ON THE INJECTIVITY RADIUS AND TANGENT CONES AT INFINITY OF GRADIENT RICCI SOLITONS

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Abstract. A lower-bound estimate of injectivity radius for complete Riemannian manifolds is discussed in a pure geometric viewpoint and is applied to study tangent cones at infinity of certain gradient Ricci solitons. We also study the asymptotic volume ratio of gradient Ricci solitons.

1. Introduction

The Ricci solitons, which are generalizations of the Einstein manifolds, are important solutions to the Ricci flow. Besides the advantage of having explicit equations, they occur in the analysis of blow-up limits near singularities. In this article, we only discuss the complete non-compact solitons, which are much more complicated than the compact ones. In the three dimensional case, the classification of shrinking solitons under some reasonable conditions leads to the performance of surgery. For higher dimensional cases, some results about the classification of solitons were obtained in the last four years, e.g. [7, 15, 17, 19, 5, 22]. These results were derived under various curvature assumptions such as locally conformally flat, constant scalar curvature, nonnegative Ricci curvature(for expanding solitons) or bounded nonnegative curvature operator(for shrinking solitons when $n = 4$.) In this article, we try to understand the geometry of solitons which are not Ricci-nonnegative.

Besides the studies on the classification, there are some results and conjectures about the non-existence. For example, B.-L. Chen and X.-P. Zhu proved that there exists no expanding soliton with nonnegative sectional curvature and ϵ-pinched Ricci curvature, i.e. $Ric \geq \epsilon Rg$, in [9]. Here, and afterwards, R stands for the scalar curvature. This existence problem is still open if we discard the condition on the sectional curvature.

On the other hand, there are some classical non-existence theorems about general complete Riemannian manifolds. For example, there exists no manifold with $Ric \geq 0$, $|Sect| \leq C \cdot dist(O, x)^{-2-\varepsilon}$ and $Vol(B_s) \geq Cs^n$ for all geodesic balls B_s with radius s and center O, where we use C to denote various constants. This was proved by S. Bando, A. Kasue and H. Nakajima in [2]. Another non-existence result due to R. E. Greene and H. Wu [13] and G. Drees [11] states that there exists no manifold with positive sectional curvature and $\lim_{dist(O,x)\to\infty} R \cdot dist(O, x)^2 = 0$ except for $n = 4$ or 8. An approach to achieve these non-existence results is to study the tangent cones at infinity of such manifolds. Indeed, we prove that if a nonflat nonsteady Ricci soliton M satisfies $|Sect| \leq C \cdot dist(O, x)^{-2-\varepsilon}$ and the *non-accumulated property* which is stated in Section 4, then each tangent cone at infinity of M is the Euclidean space $\mathbb{R}^n$. Here we assume that the soliton has only one end and is simply connected at infinity with dimension $n \geq 3$. (These were also assumed in the article of Bando, Kasue and Nakajima.) To prove this, we use a new injectivity radius estimate which is derived in

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Section 4. Such estimate is developed to avoid the usage of the nonnegativity of Ricci curvature.

We should mention that there were other results about the non-existence of Ricci solitons. For example, S. Pigola, M. Rimoldi and A. G. Setti [20] proved that a nonflat expanding soliton (M, g, f) cannot satisfy either $|\nabla f| \in L^p(M, e^{-f}dvol)$ or $0 \le R \in L^p(M, e^{-f}dvol)$ for some $1 \le p \le \infty$. Note that, up to today, the growth of f is unknown for expanding solitons unless we have some control on the Ricci curvature.

In the next section, we study the behavior of f on solitons with $|Ric| \le C \cdot dist(O, x)^{-\varepsilon}$. In Section 3, we derive lower bounds of the asymptotic volume ratio for expanding solitons with $\frac{1}{Vol(B_s)} \int_{B_s} R \ge -Cs^{-\varepsilon}$ and shrinking solitons with $\frac{1}{Vol(B_s)} \int_{B_s} R \le Cs^{-\varepsilon}$. In Section 4, we derive an lower bound estimate about the injectivity radius. Then, in Section 5 and 6, we apply this estimate to certain Ricci solitons and study the tangent cones at infinity of solitons with fast curvature decay.

