : A_1 \subset N_{X_1}(B_1, \epsilon) \text{ and } B \subset N_{X_1}(A_1), \epsilon)\}$$

which tells us that $d_{C(X)}(A, B) \geq d_{C(X_1)}(A_1, B_1)$.

The same argument works for X_2 with A_2 and B_2, and putting them together, we have

$$d_{C(X)}(A, B) \geq \max\{d_{C(X_1)}(A_1, B_1), d_{C(X_2)}(A_2, B_2)\}$$

For brevity let's call that maximum δ.

For part 2b, assume that each A_i and B_i are nonempty. By the definition of δ, $A_1 \subset N_{X_1}(B_1, \delta)$ and $A_2 \subset N_{X_2}(B_2, \delta)$. Since A and B are actually connected, each of A_i and B_i contains p, and we can therefore seamlessly union the δ-balls together to arrive at $A_1 \vee_p A_2 \subset N_X(B_1 \vee_p B_2, \delta)$. In other words, $A \subset N_X(B, \delta)$ and similarly, $B \subset N_X(A, \delta)$. So $d_{C(X)}(A, B) \leq \delta$. Combined with the opposite inequality we have $d_{C(X)}(A, B) = \delta$.

For part 3, assume $A \subset X_1$. Because A is closed, $d_{X_1}(A, p)$ is achieved at some point $a' \in A$. Let $d_{X_1}(A, p) = \delta$. Then for all $\gamma > \delta$, $p \in B_{X_1}(A, \gamma)$ and for all $\epsilon < \delta$, $p \notin B_{X_1}(A, \epsilon)$. Let $A' \in C_p(X_1)$. Because $p \in A'$, $A' \notin B_{C(X_1)}(A, \epsilon)$. This means for all $A' \in C_p(X_1)$, $d_{C(X_1)}(A, A') \geq \delta$.

Because the graphs are path-connected, there is an interval in X_1 from p to A, the length of which is d. Denote it by $[p, a']$. Consider $A^* = A \cup [p, a']$. Clearly $A \in C_p(X_1)$. Because $A \subset A^*$, for all $\epsilon > 0$ we have $A \subset N_{X_1}(A^*, \epsilon)$. We can also see that $A^* \subset N_{X_1}(A, d_{X_1}(a', p)) = N_{X_1}(A, \delta)$ and so $d_{C(X_1)}(A^*, A) \leq \delta$. Together with the above inequality, this shows that $d_{C(X_1)}(A, C_p(X_1)) = d_{X_1}(A, p)$.

For part 4, let $A, B \in C(X)$ and $A_1 \neq \emptyset$ and $B_2 \neq \emptyset$. If $A_1 \in C_p(X_1)$ and $B_2 \in C_p(X_2)$ the inequality is obviously satisfied. Suppose next that $B_2 \in C_p(X_2)$ but $A_1 \notin C_p(X_1)$. So $A \cap X_2 = \emptyset$. Then the statement reduces to $d_{C(X)}(A, B) \geq d_{C(X_1)}(A_1, C_p(X_1))$. But we know

$$\begin{aligned} d_{C(X)}(A, B) &\geq d_{C(X_1)}(A_1, B_1) \\ &= d_{C(X_1)}(A, B_1) \\ &\geq d_{C(X_1)}(A, C_p(X_1)) \end{aligned}$$

where the last inequality holds because $p \in B$ and hence $B_1 \in C_p(X_1)$.

For the final case, assume $A \cap X_2 = \emptyset$ and $B \cap X_1 = \emptyset$. We show that there exists an $a \in A$ and a $b \in B$ such that $d_{C(X)}(A,B) \geq d_X(a,b)$. This is obviously satisfied if $d_{C(X)}(A,B) = \infty$. If not, fix an $a \in A$. If $A \subset N_X(B,\epsilon)$ then there exists $b \in B$ such that $d_X(a,b) < \epsilon$. Pick a decreasing sequence of $\epsilon_n \to d_{C(X)}(A,B)$. This produces a sequence of $b_n \in B$ with $d_X(a,b_n) < \epsilon_n$. Since all the b_n are in an ϵ_1-ball around a, the sequence is bounded and therefore has a convergent subsequence, $b_{n_k} \to b$. B is closed, so $b \in B$, and by construction, $d_X(a,b) \leq \epsilon \leq d_{C(X)}(A,B)$.

The existence of such an a and b gives us

$$\begin{aligned} d_{C(X)}(A,B) &\geq d_X(a,b) \\ &= d_X(a,p) + d_X(p,b) \\ &\geq d_{X_1}(A_1,p) + d_{X_2}(p,B_2) \\ &= d_{C(X_1)}(A_1, C_p(X_1)) + d_{C(X_2)}(C_p(X_2), B_2) \end{aligned}$$

where the first equality is a result of the positions of a and b in the graph, and the last equality is from part 3 of the lemma.

For part 5, assume that $A_1 = A \cap X_1 \neq \emptyset$ and that $B \subset X_2$. We will show the inequality $d_{C(X)}(A,B) \geq d_{C(X_1)}(A_1,\{p\})$ by showing the following subset statement:

$$\{\epsilon : A \subset N_X(B,\epsilon) \text{ and } B \subset N_X(A,\epsilon)\} \subset \{\epsilon : A_1 \subset N_{X_1}(\{p\},\epsilon) \text{ and } \{p\} \subset N_{X_1}(A_1,\epsilon)\}$$

and applying infimums on both sides. Let ϵ be a member of the set on the left. Then we have $A \subset N_X(B,\epsilon)$, which implies $A_1 \subset N_X(B,\epsilon)$. This neighborhood, being around B and containing A, must therefore intersect both X_1 and X_2, and since the neighborhood is connected, we therefore know $p \in N_X(B,\epsilon)$.

Because B is contained entirely in X_2, it is obvious that for all $a \in A_1$, $d_X(a,B) \geq d_X(a,p)$. So $A_1 \subset N_X(\{p\},\epsilon)$, and by intersecting with X_1 we have $A_1 \subset N_{X_1}(\{p\},\epsilon)$.

However, since $d_X(a,p) < \epsilon$ for all $a \in A_1$, it is also true that $\{p\} \subset N_{X_1}(A_1,\epsilon)$. Hence ϵ is a member of the set on the right, and the inequality holds. □

We are now ready to prove the theorem, stated again here for convenience.

Theorem 3. *For $X_1, X_2 \in \mathcal{X}$ and the compound graph $X = X_1 \vee_p X_2$ (where p is a vertex of X_1 and X_2), then*

$$C(X) \approx \frac{C(X_1) \sqcup C_p(X_1) \times C_p(X_2) \sqcup C(X_2)}{(C_p(X_1) \sim C_p(X_1) \times \{p\} \text{ and } \{p\} \times C_p(X_2) \sim C_p(X_2))}$$

Proof: We begin by giving a function from the disjoint union to $C(X)$, which is continuous and respects the identifications. For brevity we will write $K = C(X_1) \sqcup C_p(X_1) \times C_p(X_2) \sqcup C(X_2)$ and the quotient space as $K/\sim$. We will call the quotient map π. In the quotient space we will write $[A]$ for the equivalence class of A. Let $f : K \to C(X)$ be given by:

$$f(A) = \begin{cases} A & \text{if } A \in C(X_1) \\ A_1 \vee_p A_2 & \text{if } A = A_1 \times A_2 \text{ for } A_1 \in C_p(X_1),\ A_2 \in C_p(X_2) \\ A & \text{if } A \in C(X_2) \end{cases}$$

It is clear that f is continuous on $C(X_1)$ and $C(X_2)$. As for the cross product, fix an element $A = A_1 \times A_2$ and a small ϵ. In order to ensure that for $B = B_1 \times B_2$, $f(B)$ is within ϵ of $f(A)$ in $C(X)$, it is enough to have $d_{C(X_i)}(A_i, B_i) < \epsilon$ for $i = 1, 2$, which is ensured if $d(A, B) < \epsilon$ in the cross product. Since f is continuous on each section of the disjoint union, it is therefore continuous.