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2. The growth of f on Ricci solitons with certain curvature decay

Let M be a complete non-compact expanding gradient Ricci soliton, which satisfies the following equation:

$$R_{ij} + \nabla_i \nabla_j f = -g_{ij},$$

and R be the scalar curvature of M. The following two lemmas are well-known.

Lemma 1. (R. Hamilton, [14]) *We have $R + |\nabla f|^2 + 2f = C_1$ for some constant C_1 which can be absorbed by f.*

Lemma 2. (B.-L. Chen, [7]) *We have $R \ge -C_2$ for some constant $C_2 > 0$.*

Given a fixed point $O \in M$, we set $s = dist(O, x)$ and $\gamma(s)$ be a unit-speed minimizing geodesic connecting O and x, where $x \in M$ is chosen arbitrarily. We use the notation $'$ to denote the differentiation with respect to s along $\gamma(s)$. The following proposition, which seems to appear first time in the literature in [21], is an easy consequence of Lemmas 1 and 2.

Proposition 1. *For every expanding soliton M, we have $|f'(x)| \le |\nabla f(x)| \le s + L(O)$, where $L(x) = \sqrt{C_1 + C_2 - 2f(x)} = \sqrt{C_2 + R(x) + |\nabla f(x)|^2}$. Moreover, when $Ric \ge 0$, we have $f'(x) \le -s + f'(O)$.*

Proof. Since $-C_2 + |\nabla f|^2 + 2f \le R + |\nabla f|^2 + 2f = C_1$, we have $|\nabla f| \le \sqrt{C_1 + C_2 - 2f} = L$. Combining with $\nabla L = \frac{-\nabla f}{\sqrt{C_1 + C_2 - 2f}}$, we have $|\nabla L| \le 1$.

Integrating it from the point O to some point $x = \gamma(s)$ along γ, we have

$$L(x) - L(O) = \int_0^s L' \le \int_0^s |\nabla L| \le s.$$

Hence, $|f'(x)| \le |\nabla f(x)| \le L(x) \le s + L(O)$.

When $Ric \ge 0$,

$$\int_0^s f'' \le \int_0^s Ric(\gamma', \gamma') + \int_0^s f'' = -s$$

implies that $f'(x) \leq -s + f'(O)$. □

From this proposition, it is easy to see that for every expanding gradient Ricci soliton which has nonnegative Ricci curvature, the potential function $f(x)$ must decrease quadratically in s. The following theorem shows that this property holds for expanding solitons whose Ricci curvatures may be nonpositive.

Theorem 1. *If $|Ric| \leq Cs^{-\varepsilon}, s \equiv dist(O,x)$, for some constant $\varepsilon < 1$ and some point $O \in M$, then there exists a point $p \in M$ and $C_3, C_4 > 0$ such that $|Ric| \leq C_3 \cdot dist(p,x)^{-\varepsilon}$ and f satisfies*

$$-r\left(1+\frac{C_4}{r^{\varepsilon}}\right) \leq f'(x) \leq -r\left(1-\frac{C_4}{r^{\varepsilon}}\right),$$

where $r = dist(p,x)$. As a consequence, we have

$$-\frac{1}{2}r^2\left(1+\frac{C_5}{r^{\varepsilon}}\right) + f(p) \leq f(x) \leq -\frac{1}{2}r^2\left(1-\frac{C_5}{r^{\varepsilon}}\right) + f(p).$$

Proof. From

$$-C\int_0^s s^{-\varepsilon} + \int_0^s f'' \leq \int_0^s Ric(\gamma',\gamma') + \int_0^s f'' = \int_0^s -1 \leq C\int_0^s s^{-\varepsilon} + \int_0^s f'',$$

we have

$$-s - C\int_0^s s^{-\varepsilon} \leq \int_0^s f'' \leq -s + C\int_0^s s^{-\varepsilon}$$

and hence

$$-s\left(1+\frac{C_4}{s^{\varepsilon}}\right) + f'(O) \leq f'(x) \leq -s\left(1-\frac{C_4}{s^{\varepsilon}}\right) + f'(O).$$

In order to achieve the conclusion, it is enough to show that f has a critical point p (and then repeat the calculation above.) This can be observed by considering the geodesic sphere $\partial B_s(O)$ with s very large. Since $\nabla f \cdot \nabla s$ is negative on such sphere, ∇f must point inwards. So $\nabla f = 0$ at some point p inside the ball $B_s(O)$.