It is easy to see that f respects the identifications. For instance, suppose $A \in C_p(X_1)$. Then if A is thought of as part of the first factor $C(X_1)$, we have $f(A) = A$. If A is thought of as part of the cross product, $A = A \times \{p\}$, then $f(A) = f(A \times \{p\}) = A \vee_p \{p\} = A$. Similarly for $A \in C_p(X_2)$. So the function f gives a continuous map on the quotient space, $\hat{f} : K/\sim \to C(X)$.

Now we will show that $\hat{f}$ is a bijection. First suppose $B \in C(X)$. If $B \subset X_1$ then let $A = B \in K$ and then $\hat{f}([A]) = B$. Similarly if $B \subset X_2$. If $B_i = B \cap X_i \neq \emptyset$ for $i = 1, 2$ then let $A = B_1 \times B_2 \in K$. Then $\hat{f}([A]) = f(B_1 \times B_2) = B_1 \vee_p B_2 = B$. So $\hat{f}$ is onto.

Showing $\hat{f}$ is one-to-one takes more work. We will instead show that if $f(A) = f(B)$, $[A] = [B]$.

First we look at the specific case where $f(A) = \{p\}$. Then there are three options for A from the disjoint union (and also for B): $A = \{p\} \in C(X_1), A = \{p\} \in C(X_2)$, or $A = \{p\} \times \{p\} \in C_p(X_1) \times C_p(X_2)$. But all three are identified to the same point under π, so $[A] = [B]$.

If $f(A) \in C_p(X_1)$, and $f(A) \neq \{p\}$, then we have two separate options for A (and also B): $A = f(A) \in C(X_1)$ or else $A = f(A) \times \{p\} \in C_p(X_1) \times C_p(X_2)$. But both are identified to the same spot under π so $[A] = [B]$. Similarly if $f(A) \in C_p(X_2)$, $f(A) \neq \{p\}$.

Next, suppose $f(A) \subset C(X_1) \setminus C_p(X_1)$. Then the only option for A (and also B) is that $A = f(A) \in C(X_1)$. This means $A = B$ and so $[A] = [B]$. Similarly if $f(A) \in C(X_2) \setminus C_p(X_2)$.

Finally it may be that $f(A)$ contains both points of X_1 other than p and points of X_2 other than p. Then we will write $f(A)_i = f(A) \cap X_i$ and the only option for A is that $A = A_1 \times A_2 = f(A)_1 \times f(A)_2$. Similarly for B; therefore $A = B$ and so $[A] = [B]$. This concludes the proof that $\hat{f}$ is one-to-one.

Since we are not assuming these spaces are compact, we still need to show that $\hat{f}$ has a continuous inverse. The inverse, a map from $C(X)$ to $K/\sim$, is easy to give. Let $A \in C(X)$.

$$\hat{f}^{-1}(A) = \begin{cases} [A] & \text{if } A \subset X_1 \\ [A_1 \times A_2] & \text{if } A_i = A \cap X_i \neq \emptyset \text{ for } i = 1, 2 \\ [A] & \text{if } A \subset X_2 \end{cases}$$

In order to show that the pre-images under $\hat{f}^{-1}$ of open sets are open, we must take an open set in $K/\sim$ and show it goes to an open set in $C(X)$ under $\hat{f}$. Open sets in $K/\sim$ are the images of saturated open sets in K, and $\hat{f}(V) = f(\pi^{-1}(V))$, so it is enough to show that the map $f : K \to C(X)$ takes saturated open sets in K to open sets in $C(X)$.

Let V be a saturated open set of K, namely $V = \pi^{-1}(\pi(V))$. Let $f(A) \in f(V) \subset C(X)$. There are six possibilities for the location of $f(A)$.

(1) $f(A) \in C(X_1) \setminus C_p(X_1)$. (i.e. $f(A) \subset X_1$ and $p \notin f(A)$)
(2) $f(A) \in C_p(X_1)$ but $f(A) \neq \{p\}$
(3) $\{p\} \subsetneq f(A) \cap X_1$ and $\{p\} \subsetneq f(A) \cap X_2$
(4) $f(A) = \{p\}$
(5) $f(A) \in C_p(X_2)$ but $f(A) \neq \{p\}$.
(6) $f(A) \in C(X_2) \setminus C_p(X_2)$.

Clearly Cases 1 and 6 are symmetric, as are Cases 2 and 5. We need to prove therefore four cases.

Case 1. Let $f(A) \in C(X_1) \setminus C_p(X_1)$. In other words, $p \notin f(A)$ and $f(A) \subset X_1$, and $f(A) \cap X_2 = \emptyset$. This implies that $A \subset X_1$; in fact, $A = f(A)$. By looking at the map we see that is the unique preimage of $f(A)$. Because $A \in V$ and V is open, and because $A \notin C_p(X_1)$ and $C_p(X_1)$ is closed, there exists $\delta > 0$ such that $B_{C(X_1)}(A, \delta) \subset V \cap C_p(X_1)^c$.

To show $B_{C(X)}(f(A), \delta) \subset f(V)$, take $B \in B_{C(X)}(f(A), \delta)$.

If $B \cap X_2 \neq \emptyset$, then by lemma 2 part 4,

$$\begin{aligned} d_{C(X)}(f(A), B) &\geq d_{C(X_1)}(f(A), C_p(X_1)) \\ &= d_{C(X_1)}(A, C_p(X_1)) > \delta \end{aligned}$$

Therefore $B \cap X_2 = \emptyset$. Hence $d_{C(X_1)}(A, B) = d_{C(X)}(f(A), B) < \delta$ and so $B \in B_{C(X)}(A, \delta) \subset V$. Also $B = f(B)$ and so $B \in f(V)$.

Case 2. Let $f(A) \in C_p(X_1)$ but $f(A) \neq \{p\}$. In other words, $\{p\} \subsetneq f(A) \cap X_1$ and $\{p\} = f(A) \cap X_2$. Then the preimage of $f(A)$ contains two points. Either $A = f(A) \in C(X_1)$ or $A = f(A) \times \{p\} \in C_p(X_1) \times C_p(X_2)$. We shall call these two preimages A^1 and A^2, respectively. Because V is a saturated open set, both preimages are in V. And because V is open, there must exist $\delta_1 > 0$ such that $B_{C(X_1)}(A^1, \delta_1) \subset V$ and $\{p\} \notin B_{C(X_1)}(A^1, \delta)$. Also, there must exist δ_2 and $\delta_3 > 0$ such that $B_{C(X_1)}(f(A), \delta_2) \times B_{C(X_2)}(\{p\}, \delta_3) \subset V$. Chose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.

To show $B_{C(X)}(f(A), \delta) \subset f(V)$, take $B \in B_{C(X)}(f(A), \delta)$.

If $B \subset X_2$ then by lemma 2 part 5, $d_{C(X)}(B, f(A)) \geq d_{C(X_1)}(f(A), \{p\}) > \delta_1 \geq \delta$, which is a contradiction, so $B_1 = B \cap X_1 \neq \emptyset$.

If $B_2 = B \cap X_2 = \emptyset$ or if $B_2 = \{p\}$, then by lemma 2 part 2, $d_{C(X_1)}(f(A), B) \leq d_{C(X)}(f(A), B) < \delta \leq \delta_1$, in which case, $B \in B_{C(X_1)}(A^1, \delta_1) \subset V$ and $B = f(B) \in f(V)$.