□

We recall that the potential function grows quadratically on every shrinking gradient Ricci soliton. This was proved by H.-D. Cao and D.T. Zhou in [3]. Moreover, by using the same proof in Theorem 1, we have $r\left(1-\frac{C_4}{r^{\varepsilon}}\right) \leq f'(x) \leq r\left(1+\frac{C_4}{r^{\varepsilon}}\right)$ and $\frac{1}{2}r^2\left(1-\frac{C_5}{r^{\varepsilon}}\right) + f(p) \leq f(x) \leq \frac{1}{2}r^2\left(1+\frac{C_5}{r^{\varepsilon}}\right) + f(p)$ for shrinking solitons which satisfy $R_{ij} + \nabla_i\nabla_j f = g_{ij}$ and $|Ric| \leq C \cdot dist(O,x)^{-\varepsilon}$.

Remark 1. The condition $|Ric| \leq Cs^{-\varepsilon}$ in Theorem 1 can be replaced by $|Ric(\gamma',\gamma')| \leq Cs^{-\varepsilon}$ for all γ starting from O. It is worthy to distinguish these two conditions because a cigar-like manifold may satisfy the second condition while breaks the first one.

3. Asymptotic volume ratio of Ricci solitons with ε curvature decay

It was mentioned in [8] that a complete non-compact expanding Ricci soliton with $0 \leq Ric \leq C$ must have positive asymptotic volume ratio, which was proved by Hamilton. The same result was proved in [4] by assuming the weaker condition that the scalar curvature is nonnegative. We now can weaken the curvature condition to be

$\frac{1}{Vol(B_s)}\int_{B_s} R \geq -Cs^{-\varepsilon}$, where $B_s \subset M$ always denotes the geodesic ball with central point O and radius s.

Theorem 2. *Let (M, g, f) be a complete non-compact expanding gradient Ricci soliton with scalar curvature R. If there exists $O \in M$ such that $\frac{1}{Vol(B_s)}\int_{B_s} R \geq -Cs^{-\varepsilon}$, where $\varepsilon > 0$ is a constant, then its asymptotic volume ratio is bounded below by a positive constant η.*

Proof. Taking the trace of the soliton equation $R_{ij} + \nabla_i\nabla_j f = -g_{ij}$ and integrating it on B_s, we have

$$-nVol(B_s) = \int_{B_s} R + \int_{B_s} \Delta f = \int_{B_s} R + \int_{\partial B_s} \nabla f \cdot \nabla s \geq \int_{B_s} R - \int_{\partial B_s} (s + L(O))$$
$$= \int_{B_s} R - (s + L(O))Area(\partial B_s) = \int_{B_s} R - (s + L(O))\frac{d}{dr}Vol(B_s).$$

Therefore,

$$\frac{d}{ds}\log Vol(B_s) \geq \frac{1}{(s + L(O))Vol(B_s)}\int_{B_s} R + \frac{n}{s + L(O)}$$
$$= \frac{1}{(s + L(O))Vol(B_s)}\int_{B_s} R + \frac{d}{ds}\log(s + L(O))^n$$

$$\Rightarrow \frac{d}{ds}\log\frac{Vol(B_s)}{(s + L(O))^n} \geq \frac{1}{(s + L(O))Vol(B_s)}\int_{B_s} R \geq \frac{-C}{(s + L(O))s^{\varepsilon}} \geq \frac{-C}{s^{1+\varepsilon}}$$
$$\Rightarrow \log\frac{Vol(B_s)}{(s + L(O))^n} \geq \int_{\rho}^{s} \frac{-C}{s^{1+\varepsilon}} + \log\frac{Vol(B_\rho)}{(\rho + L(O))^n} = \frac{C}{\varepsilon}s^{-\varepsilon} - \frac{C}{\varepsilon}\rho^{-\varepsilon} + \log\frac{Vol(B_\rho)}{(\rho + L(O))^n}$$

for any positive constant $\rho < s$

$$\Rightarrow \frac{Vol(B_s)}{(s + L(O))^n} \geq \left(e^{\frac{C}{\varepsilon}s^{-\varepsilon} - \frac{C}{\varepsilon}\rho^{-\varepsilon}}\right)\frac{Vol(B_\rho)}{(\rho + L(O))^n} \geq e^{-\frac{C}{\varepsilon}\rho^{-\varepsilon}} \cdot \frac{Vol(B_\rho)}{(\rho + L(O))^n}.$$

Hence,

$$\lim_{s\to\infty}\frac{Vol(B_s)}{s^n} \geq e^{-\frac{C}{\varepsilon}\rho^{-\varepsilon}} \cdot \frac{Vol(B_\rho)}{(\rho + L(O))^n} \equiv \eta > 0.$$

□

For shrinking gradient Ricci solitons, the same calculation gives the following theorem.

Theorem 3. *Let (M, g, f) be a complete non-compact shrinking gradient Ricci soliton which satisfies $R_{ij} + \nabla_i\nabla_j f = g_{ij}$. If there exists $O \in M$ such that $\frac{1}{Vol(B_s)}\int_{B_s} R \leq Cs^a$, where a is a nonzero constant, then its volume ratio is bounded from below by $C \cdot e^{\frac{-1}{a}s^a}$ for s large enough. When $\frac{1}{Vol(B_s)}\int_{B_s} R \leq \delta < n$, we have $Vol(B_s) \geq C \cdot s^{n-\delta}$ for s large enough.*

Remark 2. A similar result to the case $a = 0$ was proved by H.-D. Cao and D.-T. Zhou in [3].

4. Smooth geodesic loops and injectivity radius estimate

Given a Riemannian manifold M and $O \in M$, we denote $s := dist(O, x)$ for an arbitrary point $x \in M$ and introduce the following condition.

Smooth Loop Condition. *There are constants c_0 and d_0 such that there exists no smooth geodesic loop passing through x with length less than $c_0 \cdot s$ when $s \geq d_0$.*

Here we recall some fundamental properties of cut points. If $y \in M$ is a cut point of $x \in M$, then either y is conjugate to x or there exists a geodesic loop γ which passes through x and y. In the second case, γ is composed by two minimizing geodesics from x to y. If we assume that y is a nearest cut point of x, then the only possible singular point of γ is x. We say that such y realizes the injectivity radius of x via γ. Hence the above condition means that, if the injectivity radius of a point x is small, then there exists another point y which either is conjugate to x or realizes the injectivity radius of x via a *nonsmooth* γ.

In order to study nonsmooth geodesic loops, we develop the following notion: geodesic chains.

Definition 1. *If a (finite) sequence of points $\{x^{(i)}\}_{i=0}^m \subset M$ satisfies that each $x^{(i)}, i = 1, \ldots, m$, realizes the injectivity radius of $x^{(i-1)}$ via some geodesic loop $\gamma^{(i)}$, then such points and loops together is called a geodesic chain. We denote it as $G(x^{(0)}, \ldots, x^{(m)})$.*

A manifold M is said to satisfy the *non-accumulated property* if for all $D > 0$, there exists a positive integer n_0 such that $\frac{dist(x^{(0)}, x^{(n_0)})}{inj(x^{(0)})} > D$ for all $x^{(0)} \in M$ and all geodesic chains $G(x^{(0)}, \ldots, x^{(m)}) \subset M$ satisfying that $G(x^{(0)}, \ldots, x^{(m)}) \setminus B_{2D \cdot inj(x^{(0)})} \neq \phi$.