On the other hand, if $\{p\} \subsetneq B_2$, then $d_{C(X_2)}(\{p\}, B_2) \le d_{C(X)}(f(A) \vee_p \{p\}, B) < \delta \le \delta_3$, so $B_2 \in B_{C(X_2)}(\{p\}, \delta_3)$. We also know $d_{C(X_1)}(f(A), B_1) \le d_{C(X)}(f(A) \vee_p \{p\}, B) < \delta \le \delta_2$. Therefore $B_1 \times B_2 \in V$ and $B = B_1 \vee_p B_2 = f(B_1 \times B_2)$.

Case 3. Let $\{p\} \subsetneq f(A) \cap X_1$ and $\{p\} \subsetneq f(A) \cap X_2$. Let $A_i = f(A) \cap X_i$. Then we have a unique preimage: $f(A) = f(A_1 \times A_2)$, for $A_1 \times A_2 \in C_p(X_1) \times C_p(X_2)$. $A_1 \times A_2 \in V$ and so there exists a basic open set about $A_1 \times A_2$ in V:

$$B_{C(X_1)}(A_1, \delta_1) \times B_{C(X_2)}(A_2, \delta_2) \subset V$$

such that if $A' \in B_{C(X_i)}(A_i, \delta_i)$, then $A' \neq \{p\}$. In other words, the two balls don't touch the edges of the cross product, but remain in the interior. Let $\delta = \min\{\delta_1, \delta_2\}$.

Let $B \in B_{C(X)}(f(A), \delta)$. If $B \subset X_2$, then by lemma 2 part 5, $d_{C(X)}(f(A), B) \ge d_{C(X_1)}(A_1, \{p\}) > \delta_1 \ge \delta$. Since that would be a contradiction, $B_1 = B \cap X_1 \neq \emptyset$, and in fact, $\{p\} \subsetneq B_1$. Similarly if $B \subset X_1$, so $\{p\} \subsetneq B_2 = B \cap X_2$.

By lemma 2 part 2,

$$d_{C(X_1)}(A_1, B_1) \le d_{C(X)}(f(A), B) < \delta \le \delta_1$$

and similarly for A_2. So $B_1 \times B_2 \in B_{C(X_1)}(A_1, \delta_1) \times B_{C(X_2)}(A_2, \delta_2) \subset V$. Finally, $B = B_1 \vee_p B_2 = f(B_1 \times B_2) \in f(V)$.

Case 4. Let $f(A) = \{p\}$. In this case we have three possible preimages. Either $A = \{p\} \in C(X_1)$, $A = \{p\} \in C(X_2)$, or $A = \{p\} \times \{p\} \in C_p(X_1) \times C_p(X_2)$. Because V is saturated, all three preimages are in V. Because V is open, we can then find neighborhoods: $B_{C(X_1)}(\{p\}, \delta_1) \subset V$, $B_{C(X_2)}(\{p\}, \delta_2) \subset V$, and $B_{C(X_1)}(\{p\}, \delta_3) \times B_{C(X_2)}(\{p\}, \delta_4) \subset V$. Choose $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Let $B \in B_{C(X)}(f(A), \delta)$. B may intersect one of X_1 or X_2, or both.

Suppose $B_1 = B \cap X_1 \neq \emptyset$. Then suppose further that $B_2 = B \cap X_2 \neq \emptyset$, either. Then $B = B_1 \vee_p B_2 = f(B_1 \times B_2)$. Also,

$$d_{C(X_1)}(\{p\}, B_1) \le d_{C(X)}(\{p\}, B) < \delta \le \delta_3$$
$$d_{C(X_2)}(\{p\}, B_2) \le d_{C(X)}(\{p\}, B) < \delta \le \delta_4$$

so $B_1 \times B_2 \in B_{C(X_1)}(\{p\}, \delta_3) \times B_{C(X_2)}(\{p\}, \delta_4) \subset V$, and hence $B \in f(V)$.

Instead, assume $B_1 \neq \emptyset$ but $B_2 = \emptyset$. Then by lemma 2, part 1, $d_{C(X_1)}(\{p\}, B_1) = d_{C(X)}(\{p\}, B) < \delta \le \delta_1$. So $B \in B_{C(X_1)}(\{p\}, \delta_1) \subset V$ and $B = f(B)$, so $B \in f(V)$.