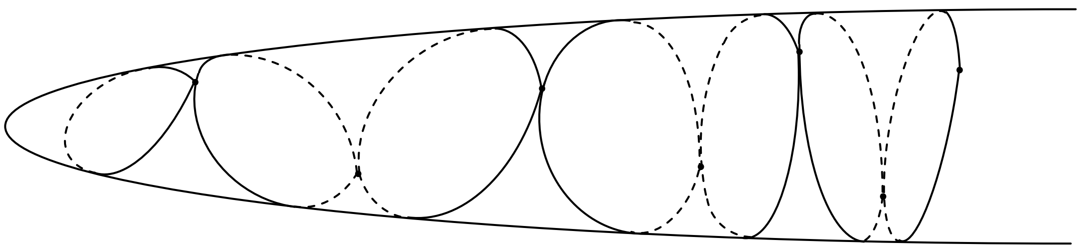

Fig. 1. A cylinder-like end does not satisfy the non-accumulated property.

Theorem 4. *Let M be a complete Riemannian manifold satisfying $|Sect| \leq C \cdot s^{-2}$, where $s := dist(O, x)$. If M satisfies the smooth loop condition and the non-accumulated property, then there exists a constant $\delta > 0$ such that $inj(x) \geq \delta \cdot s$ for all $x \in M$.*

Proof. Let $q_k \in M$ and $\lambda_k := \frac{1}{2} dist(O, q_k) \to \infty$. For $x \in B_{\lambda_k}(q_k)$, we want to show that $inj(x) \geq \delta \cdot dist(x, \partial B_{\lambda_k}(q_k))$. If this is the case, then the lemma follows by taking $x = q_k$.

We argue by contradiction. Suppose that there exist $\delta_k \searrow 0$ and $x_k \in B_{\lambda_k}(q_k)$ such that $inj(x_k) = \delta_k \cdot d_k$, where $d_k := dist(x, \partial B_{\lambda_k}(q_k))$. Furthermore, we may assume that the function $F(y) := \frac{inj(y)}{dist(y, \partial B_{\lambda_k}(q_k))}$, $y \in B_{\lambda_k}(q_k)$, achieves its minimum at x_k. Hence

$$inj(y) = F(y) \cdot dist(y, \partial B_{\lambda_k}(q_k)) \geq F(x_k) \cdot \frac{1}{2} dist(x_k, \partial B_{\lambda_k}(q_k)) = \frac{1}{2} inj(x_k)$$

for all $y \in B_{\frac{1}{2}d_k}(x_k)$.

Let $\widetilde{g}_k := (\delta_k d_k)^{-2} g$ and consider the sequence of rescaled pointed geodesic balls $\left(\widetilde{B}_{\frac{1}{2\delta_k}}(x_k), x_k, \widetilde{g}_k\right)$. Since $|\widetilde{Sect}| \leq C \cdot \lambda_k^{-2} \delta_k^2 d_k^2 \to 0$ and $\widetilde{inj} \geq \frac{1}{2}$ on $\widetilde{B}_{\frac{1}{2\delta_k}}(x_k)$, by using the harmonic coordinates, we know that the sequence converges to a complete flat manifold (B, x_∞, g_∞) in $C^{1,\sigma} \cap L^{2,p}$-topology (for all p and $\sigma \in (0,1)$). For the usage of the harmonic coordinates, one can consult, for example, [1, 18].

Notice that the flat limit manifold B is non-compact because $diam\left(\widetilde{B}_{\frac{1}{2\delta_k}}\right) \to \infty$. So it might be $\mathbb{R}^{n-1} \times \mathbb{S}^1$ or $\mathbb{R}^{n-k} \times \mathbb{F}^k$, where $\mathbb{F}^k$ is a Bieberbach manifold. ($inj(x_\infty) = 1$ implies that $B \neq \mathbb{R}^n$.) For later use, we denote D as the diameter of one slice of B, that is, $\mathbb{S}^1$ or $\mathbb{F}^k$.

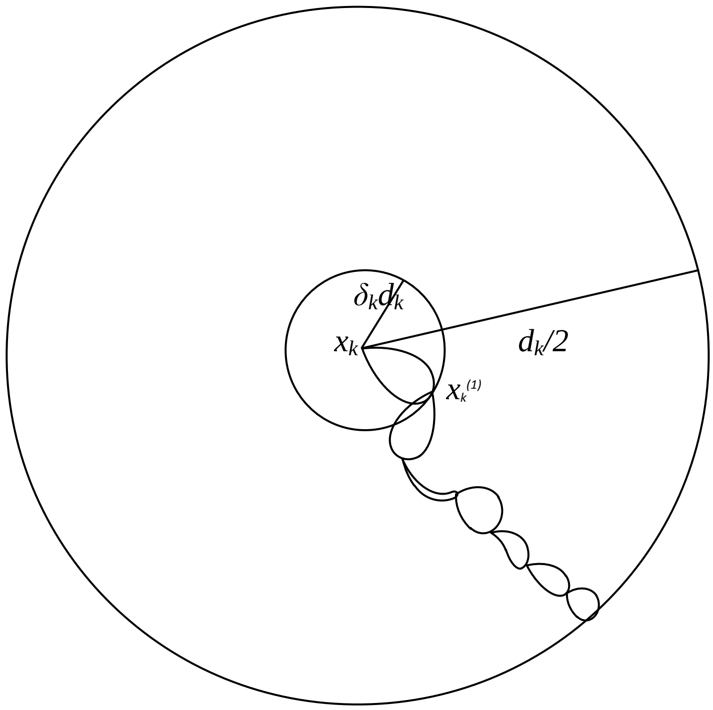

Fig. 2. Construct a geodesic chain from each $x_k^{(0)} \equiv x_k$.

In the rest of the proof, we show that none of these cases can happen, hence a contradiction arises. Consider a point $x_k^{(1)}$ which realizes the injectivity radius of $x_k^{(0)} := x_k$. By Klingenberg's lemma and the assumption on the sectional curvature, there exists a geodesic loop $\gamma_k^{(1)}$ passing through $x_k^{(1)}$ and x_k with length $2\delta_k d_k$. Since the loop is smooth at $x_k^{(1)}$, by the smooth loop condition, this loop is not smooth at x_k. This implies that $inj(x_k^{(1)}) < inj(x_k)$.

So we can find another point $x_k^{(2)}$ which realizes the injectivity radius of $x_k^{(1)}$. This process can continue until some point $x_k^{(m_k)}$ has its nearest cut point $x_k^{(m_k+1)} \notin B_{\frac{1}{2}d_k}(x_k)$.

(Note that for every real number $D > 0$ and k large enough, there is a integer $m'_k < m_k$ such that $x_k^{(m'_k)} \notin B_{D\delta_k d_k}(x_k)$. A priori, m'_k depends on k. However, the non-accumulated condition acclaims that there exists a number n_0 such that $x_k^{(n_0)} \notin B_{D\delta_k d_k}(x_k)$ for all k. We shall use this in the next paragraph.)

Now, on each rescaled ball $\widetilde{B}_{\frac{1}{2\delta_k}}(x_k)$ we have a geodesic chain $G(x_k^{(0)} \equiv x_k, \ldots, x_k^{(m_k)}, \ldots)$. Exactly, we have a finite sequence of points $\{x_k^{(i)}\}_{i=0}^{m_k}$ and geodesic loops $\{\gamma_k^{(i)}\}_{i=1}^{m_k}$ with lengthes $\left|\gamma_k^{(i)}\right| \geq \frac{1}{2}, \forall i = 1, \ldots, m_k$. We want to take a subsequential limit of these chains into B (and derive a contradiction). There are two possibilities: either there are two limit points $x_\infty^{(i-1)}$ and $x_\infty^{(i)}$ lying in different slices, or all the points accumulate to the same slice $\{x_\infty\} \times \mathbb{F}^k$ (or $\{x_\infty\} \times S^1$). By the non-accumulated condition, there is a limit point $x_\infty^{(n_0)}$ such that $dist(x_\infty^{(n_0)}, x_\infty) > 2D$ where D is the diameter of one slice of B. Hence the second case shall be ruled out.

The first case is also impossible. Indeed, if there exists a geodesic loop $\gamma_\infty^{(i)}$ which is not contained in the slice $\{x_\infty^{i-1}\} \times \mathbb{F}^k$ (or $\{x_\infty^{i-1}\} \times S^1$) of B, then we can project it to get a strictly shorter geodesic loop which is contained in $\{x_\infty^{i-1}\} \times \mathbb{F}^k$ (or $\{x_\infty^{i-1}\} \times S^1$). This contradicts the fact that $inj(x_\infty^{(i-1)}) = \frac{1}{2}\left|\gamma_\infty^{(i)}\right|$. □

5. Geodesic loops on gradient Ricci solitons

In the following theorem, we find out some Ricci solitons which satisfy the smooth loop condition.