We argue similarly if $B_1 = \emptyset$ and $B_2 \neq \emptyset$. This finishes the final case. □

6. The connected components of $(C(X), \tau_H)$

The last two examples of section 3 show that there is a relationship between the number of rays in a given graph, and the number of connected components of its hyperspace. That relationship is what we explore in this section.

We begin by developing some extra terminology to deal with ray-graphs in particular. Recall that $\mathcal{R} = \{R_1, \dots R_n\}$ denotes the set of rays in a given ray-graph. If $\#\mathcal{R} = n$, we will call X an *n-legged* graph. If a closed, nonempty subset $A \subset X$ has the property that $A \cap R_i$ is an unbounded set, we will say that A is *unbounded in direction i*. Because we are interested in those directions in which a given A is unbounded, we will also refer to the *unbounded direction set of* A, which is the set of indices $\{l_1, \dots, l_m\}$, $m \leq n$, for which A is unbounded in direction l_i. Clearly there are 2^n possible unbounded direction sets, in one-to-one correspondence with the power set of $\{1, 2, \dots, n\}$.

Lemma 4. *Let $A, B \in C(X)$ under the Hausdorff topology.*

1. *If $d_{C(X)}(A, B) < \infty$, then A and B have the same unbounded direction set.*
2. *If A and B have distinct unbounded direction sets, e.g. there exists a ray $R_i \in X$ such that A is unbounded in direction i but B is not, then there does not exist any path through $C(X)$ from A to B.*

Proof: If A is unbounded in direction i and B is not, then clearly $d_{C(X)}(A, B) = \infty$. Since any path is a continuous image of a compact set, it must have a compact image which contains A and B. If $d_{C(X)}(A, B) = \infty$, this is impossible. □

Lemma 5. *If $X \in \mathcal{X}$ is an N-legged graph, then the hyperspace $(C(X), \tau_H)$ has at least 2^N connected components.*

Proof: Let $X \in \mathcal{X}$ be a graph with N distinct rays, labelled $R_1, \dots, R_N$. Consider the power set $\mathcal{P}$ of $\{1, \dots, N\}$. Each element of the power set corresponds to an unbounded direction set. For example, if a set A is unbounded in directions $1, 3$ and 5, we would say it corresponds to the element of the power set $\{1, 3, 5\}$.

Define a map $f : C(X) \to \mathcal{P}$ by $f(A) = S$ if A is unbounded in the set of directions S. We will show that f is continuous, and therefore $C(X)$ has at least 2^N connected components.

Because $(C(X), \tau_H)$ is first countable, it is enough to show convergent sequences are mapped to convergent sequences. Let $A_n \to A$ be a convergent sequence of elements of $C(X)$, meaning that $d_{C(X)}(A_n, A) \to 0$ as $n \to \infty$. If A is unbounded in direction R_k, and A_n is not (or vice versa) we know $d_{C(X)}(A, A_n) = \infty$, so for all n greater than some n_0 we must have A_n unbounded in the same set of directions as A. Therefore $f(A_n) = f(A)$ for all $n > n_0$ and f is continuous. □

Remark: We will continue to refer to the map f from this lemma throughout the paper in order to easily describe members of $C(X)$ according to their unbounded direction set. Note that if $f(A) = f(B)$ then A and B have the same unbounded direction set.

Theorem 6. *If $X \in \mathcal{X}$ is an N-legged graph, then the hyperspace $(C(X), \tau_H)$ has exactly 2^N path-connected components.*

The previous lemma showed that $C(X)$ has at least 2^N connected components. We will now show that it has no more than that, by showing that for all $S \in \mathcal{P}$,

$\{A \in C(X) : f(A) = S\}$ is a path-connected set. This will be done by taking any element in a given component and constructing a path from it to a designated "default" element of that component.