Theorem 5. *Consider a gradient Ricci soliton M which satisfies $R_{ij} + \nabla_i\nabla_j f = \lambda g_{ij}$.*

Let $h : M \to \mathbb{R}$ be a nonnegative function such that $h(x) \to 0$ as $s \to \infty$. If one of the following three conditions holds:

(i) $\lambda = 1$ and $Ric \leq h \cdot g$;

(ii) $\lambda = 0$ and $Ric > 0$;

(iii) $\lambda = -1$ and $Ric \geq -h \cdot g$,

then M contains no smooth geodesic loop outside a compact set K (K is empty for case (ii)). In particular, M satisfies the smooth loop condition.

Proof. Suppose that there is a smooth geodesic loop $\gamma \subset M \setminus B_s(O)$ whose length is denoted by l. Integrating the equation of soliton on γ, we have

$$\lambda l = \int_\gamma \lambda|\gamma'|^2 = \int_\gamma Ric(\gamma', \gamma') + \int_\gamma f'' = \int_\gamma Ric(\gamma', \gamma').$$

This contradicts all the three conditions. □

Remark 3. It is easy to see that this theorem holds for non-gradient solitons. On the other hand, the condition $Ric \leq h \cdot g$ (resp. $Ric \geq -h \cdot g$) can be replaced by $Ric < \lambda \cdot g$ (resp. $Ric > \lambda g$) on the ends of M. Note that the condition $Ric < \lambda \cdot g$ on a shrinking soliton is equivalent to the convexity of f.

From our method developed in the previous section, we can prove some noncollapsing properties on solitons which may not have bounded curvature.

Corollary 1. *Let M be a gradient Ricci soliton satisfing $R_{ij} + \nabla_i\nabla_j f = \lambda g_{ij}$ and the non-accumulated property. Suppose $|Sect| \leq \frac{1}{r^2}$ on $B_r(x) \subset M$ for some $x \in M$ and $r > 0$. If*

(i) $\lambda > 0$ and $Ric < \lambda \cdot g$ on $B_r(x)$,
(ii) $\lambda = 0$ and $Ric > 0$ on $B_r(x)$ or
(iii) $\lambda < 0$ and $Ric > \lambda \cdot g$ on $B_r(x)$,
then $inj(x) \geq \kappa r$ for some constant $\kappa > 0$.

Remark 4. In [15], A. Naber proved that every n-dimensional shrinking soliton with bounded curvature and $n \geq 2$ is κ-noncollapsed.

6. Tangent cones at inifinity of gradient Ricci solitons

Using the estimate of the injectivity radius, we can blow down a manifold to get its tangent cone at infinity. When the sectional curvature decays faster than quadratically on a nonsteady Ricci solitons, we have the following theorem.

Theorem 6. *Let M be a complete non-compact gradient Ricci soliton which satisfies $R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$, where $\lambda = 1$ or -1, and $|Sect| \leq C \cdot dist(O, x)^{-2-\varepsilon}$. Furthermore, M is assumed to satisfy the non-accumulated property. If M is simply connected at infinity, one-ended and has dimension $n \geq 3$, then every tangent cone at infinity of M is the Euclidean space $\mathbb{R}^n$.*

Proof. Consider a tangent cone at infinity M^∞, which is a Gromov-Hausdorff limit of a sequence $(M, O, \widetilde{g}_k) := (M, O, \frac{1}{\lambda_k^2} g)$ with vertex O, where $\lambda_k \to \infty$ as $k \to \infty$. Here we use a tilde to emphasize that the metric is rescaled. Any arbitrary point $q \in M^\infty, q \neq O$ and $dist_\infty(O, q) = r_0$, is associated with a sequence $q_k \to q$, where $dist_k(O, q_k) = \lambda_k r_0 \to \infty$ as $k \to \infty$. By using our injectivity radius estimate, Hamilton's compactness theorem and Shi's estimate, the convergence is in fact in C^∞_{loc}-topology.