Proof: We begin by choosing for each $S \in \mathcal{P}$ a particular element A^*_S of $\{A \in C(X) : f(A) = S\}$. The element will consist of the complete finite-graph X_G, and all the rays which are in the unbounded direction set, and no part of the other rays. Precisely,

$$A^*_S = X_G \cup \bigcup_{k \in S} R_k$$

Given an element $A \in C(X)$ with $f(A) = S$, we will construct a path from A to A^*_S. There are three steps. First, if A is contained completely in a ray, we grow it up until it reaches X_G. Secondly, we shrink back any sections of A which lie in rays that aren't in the unbounded direction set. Finally we grow A to include all of X_G.

First, suppose $A \cap X_G = \emptyset$, and hence $A \subset R_k$ for some k. Denote by v_k the vertex of X_G where R_k is attached. Pick $a \in A$. In this case we define the first step in the path as follows. $\phi_0 : [0, 1] \to C(X)$ is given by

$$\phi_0(t) = A \cup [a, a(t - 1) + v_k t]$$

The interval on the right lives in R_k.

If, on the other hand, $A \cap X_G \neq \emptyset$, let $\phi_0(t) = A$ be the constant path. In either case, let $\hat{A} = \phi_0(1)$ and notice that $\hat{A} \cap X_G \neq \emptyset$.

Next we define a path $\phi_1 : [0, 1] \to C(X)$ by

$$\phi_1(t) = (\hat{A} \cap X_G) \cup \bigcup_{j \in S} R_j \cup \bigcup_{j \notin S \text{ and } A \cap R_j \neq \emptyset} [v_j, tv_j + (t - 1)r_j]$$

where v_j is the vertex in X_G where R_j is attached, and r_j is the endpoint of the intersection of $\hat{A}$ with the ray R_j. This is a path from $\hat{A}$ to the set which agrees with $\hat{A}$ in X_G but does not contain any section of any rays which are not in A's unbounded direction set. Call this intermediate set $A' = \phi_1(1)$.

The final step will grow A' to include all of X_G. Fix an element $a \in A' \cap X_G$. Because X_G is a graph, it is path connected, so there exists a path $\gamma : [0, 1] \to X_G$ which starts at a and whose image contains all of X_G. Define

$$\phi_2(t) = (A' \cap X_G) \cup \bigcup_{j \in S} R_j \cup \bigcup_{x \in [0,t]} \gamma(x)$$

Clearly $\phi_2(0) = A'$ and $\phi_2(1) = A^*_S$. The continuity of γ makes ϕ_2 continuous, and the construction ensures $\phi_2(t) \in C(X)$ at all times. Following ϕ_0 with ϕ_1 and ϕ_2, we have a path from A to A^*_S. Hence the set $\{A \in C(X) : f(A) = S\}$ is path-connected. □

References

[1] G. Acosta, *Continua with unique hyperspace*, Continuum theory; proceedings of the special session in honor of Professor Sam B. Nadler, Jr.'s 60 birthday. **230** (2002), 33-49.

[2] G. Acosta, *Continua with almost unique hyperspace*, Top. App. **117** (2002), 175-189.

[3] R. Duda, *On the hyperspace of subcontinua of a finite graph I*, Fundamenta Mathematicae. **62** (1968), 265–286.

[4] R. Duda, *On the hyperspace of subcontinua of a finite graph II*, Fundamenta Mathematicae. **63** (1968), 225–255.

[5] C. Eberhart, S. Nadler, *Hyperspaces of cones and fans*, Proc. Amer. Math. Soc, **77** (1979), no. 2, 279–288.

[6] A. Illanes, *The hyperspace $C_2(X)$ for a finte graph is unique*, Glasnik Matematicki, **37** (2002), 347–363.

[7] A. Illanes, *Finite graphs X have unique hyperspaces $C_n(X)$*, Top. Proc., **27** (2003), 179-188.

[8] A. Illanes, S. Nadler, *Hyperspaces: Fundamentals and Recent Advances*, Marcel Dekker, Inc., New York, 1999.

Department of Mathematics, Stonehill College, Easton, Massachusetts 02357, nesty@stonehill.edu