Noting that $\left|\widetilde{\nabla}_i \widetilde{\nabla}_j f_k\right| = \left|(\widetilde{g}_k)_{ij} - \frac{1}{\lambda \lambda_k^2} (\widetilde{Ric_k})_{ij}\right|$, together with the estimates of the growth of f and ∇f which are stated in Section 2, we know that $f_k := \frac{f}{\lambda \lambda_k^2}$ converges in C^∞_{loc}-topology to a function f^∞ with $|\nabla f| = r$ on $M^\infty \backslash \{O\}$. Moreover, $\nabla^\infty \nabla^\infty f^\infty = g^\infty$ and $f^\infty(q) = \lim_{k \to \infty} \frac{f}{\lambda \lambda_k^2}(q_k) = \frac{1}{2} r_0^2$. Since q was chosen arbitrarily, we have

$$f^\infty(x) = \frac{1}{2} r^2 \quad \text{and} \quad g^\infty = Hess\left(\frac{1}{2} r^2\right)$$

where $r(x) := dist_\infty(O, x)$ and $x \in M^\infty \setminus \{O\}$.

In [6], J. Cheeger and T. H. Colding have proven that $M^\infty \setminus \{O\}$ with $g = Hess(\frac{r^2}{2})$ must be a warped product manifold and $g = dr^2 + cr^2 \bar{g}$ for some $c > 0$, where $\bar{g}$ is the metric of $N := \{x \in M^\infty | r(x) = 1\}$. In order to prove that M^∞ is isometric to $\mathbb{R}^n$, we only need to show that N is the standard sphere with sectional curvature c. (Because the standard metric on $\mathbb{R}^n$ can be written as $g_{Eucl} = dr^2 + Cr^2 g_{\mathbb{S}^{n-1}(C)}$ for any given $C > 0$ and $g_{\mathbb{S}^{n-1}(C)}$ denotes the standard metric on sphere with sectional curvature C.)

Since $|\nabla r| \neq 0$, we can extend the normal coordinate $\{x^i\}_{i=2,\ldots,n}$ around $p \in N$ to be a local coordinate $\{r, x^i\}_{i=2,\ldots,n}$ in M such that

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n2} & \cdots & g_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & cr^2 \bar{g}_{22} & \cdots & cr^2 \bar{g}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & cr^2 \bar{g}_{n2} & \cdots & cr^2 \bar{g}_{nn} \end{pmatrix}.$$

Hence, for all $i, j = 2, \ldots, n$ and $i \neq j$, we have $\Gamma^r_{jj}(p) = -c$ and $\Gamma^r_{ij}(p) = 0$. Moreover, $\frac{\partial}{\partial x^j}(g(\frac{\partial}{\partial r}, \frac{\partial}{\partial x^j})) = 0$ implies that $\Gamma^j_{jr}(p) = -\frac{1}{c}\Gamma^r_{jj}(p) = 1$. When $n \geq 3$, we can compute the curvature of N at p by using

$$0 = R^i_{ijj} = \bar{R}^i_{ijj} + \Gamma^i_{ir}\Gamma^r_{jj} = \bar{R}^i_{ijj} - c.$$

Because M is simply connected at infinity, N must be the standard sphere with all its sectional curvatures equal c. □

Remark 5. For the two-dimensional case, there exists no nonflat shrinking soliton with $|Sect| \leq C \cdot dist(O, x)^{-2-\varepsilon}$. This is an easy consequence of L. Ni's theorem [16] which states that the scalar curvature R of a shrinking soliton with nonnegative Ricci curvature must have a positive lower bound. Even in the three-dimensional case, Ni's result works because all the three-dimensional shrinking solitons have nonnegative sectional curvatures. (This was proved by B.-L. Chen [7].) On the other hand, there exists a two-dimensional counter-example for the expanding case, i.e. an expanding soliton which has faster-than-quadratic-decay curvature and a tangent cone at infinity which is not an Euclidean plane. Such soliton was constructed in [8] by smoothly extending a cone manifold which had been conceived in [12].

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